Appendix B – Reliability Theory

One easy means for reliability analysis is to decompose a system into functional blocks in which series and parallel structures are very likely.

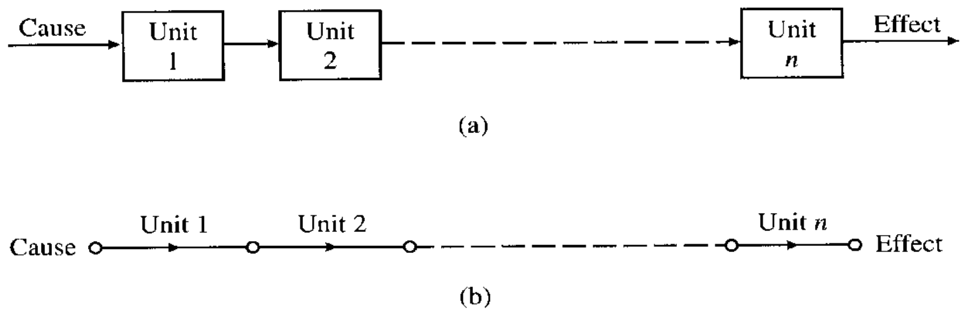


Figure B1 Series reliability configuration: (a) reliability block diagram (RBD); (b) reliability graph.

Single path, all units must operate successfully for P_s – system success

The no-assumption formulation for a series configuration contains conditional probabilities presuming that units do interact in terms of reliability

$P_s = P(x_1)P(x_2|x_1)P(x_3|\, x_1\, x_2) \,\ldots\ldots\, P(x_n|\, x_1\, x_2 \ldots\ldots x_{n-1})$ $\qquad P_{success} = P(A_1A_2 \ldots A_n)$ intersection

$P_f = 1 - P_s = P(\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \,\ldots\ldots + \overline{x}_n)$ $\qquad P_{failure}$

Note that system reliability can be viewed from P_f or P_s and in complex structures both approaches may be used at different stages of analysis. In this situation, P_f is much simpler than P_s but still expands into conditional probabilities if individual $P(x_i)$ are to be evaluated.

But if units do not interact and the **failures are independent** then

$P_s = P(x_1)P(x_2)P(x_3) \,\ldots\ldots\, P(x_n)$ Note: $P_s <$ then smallest P(xi) → the weakest link in the chain then

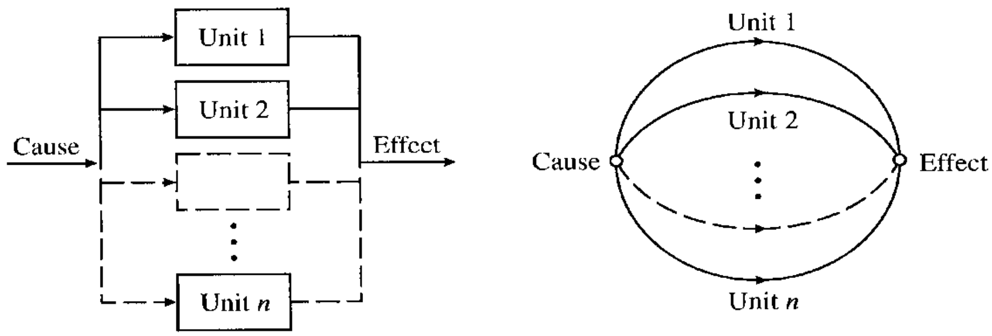
$$R(t) = P(x_1)P(x_2)P(x_3) \,\ldots\ldots\, P(x_n) = \prod_{i=1}^{n} R_i(t)$$

which for constant failure rates (λ)

$$R(t) = \prod_{i=1}^{n} e^{-\lambda t} = \exp\left(-\sum_{i=1}^{n} \lambda_i t\right)$$

BUT three big assumptions: series configuration, independence, constant-hazard model
Equation B58 → linearly increasing hazard Equation B59 → combination of both

The series reliability structure is the worst-case or lower-bound configuration.



(a) (b)

Figure B2 Parallel reliability configuration: (a) reliability block diagram; (b) reliability graph.

For a system failure, all units must fail thus $P_f = P(x_1 + x_2 + x_3 \ldots\ldots + x_n)$ P(Union)

$P_s = 1 - P_f = 1 - P(x_1 + x_2 + x_3 \ldots\ldots + x_n) = 1 - P(\bar{x}_1 \bar{x}_2 \bar{x}_3 \ldots\ldots \bar{x}_n)$

which results in a large number of conditional probabilities when the intersection terms are expanded as demonstrated in the series configuration shown above.

But if the unit failures are **independent** then

$P_s = 1 - P(\bar{x}_1)P(\bar{x}_2)P(\bar{x}_3) \ldots\ldots P(\bar{x}_n)$ seeing that $P(\bar{x}_i)\ldots P(\bar{x}_n) = P_f = 1 - R(t)$

$P_s = 1 - P_f = 1 - [1 - R(t)] = 1 - \prod_{i=1}^{n} [1 - R_i(t)]$ which for constant-hazard $= 1 - \prod_{i=1}^{n} (1 - e^{-\lambda t})$

Equation B63 → for linearly increasing hazard Equation B64 → for the general case

Summary for independent failures:

Series $R(t) = \prod_{i=1}^{n} R_i(t)$ Parallel $R(t) = 1 - \prod_{i=1}^{n} [1 - R_i(t)]$

a short cut for just 2 units in Parallel:
$R = R_1 + R_2 - R_1 R_2$

Constant failure rates (hazards) $z(t) = \lambda$ and $R(t) = e^{-\lambda t}$
Linearly Increasing Hazards $z(t) = Kt$ and $R(t) = e^{-kt^2/2}$
Weibull (all of the above + deceasing) $z(t) = Kt^m$ for $m > -1$ and $R(t) = e^{-kt^{m+1}/(m+1)}$

r-out-of-n System Configuration

A system configuration where r units out of n total units must operate for system success.

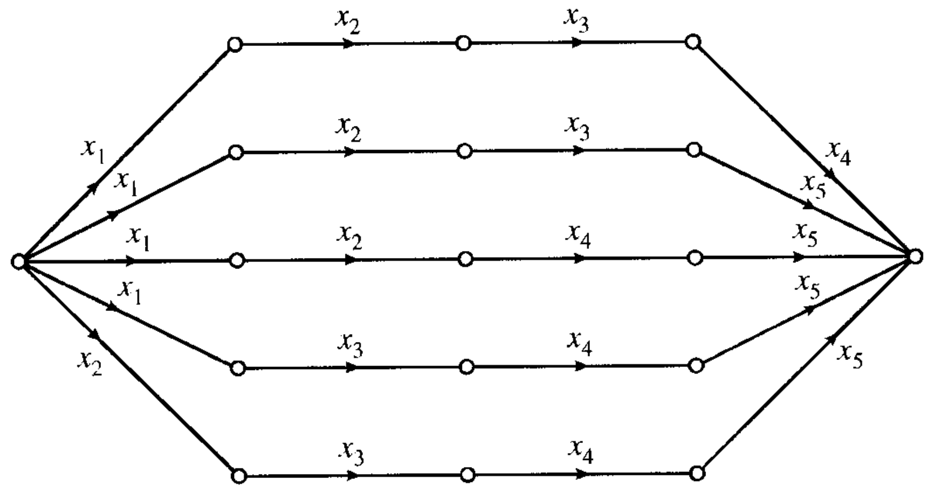


Figure B3 Reliability graph for a 4-out-of-5 system.

