Fourier's Story

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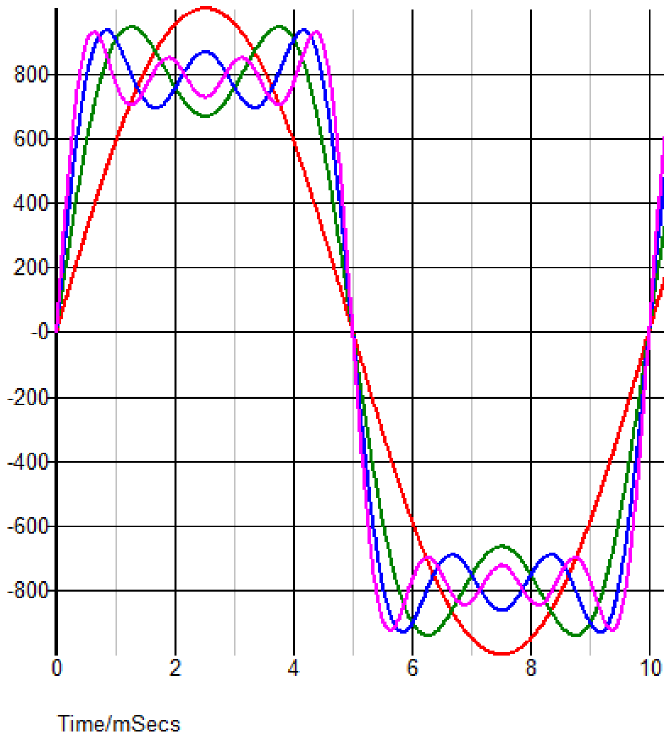


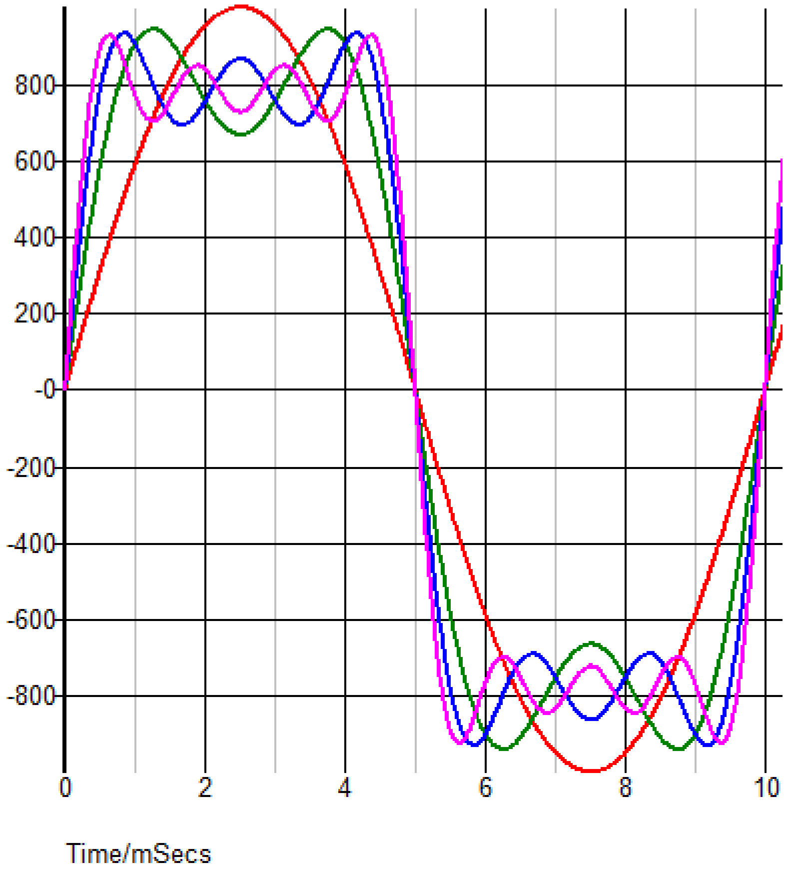
TLH 3315 FOURIER PRESENTATION 2 FS, FT, DFT

Born	21 March 1768 Auxerre, Burgundy, Kingdom of France (now in Yonne, France)
Died	16 May 1830 (aged 62) Paris, Kingdom of France
Nationality	French
Alma mater	École Normale Supérieure
Known for	(see list) Fourier number Fourier series Fourier transform Fourier's law of conduction Fourier–Motzkin elimination Greenhouse effect
Scientific career	
Fields	Mathematician, physicist, historian
Institutions	École Normale Supérieure École Polytechnique
Academic advisors	Joseph-Louis Lagrange
Notable students	Peter Gustav Lejeune Dirichlet Claude-Louis Navier Giovanni Plana

