CENG 3315 Review2_Euler.pptx

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

Euler's identity is an equality found in mathematics that has been compared to a Shakespearean sonnet and described as "the most beautiful equation.

$$e^{i\pi}+1=0$$

https://en.wikipedia.org/wiki/Leonhard_Euler

Euler was one of the most eminent mathematicians of the 18th century and is held to be one of the greatest in history. A statement attributed to Pierre-Simon Laplace expresses Euler's influence on mathematics: "Read Euler, read Euler, he is the master of us all."[6][7] He is also widely considered to be the most prolific, as his collected works fill 92 volumes,[8] more than anyone else in the field. He spent most of his adult life in Saint Petersburg, Russia, and in Berlin, then the capital of Prussia.

Amongst his many discoveries and developments, Euler is credited for popularizing: The Greek letter π (lowercase pi) to denominate the Archimedes' constant (the ratio of a circle's circumference to its diameter); First employing the term $f(x)$ to describe a function's y-axis, the letter i to express the imaginary unit equivalent to $\sqrt{-1}$, and the Greek letter Σ (uppercase sigma) to express summations; For developing "**e**", a new mathematical constant (commonly known as Euler's Number), roughly equivalent to 2.71828, to represent a logarithm's natural base which has several applications, such as calculating compound interest in financial engineering.[9]

Complex Exponentials

January 22, 2019

The exponential function $e^{\lambda t}$ is an *eigenfunction* of various operations, such as differentiation, integration, and time shifting. Thus, when these operations are performed on $e^{\lambda t}$, the result is a constant times the exponential function.

Since for any constant λ,

$$\frac{d}{dt}(e^{\lambda t}) = \lambda(e^{\lambda t}), \tag{1}$$

the exponential is an eigenfunction of the differentiation operator.

How about an example:

Let $z = re^{j\omega t}$, So that

$$\frac{d}{dt}(e^{j\omega t}) = j\omega(e^{j\omega t}) \tag{2}$$

Notice the shift by j which is 90 degrees or $e^{j\pi/2}$.

If the time shift operator is defined as $\mathcal{T}_s[f(t)] = f(t - t_0)$, then

$$\mathcal{T}_s[e^{j\omega t}] = e^{j\omega(t-t_0)} = e^{-j\omega t_0}\, e^{j\omega t}. \tag{3}$$

Since the first term in the product is constant, setting $\lambda = e^{-\alpha t_0}$ results in the eigenvalue equation

$$\mathcal{T}_s[e^{\alpha t}] = \lambda e^{\alpha t}. \tag{4}$$

These relationships and the result for the integral of $z = re^{j\omega t}$ are the reasons for using the exponential form of the sinusoids.

Another reason for using expontentials is that the exponentials are simple to multiply. Just multiply the magnitudes and add the angles.

LECTURE OBJECTIVES

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids
  - Take Real or Complex part

$$\boxed{A\cos(\omega t + \varphi) = \Re\{(Ae^{j\varphi})e^{j\omega t}\}}$$

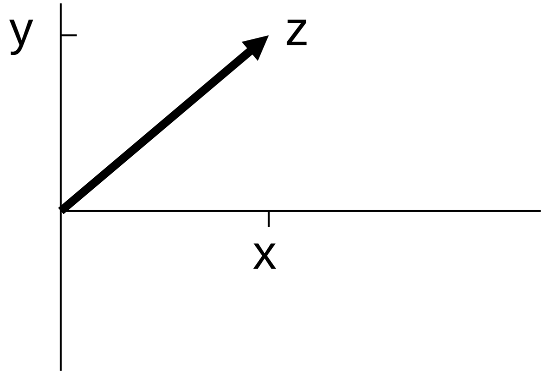
WHY? What do we gain?