The above reliability graph shows five paths where 4 of the 5 units must be operational for the system to be successful (P_s).

$P_s = P(x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4\,x_5)$ (Equation B17)

Solving B17 requires simplification of redundant terms see B18

If the n units are **identical** and their failures are independent, then the binomial distribution is applicable (Equation B68):

$$P_s = \sum_{k=r}^{n} \binom{n}{k} [R(t)]^k\,[1 - R(t)]^{\,n-k}$$

where the R(t) is given on the previous page for the failure rates of constant, linearly increasing and all possible rates (+, -, constant). R(t) may be a number such as the probability of success or as inferred in this equation, a function of time.

When r = 1 (one unit must operate for system success) the configuration becomes a parallel system (*upper bound*). When all units must operate (r = n), then the configuration becomes a series system (*lower bound*). Thus both the series and parallel systems are subclasses of the r-out-of-n system.

If the r-out-of-n model requires r modules for success, then the system can tolerate n – r failures. Example: a triple modular redundancy (TMR) system requires two out of three modules to function in order for the system to operate. A TMR system can thus tolerate a single module failure out of the three modules in the system; it continues to work if all three modules or just two of the three modules function.

Fault-Tree Analysis – root node is the top undesired event and the diagram 'branches' are the secondary undesired events leading to the top node. The process is continued until basic events are reached which are called elementary faults.

A fault tree for a brake system is shown in Figure B13

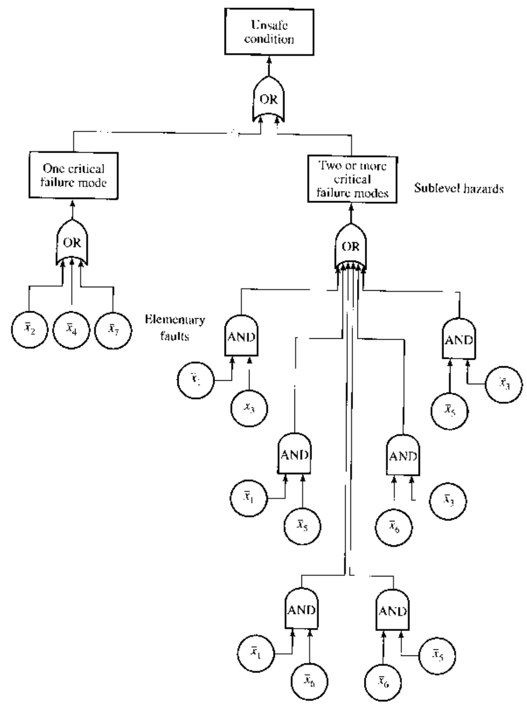


Figure B13 Safety fault tree for simplified brake example. Presence of any x_i means failure mode i *does* occur, indicating failure of element i.

The safety analysis consists of:

1. Decomposing the system into subsystems or piece-parts.
2. Drawing a safety block diagram (SBD) or fault tree (FT) (computer programs are available for this purpose). See Appendix D.
3. Computation of the probability of safety from the Safety Block Diagram (SBD) or Fault Tree (FT) (computer programs are also available for this purpose).
4. Determining the failure rates of each component element. This is a data collection and estimation problem.
5. Substitution of failure rates into the expression obtained in step 3 (also done by computer programs).

Failure Mode and Effect Analysis (FMEA) – a systematic procedure for identifying possible failure modes and evaluating their consequences. The basic questions:

1. How can each component or subsystem fail? What is the failure mode?
2. What cause might produce this failure? What is the failure mechanism?
3. What are the effects of each failure if it does occur.

A FMEA for the brake system is shown in Table B3 on page 443.

Cut-Set and Tie Set Methods

Tie Set – a group of branches which forms a connection between input and output traversed in a given direction (→). Minimal tie sets contain a minimum number of elements between input/output (cause/effect) for system success (P_s).

System Reliability R(t) from minimal Tie Set $R = P(T_1 + T_2 + \ldots\ldots + T_3)$

Cut Set – a set of branches when removed from the graph interrupts any possible connection between the input and output. The probability of system failure is the probability that at least one minimal cut set fails.

System Failure $P_f = P(\overline{C}_1 + \overline{C}_2 + \overline{C}_3 + \ldots\ldots.. + \overline{C}_j)$

System Reliability from cut set $R = 1 - P_f = 1 - P(\overline{C}_1 + \overline{C}_2 + \overline{C}_3 + \ldots\ldots.. + \overline{C}_j)$

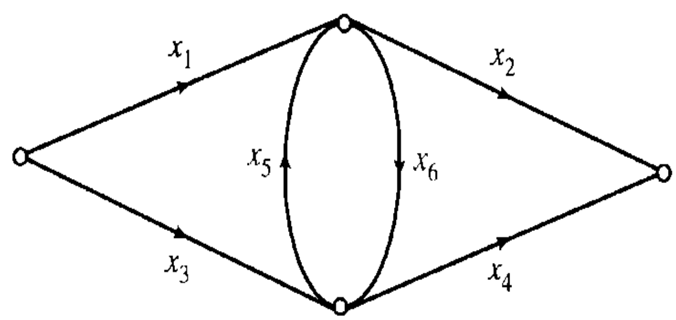
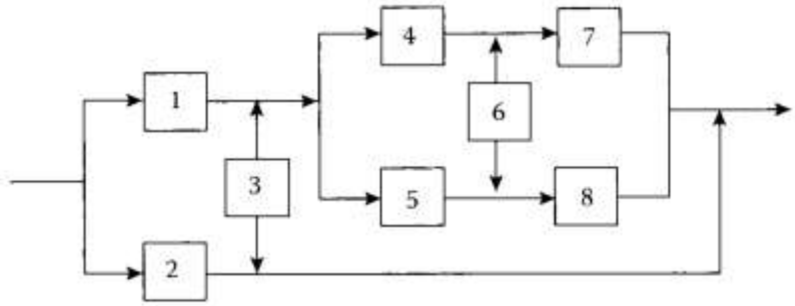


Figure B4 Reliability graph for a six-element system.

Some Tie Sets $T_1 = x_1x_2$ $T_2 = x_3x_4$ (both minimal) $T_5 = x_1x_6x_5x_2$ not minimal since T_5 traverses top node twice

Some Cut Sets $C_1 = x_1x_3$ $C_4 = x_1x_5\,x_4$ (both minimal) $C_5 = x_3x_6x_1$ not minimal since C_5 contains C_1

An example of a complex system that can't be reduced to just parallel and series representations (since the paths are not independent).



Complex nonparallel-series system.

You could use the decomposition method to analyze the system which relies on the conditional probability concept to decompose the system. This is that R(t) is equal to the reliability of the system given a chosen unit's success multiplied by the reliability of the unit **plus** the reliability of the system given the unit's failure multiplied by the unreliability of the unit. In the above case, it is Unit 3 and Unit 6 that don't allow reduction to simple parallel and/or series paths.

A computationally intensive method but one that allows use of computers (brute force methods) is path-tracing (use of tie-set and cut-set methods to examine all paths).