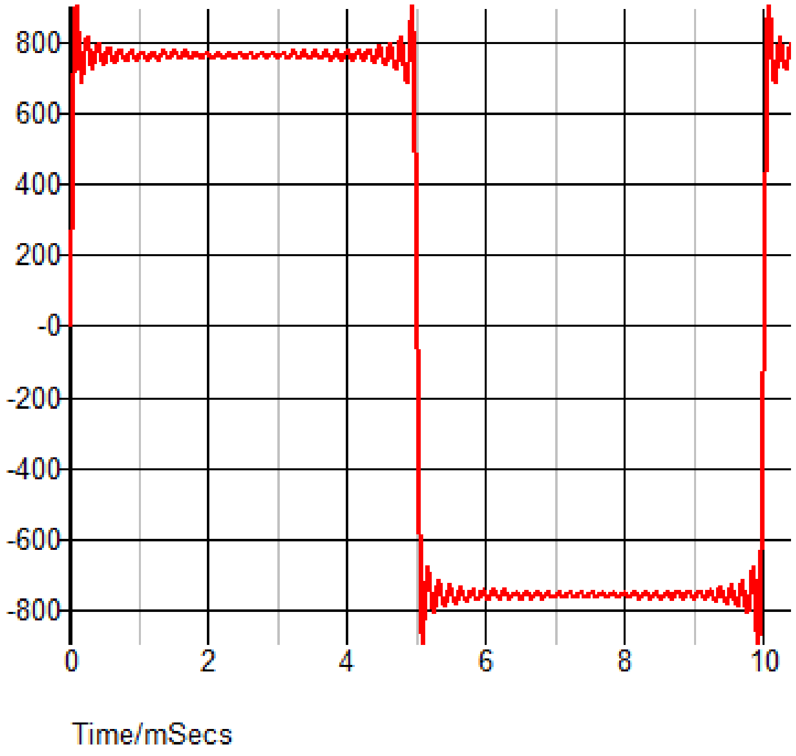
TABLE 11.2 *Table of Fourier techniques*

Name	*Characteristics*	*Typical use*
Fourier series	$f(t)$ continuous $F(\omega_i)$ discrete	Analysis of periodic functions and signals
Fourier transform	$f(t)$ continuous $F(i\omega)$ continuous	Frequency analysis of signals and systems
Discrete Fourier transform (DFT)	$f(t_i)$ discrete $F(\omega_i)$ discrete	Computation of other transforms Analysis of sampled signals
Fast Fourier transform (FFT)	$f(t_i)$ discrete $F(\omega_i)$ discrete	Algorithm to compute the DFT

FOURIER SERIES



100Hz square wave to 700 Hz



32 frequencies (i.e. 100Hz up to 6300Hz components)

https://www.eeworldonline.com/the-importance-of-fourier-transforms/

Approximation of Periodic Signals by Sinusoids

- Any **periodic signal** can be approximated by a sum of many sinusoids at harmonic frequencies of the signal (kf_0) with appropriate amplitude and phase.
- The more harmonic components are added, the more accurate the approximation becomes.
- Instead of using sinusoidal signals, mathematically, we can use the **complex exponential** functions with both positive and negative harmonic frequencies