A path or tie-set merely represents a 'path' through the graph. A minimal path-set or tie-set is a path/tie set containing the minimum number of units needed to guarantee a connection between the input and output points. For example: Path = (1,4,7) is a minimal path but Path = (2,3,4,7) is not since Unit 1 is sufficient to provide a path to Units 4,7 or Units 5,6.

The minimal path sets are: P1 = (2) P2 = (1,3) P3 = (1,4,7) P4 = (1,5,8)
P5 = (1,4,6,8) P6 = (1,5,6,7)

A cut-set is a set of units that interrupt all possible connections between the input and output points. A minimal cut-set is the smallest set of units needed to guarantee an interruption of flow. A Minimal cut-set shows a combination of unit failures that cause a system to fail.

The minimal cut-sets are: C1 = (1,2) C2 = (4,5,3,2) C3 = (7,8,3,2)
C4 = (4,6,8,3,2) C5 = (5,6,7,3,2)

These formulations can be used to determine the system reliability. The minimal Tie-Sets produce R(t) whereas the minimal Cut-Sets produce 1 – R(t) = $P_{failure}$

Failure-Rate Models (how to handle observed failure data)

Failure data for systems/components can be obtained from items in a population placed in a life test or using the field data from repair reports for the operating hours of the replaced parts. Both of these methods and most methods of collecting failure data result in **discrete** data points that would lead to piecewise-continuous failure density $f_d(t)$ and hazard-rate $z_d(t)$ functions in terms of the collected failure data.

From calculus, these discrete functions approach continuous functions in the limit as the data points become large and $\Delta t \rightarrow 0$

Data **Density Function $f_d(t)$** - the ratio of the number of failures occurring in the interval [n at t_i - n at $(t_i + \Delta t_i)$] to the size of the original population N.

$$f_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i] / N}{\Delta t_i} \quad \text{for } t_i < t \leq \Delta t_i$$

(The **overall** speed at which failures are occurring.)

Data **Hazard Rate $z_d(t)$** – the failure rate over the interval as the ratio of the number of failures occurring in the interval $(t_i + \Delta t_i)$ to the # of *survivors* at the **beginning** of the time interval $n(t_i)$

$$z_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)] / n(t_i)}{\Delta t_i} \quad \text{for } t_i < t \leq \Delta t_i$$

(The **instantaneous** speed of failures.)

Section B3.4 - *Reliability in Terms of Hazard Rate and Failure Density* takes the above discrete equations and develops continuous equations B37 and B39 for z(t) and R(t) using the fundamental theorem of calculus, the limit as $\Delta t \rightarrow 0$

$z(t) = f(t) / R(t)$ Equation B37 Proceeding in a ***different*** manner from page 431

knowing that $R(t) = 1 - F(t) = 1 - \int f(t)\, dt$ then $dR(t) / dt = - f(t)$

$$z(t) = -\frac{dR(t)}{dt} \Big/ R(t)$$

integrating both sides (knowing $\ln x = \int dx/x$)

$$\ln\{R(t)\} = -\int z(t)\, dt$$

then exponentiating both sides of the equation results in

$$R(t) = e^{-\int z(t)\, dt}$$

Equation B39 which relates reliability to the hazard rate z(t)

The textbook has some failure data for a life test on a group of 10 electronic components. The author takes the stance that the observed failure occurred just before the end of the interval; I would have taken the failures to have occurred at the midpoint of each interval.

TABLE B1 Failure Data for 10 Hypothetical Electronic Components

Failure Number	Operating Time, *h*
1	8
2	20
3	34
4	46
5	63
6	86
7	111
8	141
9	186
10	266

The computations for failure density $f_d(t)$ [overall] and Hazard Rate z_d (t) [instantaneous] are:

TABLE B2 Computation of Data Failure Density and Data Hazard Rate

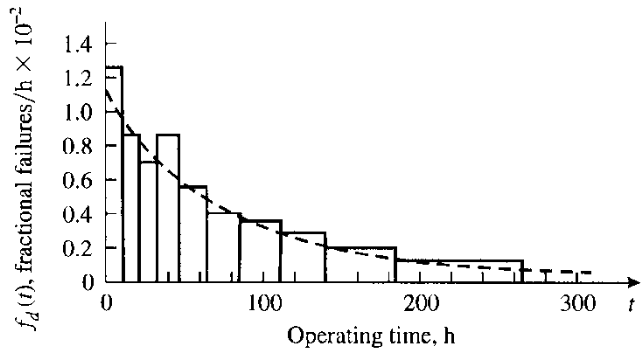
Time Interval, *h*	Failure Density per Hour, $f_d(t)(\times 10^{-2})$	Hazard Rate per Hour, $z_d(t)(\times 10^{-2})$
0–8	$\frac{1}{10 \times 8} = 1.25$	$\frac{1}{10 \times 8} = 1.25$
8–20	$\frac{1}{10 \times 12} = 0.84$	$\frac{1}{9 \times 12} = 0.93$
20–34	$\frac{1}{10 \times 14} = 0.72$	$\frac{1}{8 \times 14} = 0.96$
34–46	$\frac{1}{10 \times 12} = 0.84$	$\frac{1}{7 \times 12} = 1.19$
46–63	$\frac{1}{10 \times 17} = 0.59$	$\frac{1}{6 \times 17} = 0.98$
63–86	$\frac{1}{10 \times 23} = 0.44$	$\frac{1}{5 \times 23} = 0.87$
86–111	$\frac{1}{10 \times 25} = 0.40$	$\frac{1}{4 \times 25} = 1.00$
111–141	$\frac{1}{10 \times 30} = 0.33$	$\frac{1}{3 \times 30} = 1.11$
141–186	$\frac{1}{10 \times 45} = 0.22$	$\frac{1}{2 \times 45} = 1.11$
186–266	$\frac{1}{10 \times 80} = 0.13$	$\frac{1}{1 \times 80} = 1.25$

N = 10

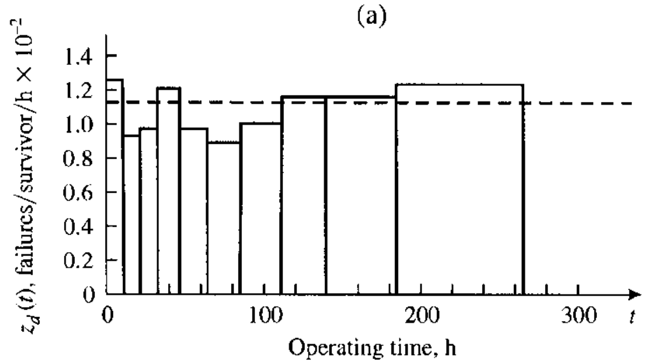
$n(t_i)$ goes from 10 to 1 working items

Note that the 'instantaneous' population $n(t_i)$ is adjusted at the start of each period to calculate the Hazard Rate $z_d(t)$ - failure rate.