If $x(t)$ is periodic with period T, it is also periodic with period nT, that is:

$$x(t) = x(t + nT).$$

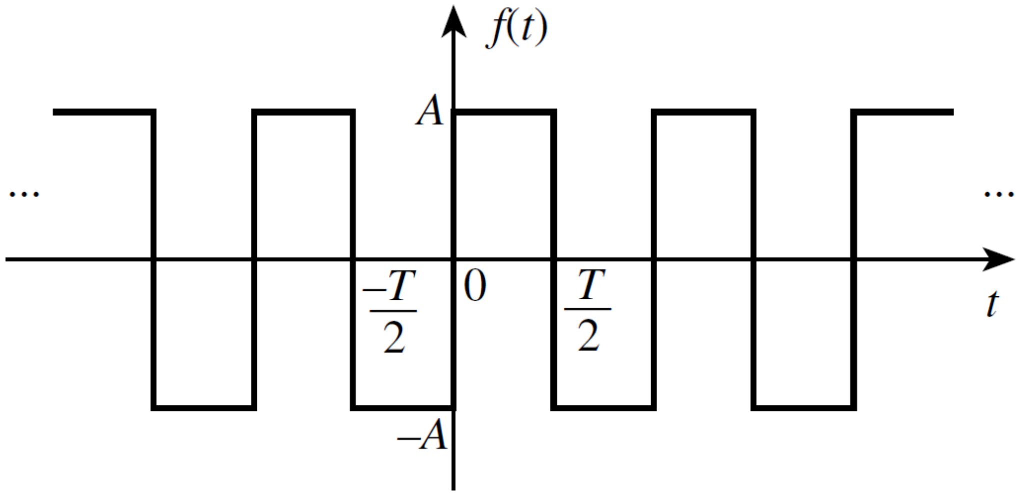


FIGURE 8.4 *Square wave of Example 8.4*

The first observation is that $f(t)$ is odd, which yields the result that $a_0 = 0$ and $a_i = 0$ for every coefficient of the cosine terms. Letting $\omega_0 = 2\pi/T$, the coefficients b_n are

$$b_n = 2\left(\frac{2}{T}\right)\int_0^{T/2} A\sin(n\omega_0 t)\,dt.$$

The result is

$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

where $(2n-1)$ is introduced to assure that only odd terms are included in the summation. The sine waves that make up the Fourier series for the odd square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right],$$

Complex Series Square Wave Example

Consider the odd square wave of Example 8.4 and the complex Fourier coefficients

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{0} (-A)e^{-in\omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2} (A)e^{-in\omega_0 t}\,dt, \tag{8.29}$$

which leads to the series

$$f(t) = \frac{2A}{i\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}, \tag{8.30}$$

as defined in Equation 8.23.

Each coefficient has the form

$$\alpha_n = \frac{2A}{in\pi} = \frac{2A}{n\pi} e^{-i\pi/2}, \qquad n = \pm 1, \pm 3, \ldots,$$

and the coefficients for even values, $n = 0, \pm 2, \ldots$, are zero. Notice that the coefficients decrease as the index n increases. The use of these coefficients to compute the *frequency spectrum* of $f(t)$ is considered later.

The trigonometric series is derived from the complex series by expanding the complex series of Equation 8.30 as

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t} \\ &= \cdots - \frac{2A}{3\pi i} e^{-i3\omega_0 t} - \frac{2A}{\pi i} e^{-i\omega_0 t} + \frac{2A}{\pi i} e^{i\omega_0 t} + \frac{2A}{3\pi i} e^{i3\omega_0 t} + \cdots \end{aligned}$$