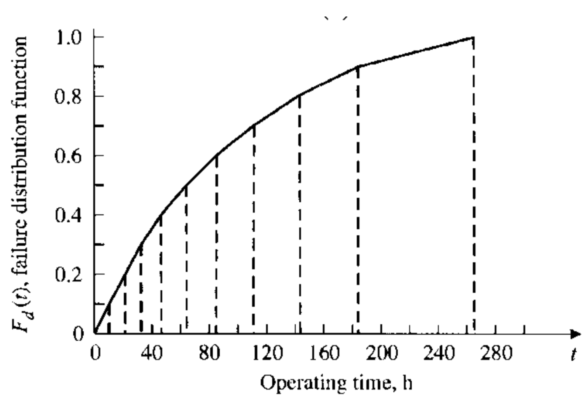
For discrete data, histograms are plotted (for a large N you would group failures):



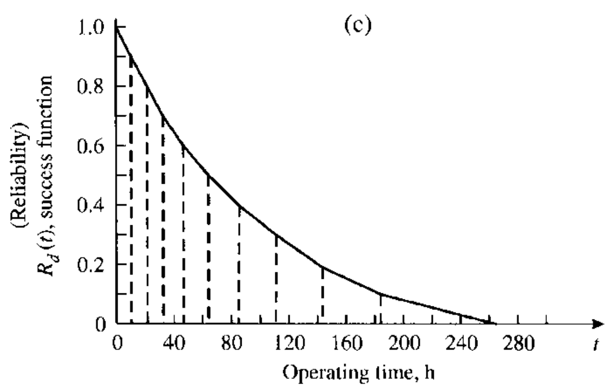
(a)



Integrating $f_d(t)$ the failure density function by taking the area of the appropriate histogram/rectangle results:



(c)



(d)

$R_d(t) = 1 - F_d(t)$

By examining the definition for $f_d(t)$ and noting that $R_d(t) = 1 – F(t) = 1 - \int f_d(t)\, dt$

$R_d(t_i) = n(t_i) / N$ where $n(t_i)$ is the # of operating components at t_i for entire
N = population [thus R(t) is normalized over the population]

The continuous representations can be drawn by 'eyeballing' the given discrete data points (or a least-squares-fit).

For discrete failure points, for n items, the Mean Time To Failure is

$$MTTF = (1/n) \sum_{i=1}^{n} t_i$$ for 1 failure per interval with the failure at the end of the interval although one can normalize the data points to the midpoint of the interval

MTTF is not the same as MTBF (mean time between failures) since MTBF implies repair or replacement situations, although the terms are used interchangeably incorrectly

For the continuous data $MTTF = \int_0^\infty R(t)\, dt$

Thus for a constant hazard rate (constant failure rate λ)

exponential density → Poisson distribution → constant hazard → constant failure rate

which is the simplest case we use a lot BUT don't forget the assumptions.

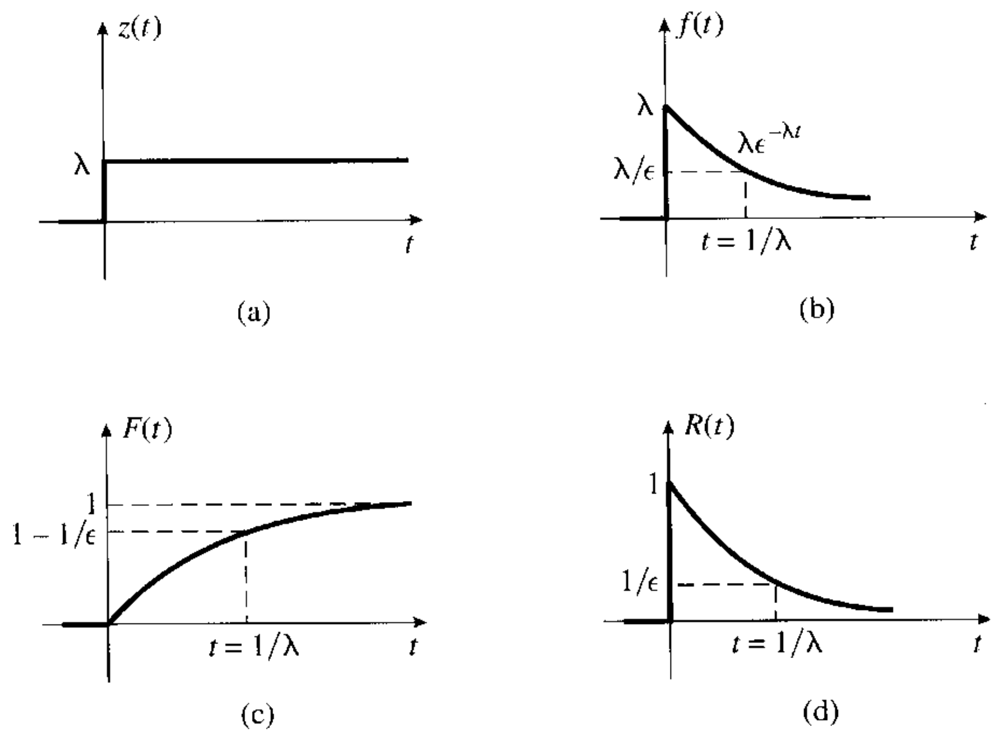


Figure B8 Constant-hazard model: (a) constant hazard; (b) decaying exponential density function; (c) rising exponential distribution function; and (d) decaying exponential reliability function.

$z(t) = \lambda$ Failure or Hazard Rate

$f(t) = \lambda\ e^{-\lambda t}$ Failure Density Function (PDF)

$F(t) = 1 - e^{-\lambda t}$ Cumulative Distribution Function (CDF)

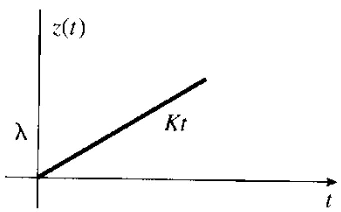
F(t) can also be called the failure distribution function $F(t) = \int f(t)\,dt$

$R(t) = e^{-\lambda t} = 1 - F(t)$ Reliability $[\, 1 - \int f(t) \,]$

$MTTF = 1 / \lambda$ Mean-Time-To-Failure (average time to failure)

The number of failures is the same during the 1st hour of operation as the number of failures between 1,000 to 1,001 hours given the same population size at the start of the period under consideration. Thus if $\lambda = 0.1$ failures/hour for N = 100 items, then we'll have 10 failures during the 1st hour of operation and 10 failures between 1,000 to 1,001 hours given that a similar population of N = 100 items have already survived 1,000 hours.

For **linearly increasing** hazard rate which results in the Rayleigh distribution (page 434)



$z(t) = Kt$

$f(t) = Kt\ e^{-Kt^2/2}$

$R(t) = e^{-Kt^2/2}$

$MTTF = [\, \pi / 2K \,]^{1/2}$

The Weibull Model (two parameter model, wide range of hazard curves), the density function (B47), reliability function (B48) and MTTF (B54), pages 435 – 437.

$z(t) = Kt^m$ for $m > 1$

$f(t) = Kt^m\ e^{-K t^{m+1}/(m+1)}$

$R(t) = e^{-K t^{m+1}/(m+1)}$

Choosing different values for K and m allows approximation of a wide range of hazard curves. K is the ***scale*** parameter and m is the ***shape*** parameter. The estimation process is more complex because it is a two-parameter model. There is even a three-parameter model produced by replacing t with $t - t_o$ where t_o is called the location parameter.

Textbook Section B6 - Markov Reliability and Availability Models

Markov models work very well for constant failure rates λ [hazard rates z(t)] and constant repair rates μ [repair rates w(t) will be fully discussed in Chapter 3]. If these rates are time dependent, the math get very complicated. The Markov models we'll deal with are very simple. In the real world, Markov computer modeling programs are commonly used (to be discussed in Appendix D).

The Markov property - the probability that a component fails in the small interval Δ t is proportional to the length of the interval.

The formulation for *our* Markov model will be for discrete-states leading to continuous-time differential equations (easily solvable). The states of the system at t = 0 are the called the *initial states* and those states representing the final or equilibrium state are called *final states*. The Markov state equations describe the probabilistic transitions (over a period of Δ t) from the initial to the final state. These transition probabilities must obey:

1. z(t) Δt – the probability of transition from one state to another in Δ t.
   If z(t), the hazard rate is constant, z Δt = λ and the model is then *homogeneous* (as compared to time dependent → non-homogeneous).
2. No more than one transition in Δ t (neglect higher order terms).

State-transition equations (difference equations) can be formulated, which result in first-order linear differential equations (simultaneous differential equations) which can be solved by classical differential equation theory using known initial conditions.

The text on pages 447 – 449 solves for R(t) for a two state system where $P_{so}(t)$ = the probability of being in state 0 at time t and $P_{s1}(t)$ = the probability of being in the *failed* state 1 at another time t where the transition probability z(t) Δt is the probability of failure (the change from s_o to s_1) with the system working at t = 0 [initial conditions $P_{so}(0) = 1$ & $P_{s1}(0) = 0$]. Solving the differential equation (B73) for $P_{so}(t)$ the *working* state results in

$$R(t) = P_{s_0}(t) = \exp\left[-\int_0^t z(\xi)\, d\xi\right]$$

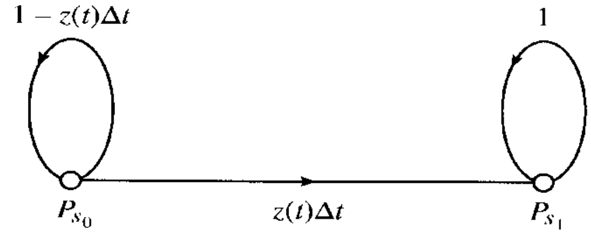
or for a constant hazard rate $P_{so}(t) = e^{-\lambda t} = R(t)$ → the reliability equation

Solving differential equation B74 for $P_{s1}(t)$ results in a solution for the *failed* state

$$P_{s_1}(t) = 1 - \exp\left[\int_0^t z(\xi)\, d\xi\right]$$

Again for a constant hazard rate $P_{s1}(t) = 1 - e^{-\lambda t}$ obvious since $P_{so}(t) + P_{s1}(t) = 1$

Markov Graphs – nodes represent system states (signal sources) and branches are state transition probabilities (transmissions). A Markov graph for a single non-repairable element is as follows:



P_{S0} – Probability of being in state 0 (working)
P_{S1} – Probability of being in state 1 (failed)

Figure B14 Markov graph for a single nonrepairable element.

For this Markov model, the <u>discrete</u> time matrix equation would be $\mathbf{P}(t+\Delta t) = \mathbf{P}(t)\cdot\mathbf{T}$

For this two state model, the T (transition) matrix would have the following elements:

$$\mathbf{T} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix} = \begin{bmatrix} 1 - z(t)\,\Delta t & z(t)\,\Delta t \\ 0 & 1 \end{bmatrix}$$

Table B4 page 448 for the discrete time (t +Δ t) **T** matrix at the top of page just before Eq (B73)

With $\mathbf{P}(t+\Delta t) = \mathbf{P}(t)\cdot\mathbf{T}$ the expansion of this matrix equation produces the following equations:

$$P_{S0}(t+\Delta t) = \{1 - z(t)\,\Delta t\}\,P_{S0}(t) + 0\,P_{S1}(t) \qquad \{P_{S0}(t+\Delta t) - P_{S0}(t)\}/\Delta t = -z(t)\,\Delta t\}\,P_{S0}(t)$$
$$P_{S1}(t+\Delta t) = \{z(t)\,\Delta t\}\,P_{S0}(t) + 1\,P_{S1}(t) \qquad \{P_{S1}(t+\Delta t) - P_{S1}(t)\}/\Delta t = z(t)\,\Delta t\}\,P_{S0}(t)$$

Passing to the limit as Δt becomes small results in the **continuous time** differential equations, Equations B73 & B74 on page 448

$$d\,P_{S0}(t)/dt = -z(t)\,P_{S0}(t) \qquad \text{(B73)}$$
$$d\,P_{S1}(t)/dt = z(t)\,P_{S0}(t) \qquad \text{(B74)}$$

Since the sum of all probabilities must be equal to 1, the sum of transition probabilities means that all the branches *leaving* each node must sum to unity (1), that is ∑ rows = 1 in the T matrix for the discrete time Markov Model.

[It will be confusing at this point but note below that for the simplified continuous time algorithm, the ∑ rows = 0 in the T matrix but for the textbook self-loop method ∑ rows = 1 in the T matrix.]

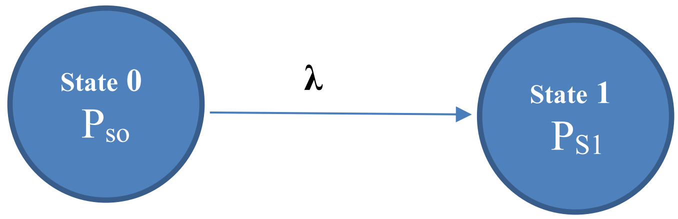
For this single non-repairable element, the probability of remaining in state S_1 is 1 (the failed state). This final state is called the ***absorbing state*** or ***trapping state***.

The same equations (B73 and B74) can be generated using a ***simplified*** continuous time algorithm for writing the state-transition equations by inspection. One equates the derivative of the probability at any node to the sum of the transmissions coming into the node. The unity gains of the self-loops must be set to zero and the Δt factors are dropped by allowing Δ t to approach 0, which results in a **continuous time** Markov Model.

See the top paragraph on pg 450 for the descriptive of this simplified algorithm method.

With constant failure rates $z(t) = \lambda$, using this simplified algorithm by setting the unity (1) gain terms of the self-loops to zero and dropping the Δ t's by setting $\Delta t \rightarrow 0$ results in a simplified Figure B14:

Simplified State Diagram



This equates to the sum of the transitional rates becoming zero since the self-loop on P_{S1} disappears and for P_{S0} the self-loop becomes $- z = - \lambda$ for this constant failure case.