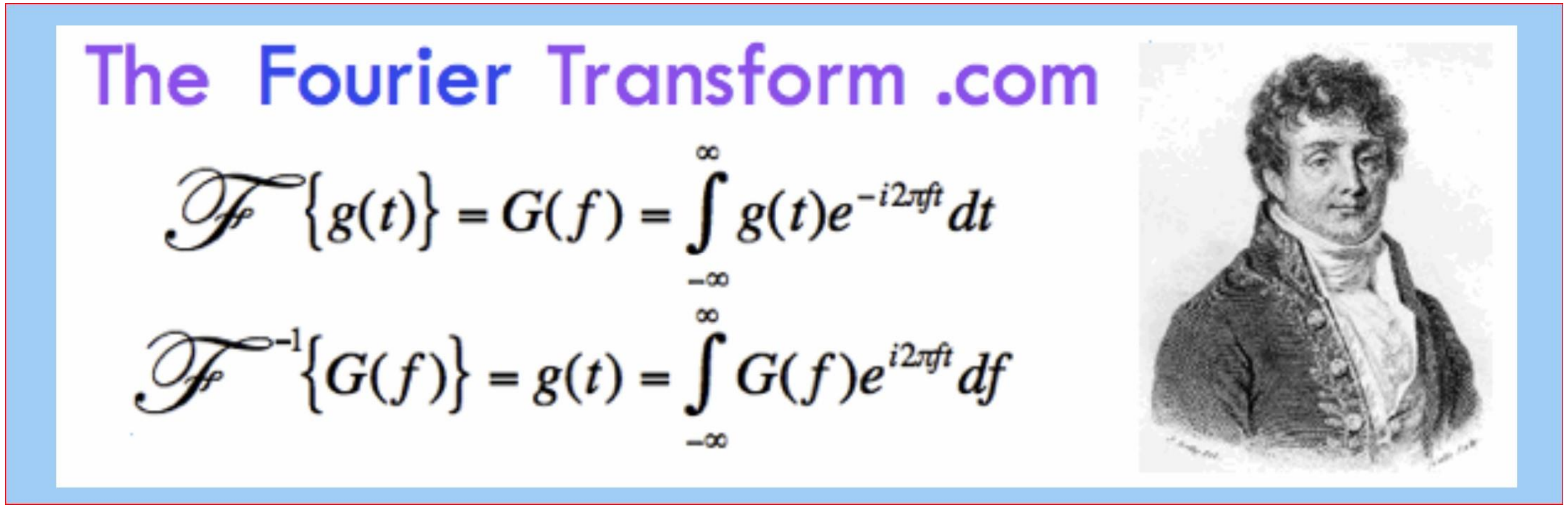
COMBINE THE EXPONENTIAL TERMS FOR EACH FREQUENCY

and recognizing the sum of negative and positive terms for each n as $2\sin(n\omega_0 t)$. The trigonometric series becomes

$$f(t) = \frac{4A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

which is the result of Example 8.4.

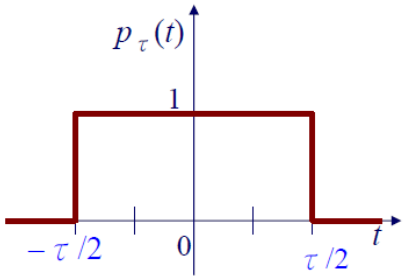
$\square$



As defined in Chapter 8, the continuous Fourier transform is

$$\mathcal{F}[f(t)] = F(f) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi ft}\,dt. \tag{11.8}$$

The frequency f in hertz is used as the parameter in this integral. The function $F(i\omega)$, where $\omega = 2\pi f$ is the frequency in radians per second,



☐ **EXAMPLE 8.11** ***Fourier Pulse Example***

The even rectangular pulse of height A and width τ is defined as

$$P(t) = \begin{cases} A, & -\dfrac{\tau}{2} \le t \le \dfrac{\tau}{2}, \\ 0, & |t| > \dfrac{\tau}{2}. \end{cases}$$

The Fourier transform is

$$\begin{aligned} \mathcal{F}[P(t)] &= \int_{-\infty}^{\infty} P(t)e^{-i\omega t}\,dt = A\int_{-\tau/2}^{\tau/2} e^{-i\omega t}\,dt \\ &= -A\frac{e^{-i\omega t}}{i\omega}\bigg|_{-\tau/2}^{\tau/2} = -A\frac{e^{-i\omega\tau/2} - e^{i\omega\tau/2}}{i\omega}. \end{aligned}$$

This result expressed in terms of the sine function is $(2A/\omega)\sin(\omega\tau/2)$. Multiplying the numerator and denominator by $\tau/2$ yields the Fourier transform as

$$P(i\omega) = A\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2} = A\tau\,\mathrm{sinc}(\omega\tau/2). \tag{8.51}$$

MATLAB Pulse Example

Figure 8.10 shows the Fourier transform for two pulses as described in Example 8.11. The positive frequencies of the transform are shown for different pulse widths. The accompanying MATLAB script was used to plot the transforms for the two pulses. Each pulse has amplitude $A = 1$. One pulse has a pulse width of 16 seconds and the other 4 seconds.