State Transition Matrix $\mathbf{T}$ = $\begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$ where now $\sum$ rows = 0

simplified model, continuous time

Continuous Time

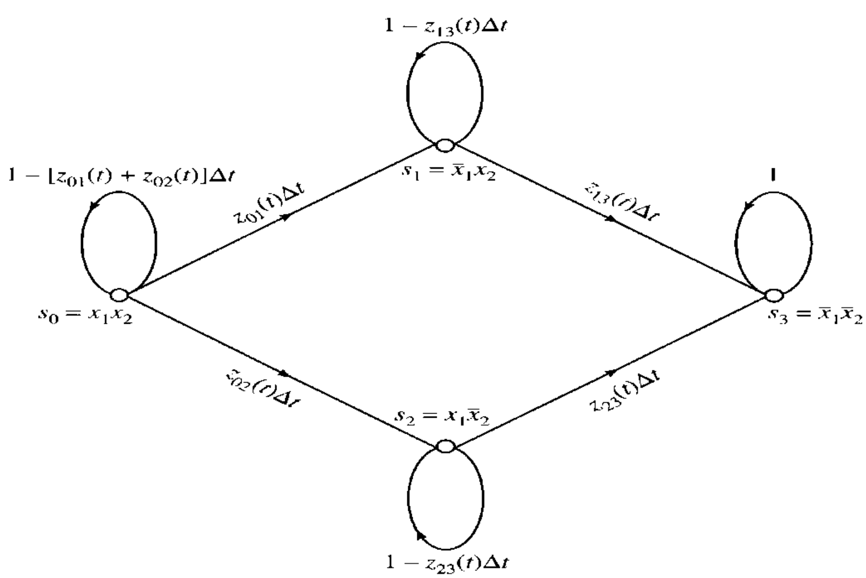
Differential equation in matrix form $\overline{\mathbf{P'}}(t) = \overline{\mathbf{P}}(t) \cdot \overline{\mathbf{T}} = [\, P_{S0}(t) \;\; P_{S1}(t) \,] \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$

From inspection, writing differential equations from the simplified model using the state transition continuous time matrix **T** can come directly from the columns of **T**.

$d\,P_{S0}(t) / dt = -\lambda P_{S0}(t) + 0\, P_{S1}(t)$ from the 1st column of **T** (~ Eq B73 for $z(t) = \lambda$)

$d\,P_{S1}(t) / dt = \lambda P_{S0}(t) + 0\, P_{S1}(t)$ from the 2nd column of **T** (~ Eq B74 for $z(t) = \lambda$)

Another example - a two element nonrepairable system (e.g., parallel µPs) Section B6.4



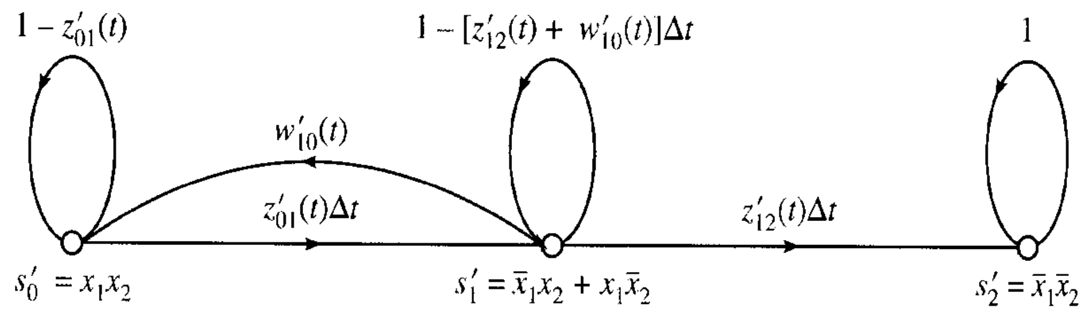
The node feedback loop (self loop) is a graphical representation of staying in the state or (1 – probability of leaving)

Figure B15 Markov graph for two distinct nonrepairable elements.

T State Transition Matrix = [Four States]	$$\begin{pmatrix} P_{00} & P_{01} & P_{02} & P_{03} \\ P_{10} & P_{11} & P_{12} & P_{13} \\ P_{20} & P_{21} & P_{22} & P_{23} \\ P_{30} & P_{31} & P_{32} & P_{33} \end{pmatrix}$$	no failures 1 failure 1 failure 2 failures	The T matrix (discrete time) entries shown in Table B5 on page 451

This example represents all possible configurations of a two element system. At the poorest reliability end, it could represent a series system [the only state representing success is R(t) for no failures = $P_{so}(t)$ → Eq B86]. Best case → a parallel system [2 elements in parallel where one failure could be tolerated] thus R(t) has three mutually exclusive states [P_{S0} P_{S1} P_{S2}] → Eq B87

Collapsing the states (S1 and S2 into S1') requires that the transition rates (the failure rates of the devices) z_{13} and z_{23} must be equal. Normally S1 and S2 are identical devices but not necessary, it's the **z(t)'s that must be equivalent** . If we provide for repair [w(t)] using the same repair rate for the two (identical) devices, then the collapsed model with repair rate **w(t)** is shown in Figure B17:



Repairable Systems (normally with a Repair Rate **μ)**

Repair is normally a very cost effective enhancement to the MTTF of a system.

Since R(t) is the probability that a system has operated over the interval 0 to t, repair is not a factor since once a failure has occurred, the interval of operation is terminated even if for a short period to effect repair [R(t) possibly not a useful term in the real world].

Availability A(t) – the probability that a system is operational at time t.

In general, $A(t) \geq R(t)$ The failed state is called the absorbing or trapping state.

In systems with repair, the absorbing or trapping state is still a possibility since in the long run, a lengthy repair will be in progress when the alternate/backup unit fails, causing a total system failure.

Normally with a constant repair rate u(t) ≥ λ(t) and/or MTTR ≤ MTTF. If the repair process restores a failed unit to usefulness before the backup unit(s) fail, the system will continue to operate (numerous assumptions including 100% failure detection, etc.)

Discrete-time Markov Model for a single unit with ***Repair***

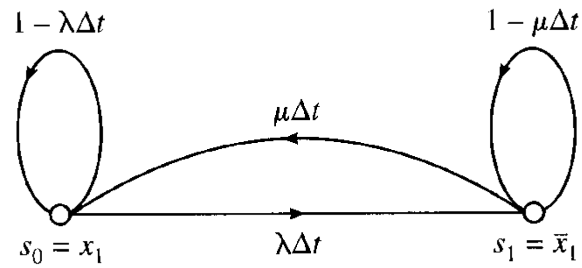


Figure B19 Markov graph for the *availability* of a single component with repair.

constant Failure Rate **λ** and constant Repair Rate **μ** → state transition rates

λ Δ t, μ Δ t → State Transition Probabilities (probability of changing states)

Section B7.3 – deriving a definition for A(t) different from the textbook using the Simplified Method

STATE TRANSITION MATRIX $\vec{T} = \begin{bmatrix} 1-\lambda\Delta t & \lambda\Delta t \\ \mu\Delta t & 1-\mu\Delta t \end{bmatrix}$ $\quad \begin{matrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{matrix}$

(TRANSITION PROBABILITIES)

WRITE EQUATIONS FOR PROBABILITY OF BEING IN STATE 0 OR 1

$$\left[p_0(t+\Delta t),\ p_1(t+\Delta t)\right] = \left[p_0(t),\ p_1(t)\right]\left[\vec{T}\right]$$

MULTIPLYING, REARRANGING, DIVIDING BY Δt AND THEN taking limit $\Delta t \to 0$

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \mu p_1(t)$$

CHAPMAN-KOLMOGOROV EQUATIONS (SIMULTANEOUS DIFFERENTIAL EQUATIONS)

$$\frac{dp_1(t)}{dt} = \lambda p_0(t) - \mu p_1(t)$$

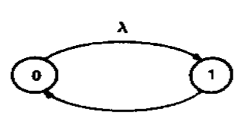
$$\left[p_0'(t),\ p_1'(t)\right] = \left[p_0(t),\ p_1(t)\right]\begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

STATE-TRANSITION RATE MATRIX

Σ ROW = Ø

(CAN BE WRITTEN BY INSPECTION)

$$\vec{P'(t)} = \vec{P(t)} \times \vec{T}$$



λ—Failure rate
μ—Repair rate

c. Two-state continuous-time Markov model

NOW SOLVE THE DIFF EQ'S

USE LAPLACE TRANSFORM TO TRANSFORM LINEAR DIFF EQ'S
INTO ALGEBRAIC EQUATIONS, SOLVE ALGEBRAIC EQUATIONS
AND THEN TRANSFORM THESE BACK INTO THE TIME DOMAIN

RESULTS IN A PARTIAL FRACTION EXPANSION

INVERSE TRANSFORM $\rightarrow t$

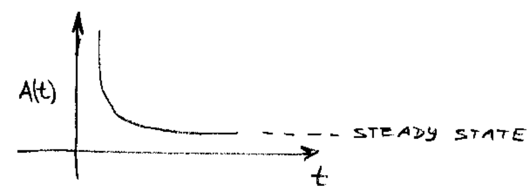
$$p_0(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\,\epsilon^{-(\lambda+\mu)t}$$

STEADY STATE + TIME VARIANT

$$p_1(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu}\,\epsilon^{-(\lambda+\mu)t}$$

$p_0(t)$ = OPERATIONAL STATE $f(t)$ $\xrightarrow[\text{DEFINED AS}]{\text{ALSO}}$ $A(t)$ AVAILABILITY
for proper system operation

SEE FIGURE B20 PG 458



FOR ONLY THE
STEADY STATE EQUATIONS $\frac{dp(t)}{dt} = \emptyset$ APPLIED TO Chapman-Kolmogorov Equation

$$0 = -\lambda p_0 + \mu p_1$$
$$0 = \lambda p_0 - \mu p_1$$

$$\Longrightarrow p_0 = \frac{\mu}{\lambda+\mu}$$

KNOWING THAT $p_0 + p_1 = 1$

WHICH IS THE STEADY-STATE AVAILABILITY $A(t)$

CAN REPRESENT RELIABILITY FUNCTION $R(t)$ AS A MARKOV MODEL
WITH FAILED STATE p_1, A TRAPPING STATE ONCE YOU'VE FAILED YOU'RE DONE

$$0 \xrightarrow{\lambda} 1$$

$$T = \begin{array}{cc} \phi & 1 \\ \end{array}$$
$$T = \begin{bmatrix} -\lambda & \lambda \\ 0 & 0 \end{bmatrix}$$

STATE-TRANSITION RATE MATRIX

by INSPECTION
(−) leaving STATE
(+) CONNECTED TO TRANSISTION TO or

$p_0(t) = \epsilon^{-\lambda t}$ OPERATIONAL STATE

$p_1(t) = 1 - \epsilon^{-\lambda t}$ FAILED STATE

Note: If one uses the textbook method for the discrete time Markov Model, then to write the continuous time differential equations, the unity gain factors of the self-loops are set to zero (1 → 0) and the Δ t factors are dropped from the branch gains. The differential equations are the continuous time derivative of the probability of being in a state which is equal to the 'flows' from the other nodes. The easier Simplified Markov Model (without the self-loops and Δ t's) produces the differential equations directly from inspection of **T**.

Derivation of A(t) = P_0(t) for a two-state S_0 and S_1 Markov Model using Laplace Transforms.

Laplace Transforms – converts differential equations into algebraic equations, which are easier to solve (albeit in the S domain). Once solved algebraically in the S domain, then the inverse Laplace transforms produce the solution in the time domain t [f*(s) → f(t)]

The Laplace transform of f(t) is defined by the integral (Eq B97):

$$\mathcal{L}\{f(t)\} = F(s) = \{f(t)\}^* = f^*(s) = \int_0^\infty f(t)e^{-st}\,dt$$

A graphical representation of the Laplace transformation techniques (Figure B24)

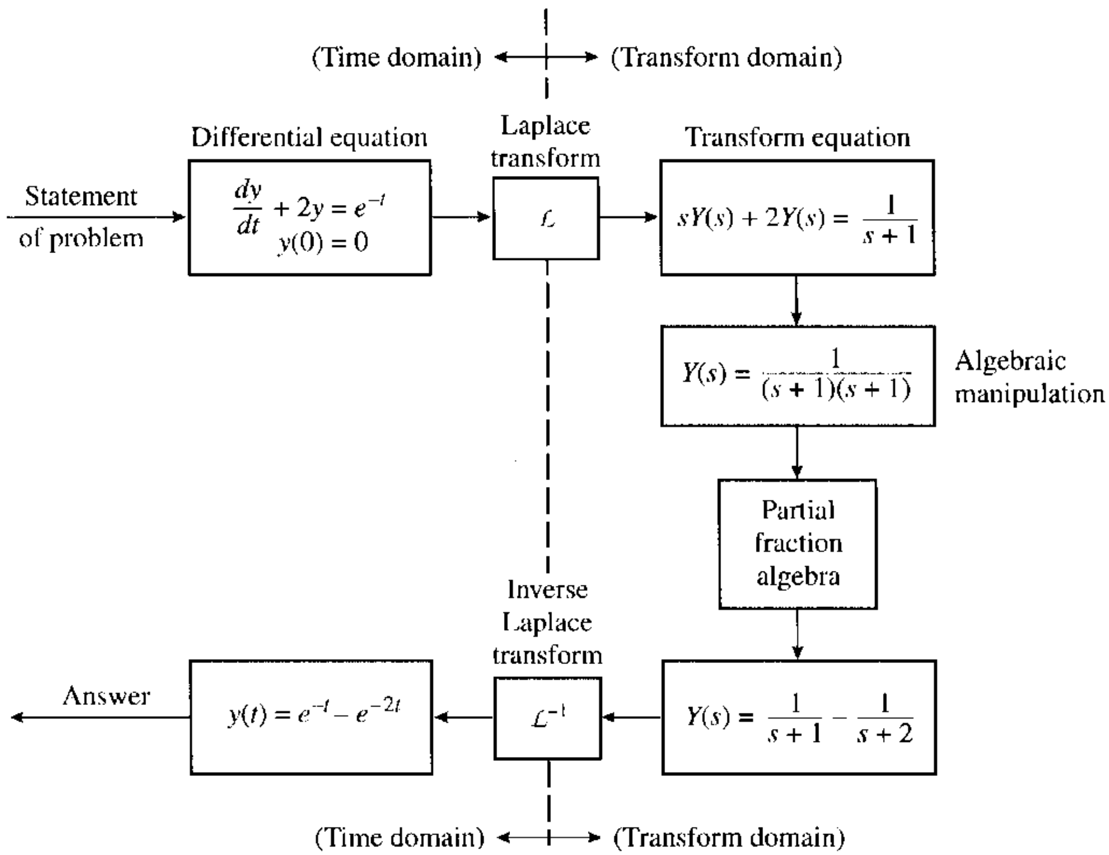


Figure B24 The solution of a differential equation by Laplace transform techniques.

Table B6 on page 465 and Table B7 on page 468 list some primary Laplace Transforms.