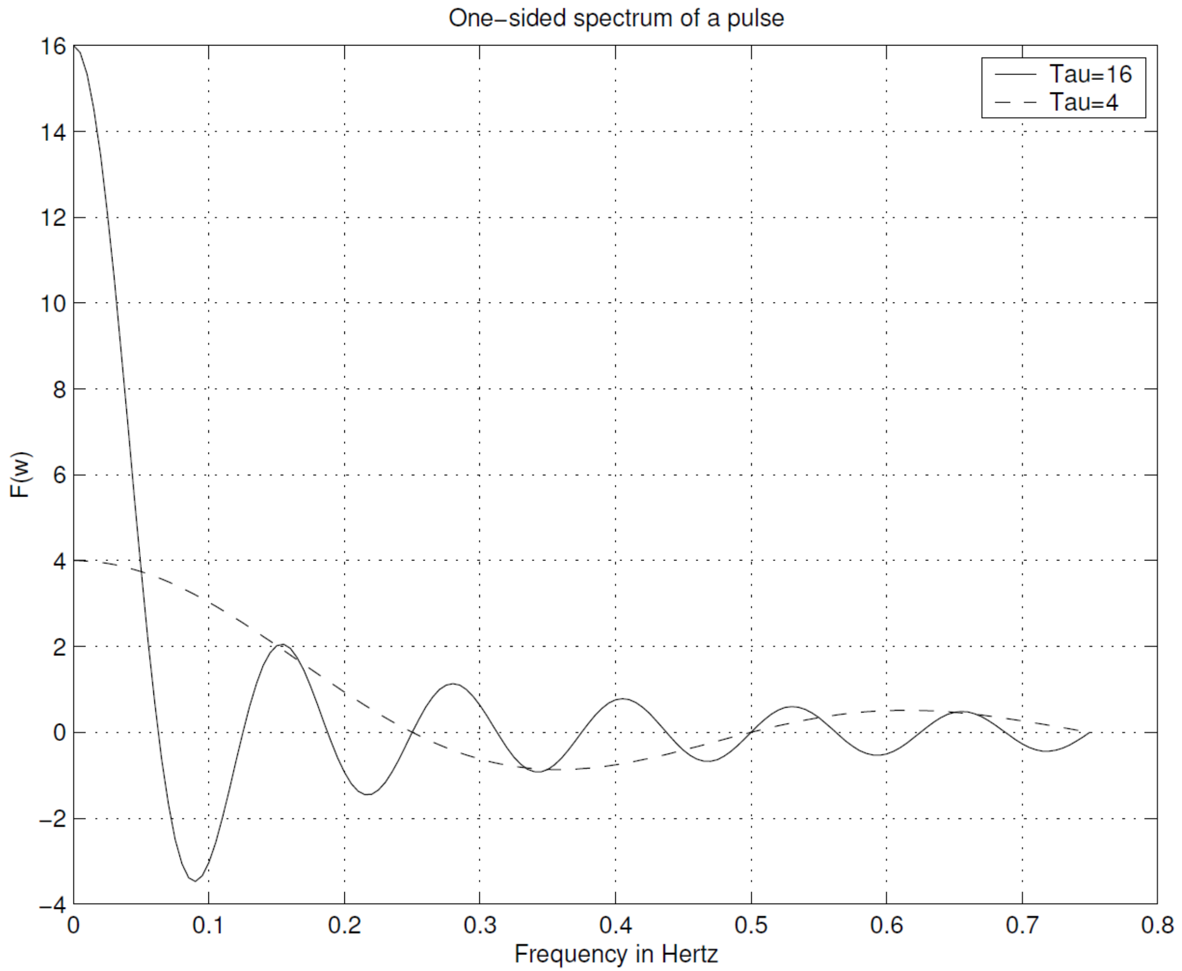


FIGURE 8.10 *Fourier transform of two pulses with different pulse widths*

RELATIONSHIP TO FOURIER SERIES

Comparing the coefficients of the Fourier series of Example 8.7 for a periodic pulse train of rectangular pulses and the Fourier transform of Example 8.11 for a single pulse shows that the series coefficients are

$$\alpha_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} f(t)e^{-in\omega_0 t}\,dt = \frac{A\tau}{T}\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}$$

and the transform is

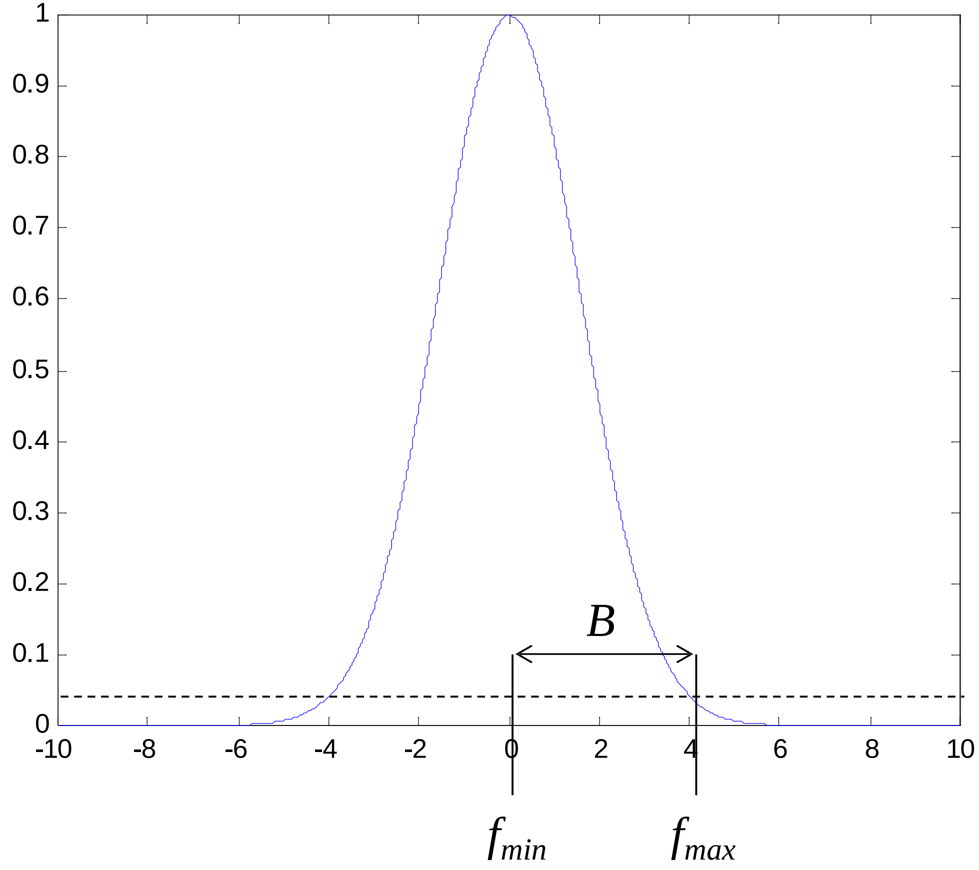
$$\mathcal{F}[f(t)] = F(i\omega) = \int_{-\tau/2}^{\tau/2} f(t)e^{-i\omega t}\,dt = A\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}.$$

By comparing the two results, it is clear that designating the transform $F(i\omega) = \mathcal{F}[f(t)]$,

$$\frac{F(n\omega_0)}{T} = \frac{A\tau}{T}\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}.$$

Thus, we conclude that the Fourier series coefficients are obtained by *sampling* the Fourier transform at the points $n\omega_0$ and dividing by the period T. However, the Fourier series itself is a continuous function of time, but the Fourier transform is a function of ω in the frequency domain.

(Effective)Bandwidth



- f_{min} (f_{ma}): lowest (highest) frequency where the FT magnitude is above a threshold
- Bandwidth:

  $B = f_{max} - f_{min}$

- The threshold is often chosen with respect to the peak magnitude, expressed in dB
- dB=10 log10(ratio)
- 10 dB below peak = 1/10 of the peak value
- 3 dB below=1/2 of the peak

More on Bandwidth
WHY WE CARE!

- Bandwidth of a signal is a critical feature when dealing with the transmission of this signal
- A communication channel usually operates only at certain frequency range (called channel bandwidth)
  - The signal will be severely attenuated if it contains frequencies outside the range of the channel bandwidth
  - To carry a signal in a channel, the signal needed to be modulated from its baseband to the channel bandwidth
  - Multiple narrowband signals may be multiplexed to use a single wideband channel

How to Observe Frequency Content from Waveforms?