One key for the solution of differential equations is the Laplace transform of df(t)/dt

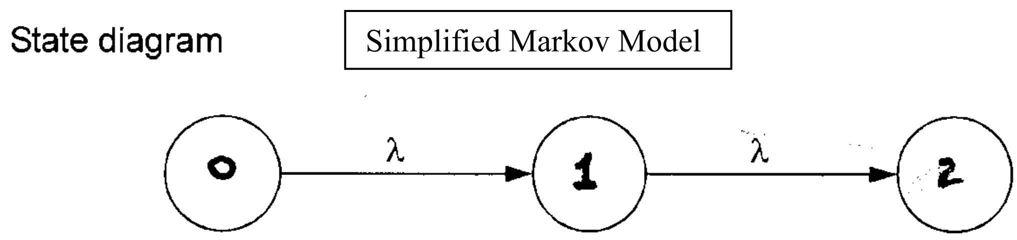
$$\left\{\frac{df(t)}{dt}\right\}^* = sf^*(s) - f(0) \qquad \text{(B101)}$$

Cold Standby System – solving for R(t) and MTTF using Laplace Transforms

The textbook has a good summary of the transform techniques; however, lets solve another simple Markov Model (Cold Standby System) for R(t) and MTTF.

A cold standby system is nothing more than a parallel configuration with one unit turned off awaiting the detection of a failure at which time this cold spare (backup) unit is switched into operation. For simplicity, we'll assume a perfect switch, constant failure rates, etc.

Simplified Markov Model for two modules in parallel, initially (at t = 0) one module is operating (the on-line or primary module) with the second module OFF, which will be the cold spare or backup.



State 0	1 module working, 1 module used as a backup, both with a failure rate of λ
State 1	One module failed, switch to standby module after it is turned ON
State 2	Both modules failed (trapping state or absorbing state)

Assumptions:

1. Both modules are identical with the same constant failure rate λ ($\lambda_1 = \lambda_1 = \lambda$)

2. While in standby (turned off), the failure rate of the spare is zero. It always works when switched into the system and turned on.

3. Perfect switch, $\lambda_s = 0$.

4. The failure of the first module is always detected, perfect coverage C = 1.

5. No repair.

Writing the state transition matrix **T** for this *simplified* model (continuous time)

$$\bar{T} = \begin{bmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{matrix}$$

$$P_0'(t) = -\lambda P_0(t)$$
$$P_1'(t) = \lambda P_0(t) - \lambda P_1(t)$$
$$P_2'(t) = \lambda P_1(t)$$

Differential equations for the cold stand-by system (derived from the columns of **T)**

MATRIX FORM

$$\bar{P}'(t) = \bar{P}(t) \cdot \bar{T}$$

$$\bar{P}'(t) = \left[P_0'(t) \; P_1'(t) \; P_2'(t)\right] \qquad \bar{P}(t) = \left[P_0(t) \; P_1(t) \; P_2(t)\right]$$

FROM LAPLACE TRANSFORM EQ B101

$$f'(t) \rightarrow s\,\overset{*}{f}(s) - f(0)$$

TAKING THE LAPLACE TRANSFORM OF THE MATRIX EQUATION

$$s\,\overset{*}{\bar{P}}(s) - \bar{P}(0) = \overset{*}{\bar{P}}(s) \cdot T$$

WHERE $\bar{P}(0) = \left[1 \; 0 \; 0\right]$ OR $P_0(0) = 1 \quad P_1(0) = 0 \quad P_2(0) = 0$

INITIAL CONDITIONS

SO OUR EQUATIONS TRANSFORMED BECOME

$$s\,\overset{*}{P}_0(s) - 1 = -\lambda \overset{*}{P}_0(s)$$
$$s\,\overset{*}{P}_1(s) - 0 = \lambda \overset{*}{P}_0(s) - \lambda \overset{*}{P}_1(s)$$
$$s\,\overset{*}{P}_2(s) - 0 = \lambda \overset{*}{P}_1(s)$$

SOLVING FOR $\overset{*}{P}_0(s)$ DIVIDE BOTH SIDES OF 1^{st} EQUATION BY $\overset{*}{P}_0(s)$

$$s - \frac{1}{\overset{*}{P}_0(s)} = -\lambda \quad \text{thus} \quad \overset{*}{P}_0(s) = \frac{1}{s+\lambda}$$

$$f(t) = \epsilon^{-\lambda t} \xrightarrow{\text{TRANSFORMED}} \overset{*}{f}(s) = \frac{1}{s+\lambda}$$

THUS $P_0(t) = \epsilon^{-\lambda t}$ THEN SOLVING FOR $\overset{*}{P}_1(s)$

$$\overset{*}{P}_1(s) = \frac{\lambda \overset{*}{P}_0(s)}{(s+\lambda)} = \frac{\lambda}{(s+\lambda)^2}$$

$$f(t) = t\,\epsilon^{-\lambda t} \xrightarrow{\text{TRANSFORMED}} \overset{*}{f}(s) = \frac{1}{(s+\lambda)^2}$$

THUS $P_1(t) = \lambda t\,\epsilon^{-\lambda t}$

THE RELIABILITY OF THE SYSTEM $R(t)$

$R(t) = P_0(t) + P_1(t)$ SINCE THE SYSTEM HAS FAILED AT $P_2(t)$, THE TRAPPING OR ABSORBING STATE

$$R(t) = P_0(t) + P_1(t) = \epsilon^{-\lambda t} + \lambda t\,\epsilon^{-\lambda t}$$

$$R(t) = (1+\lambda t)\,\epsilon^{-\lambda t}$$

$$MTTF = \int_0^\infty R(t)\,dt = \int_0^\infty (1+\lambda t)\,\epsilon^{-\lambda t}\,dt = \int_0^\infty \epsilon^{-\lambda t}\,dt + \int_0^\infty \lambda t\,\epsilon^{-\lambda t}\,dt$$

KNOWING $\displaystyle\int_0^\infty \epsilon^{-\lambda t} = \left.\frac{-\epsilon^{-\lambda t}}{\lambda}\right|_0^\infty$ AND

$$\int_0^\infty \lambda t\,\epsilon^{-\lambda t}\,dt = \lambda \int_0^\infty t\,\epsilon^{-\lambda t}\,dt = \lambda\left[\frac{\epsilon^{-\lambda t}}{\lambda^2}(\lambda t+1)\right] = \left.\frac{(\lambda t+1)\,\epsilon^{-\lambda t}}{\lambda}\right|_0^\infty$$

$$MTTF = \left.\frac{-\epsilon^{-\lambda t} + \lambda t\,\epsilon^{-\lambda t} + \epsilon^{-\lambda t}}{\lambda}\right|_0^\infty = \frac{2}{\lambda}$$

Summary for Cold Standby System:

$$P_f(t) = P_{S2}(t) = 1 - [\, P_{S0}(t) + P_{S1}(t) \,]$$

$$P_{S0}(t) = e^{-\lambda t}$$

$$P_{S1}(t) = \lambda\, t\, e^{-\lambda t}$$

$$R(t) = P_{S0}(t) + P_{S1}(t) = (1 + \lambda t)\, e^{-\lambda t}$$

$$\mathrm{MTTF} = 2 / \lambda$$