- A constant -> only zero frequency component (DC compoent)
- A sinusoid -> Contain only a single frequency component
- Periodic signals -> Contain the fundamental frequency and harmonics -> Line spectrum
- Slowly varying -> contain low frequency only
- Fast varying -> contain very high frequency
- Sharp transition -> contain from low to high frequency
- Music: contain both slowly varying and fast varying components, wide bandwidth
- Highest frequency estimation?
  - Find the shortest interval between peak and valleys

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DISTRETE FOURIER TRANSFORM

DFT PRESENTATION 1

Using Harman Chapter 11

$$F_k = F\left(\frac{k}{NT_s}\right) = \sum_{n=0}^{N-1} f(nT_s)e^{-i2\pi nk/N} \qquad \textbf{(11.5)}$$

Definition of DFT and IDFT Assume that a function $f(t)$ is defined at a set of N points, $f(nT_s)$ for $n = 0, \ldots, N-1$ values, as shown in Figure 11.3. The DFT yields the frequency spectrum at N points by the formula

$$F_k = F\left(\frac{k}{NT_s}\right) = \sum_{n=0}^{N-1} f(nT_s)e^{-i2\pi nk/N} \tag{11.5}$$

for $k = 0, \ldots, N-1$. Thus, N sample points of the signal in time lead to N frequency components in the discrete spectrum spaced at intervals $f_s = 1/(NT_s)$. The Inverse DFT (IDFT) is defined as

$$f_n = f(nT_s) = \frac{1}{N}\sum_{k=0}^{N-1} F\left(\frac{k}{NT_s}\right)e^{i2\pi nk/N} \tag{11.6}$$

for $n = 0, \ldots, N-1$. The IDFT is used to re-create the signal from its spectrum.

LECTURE OBJECTIVES

- **D**iscrete **F**ourier **T**ransform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

- DFT from DTFT by **frequency sampling**

- DFT computation (FFT)

- DFT pairs and properties
  - Periodicity in DFT (time & frequency)

DIFFERENCES   TLH AND DSPF

$$F_k = F\left(\frac{k}{NT_s}\right) = \sum_{n=0}^{N-1} f(nT_s)e^{-i2\pi nk/N} \qquad (11.5)$$

Ts is the time between samples   S = 1/Ts samples/sec

Thus, Fk values are frequencies and f values are given at a specific time nTs

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

Here n and k are indices –

NO relation to physical time of frequency

The DFT can be used to approximate the continuous Fourier transform. As defined in Chapter 8, the continuous Fourier transform is

$$\mathcal{F}[f(t)] = F(f) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi ft}\,dt. \tag{11.8}$$

The frequency f in hertz is used as the parameter in this integral. The function $F(i\omega)$, where $\omega = 2\pi f$ is the frequency in radians per second, Using the sampled $f(t)$ with $t = nT_s$ and replacing f by the discrete frequencies $f_s = k/(NT_s)$ leads to the approximation of the Fourier transform as

$$F\left(\frac{k}{NT_s}\right) = T_s \sum_{n=0}^{N-1} f(nT_s)e^{-i2\pi nk/N} \qquad \textbf{(11.9)}$$

for $k = 0, \ldots, N-1$. The factor $\Delta t = T_s$ replaced dt in the integral and is used as a multiplier of the DFT defined by Equation 11.5 in order to approximate the continuous Fourier transform. Problem 11.2 presents another derivation of the DFT approximation to the Fourier transform.

WATCH THE NOTATION: f_s here is the Resolution in frequency in Hertz.
The Sampling Rate is given by 1/T_s
Note: f_max is at index N/2 so F[(N/2)/(NT_s)] = F[1/(2T_s)] as Expected!

DFT summary Table 11.4 summarizes the DFT or FFT parameters when a real signal is sampled every T_s seconds for $(N-1)T_s$ seconds.

TABLE 11.4 *DFT parameters*

Parameter	*Notation*	
Time domain:		
Sample interval	T_s (s)	$= \Delta t = 1/S$
Sample size	N points	
Length	$(N-1)T_s$ (s)	
Period (from IDFT)	$T = NT_s$ (s)	
Frequency domain:		
Frequency Spacing	$f_s = \dfrac{1}{T} = \dfrac{1}{NT_s}$ (Hz)	$= \Delta f$
Spectrum size	N components	
Maximum frequency	$\dfrac{N}{2} f_s = F_{\max}$ (Hz)	
Frequency period	$F_p = N f_s = \dfrac{1}{T_s}$ (Hz)	