- Sinusoids are the basis of DSP,
  - but trig identities are very tedious

- Abstraction of complex numbers
  - Represent cosine functions
  - Can replace most trigonometry with **algebra**
- **Avoid (Most) all Trigonometric manipulations**

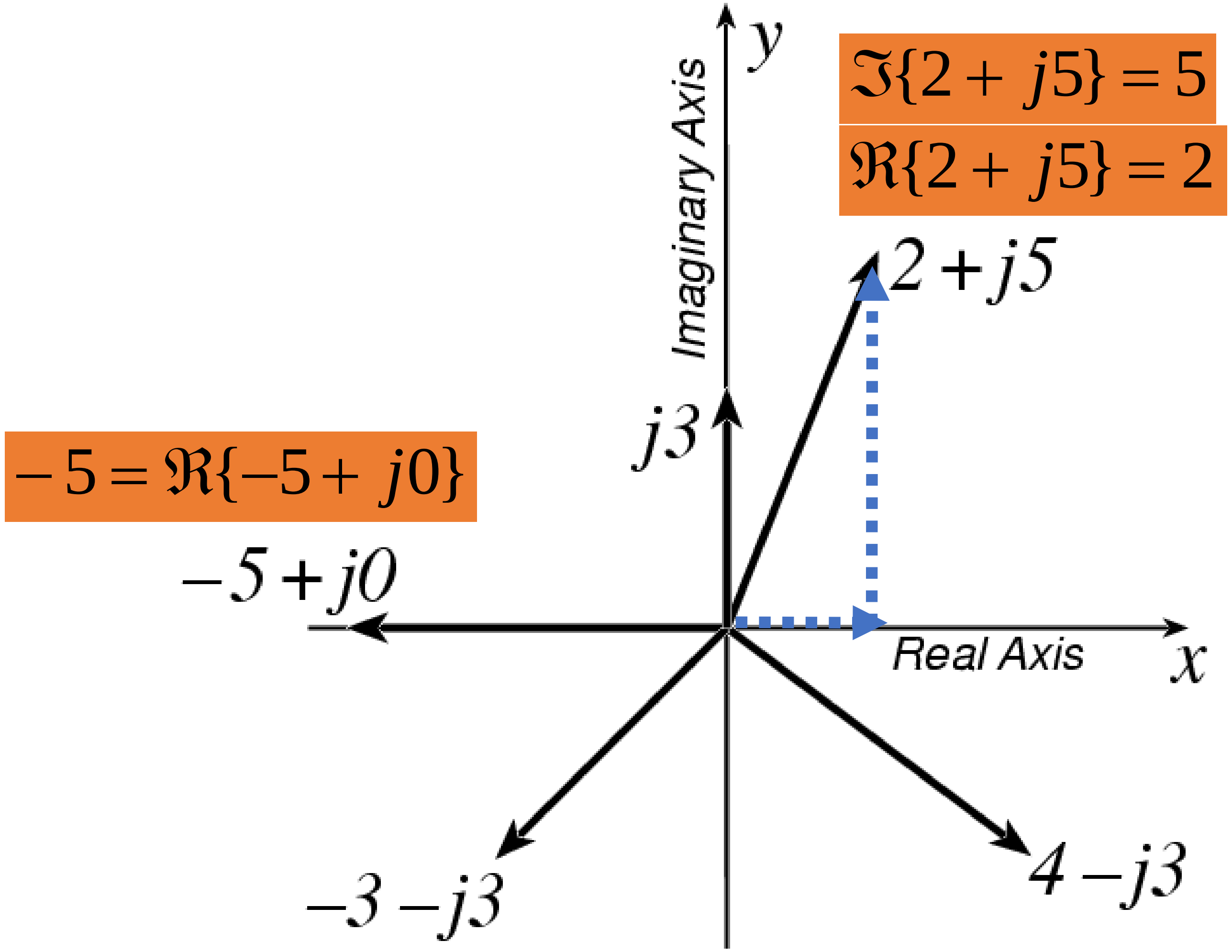
COMPLEX NUMBERS

- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$



Cartesian coordinate system

PLOT COMPLEX NUMBERS



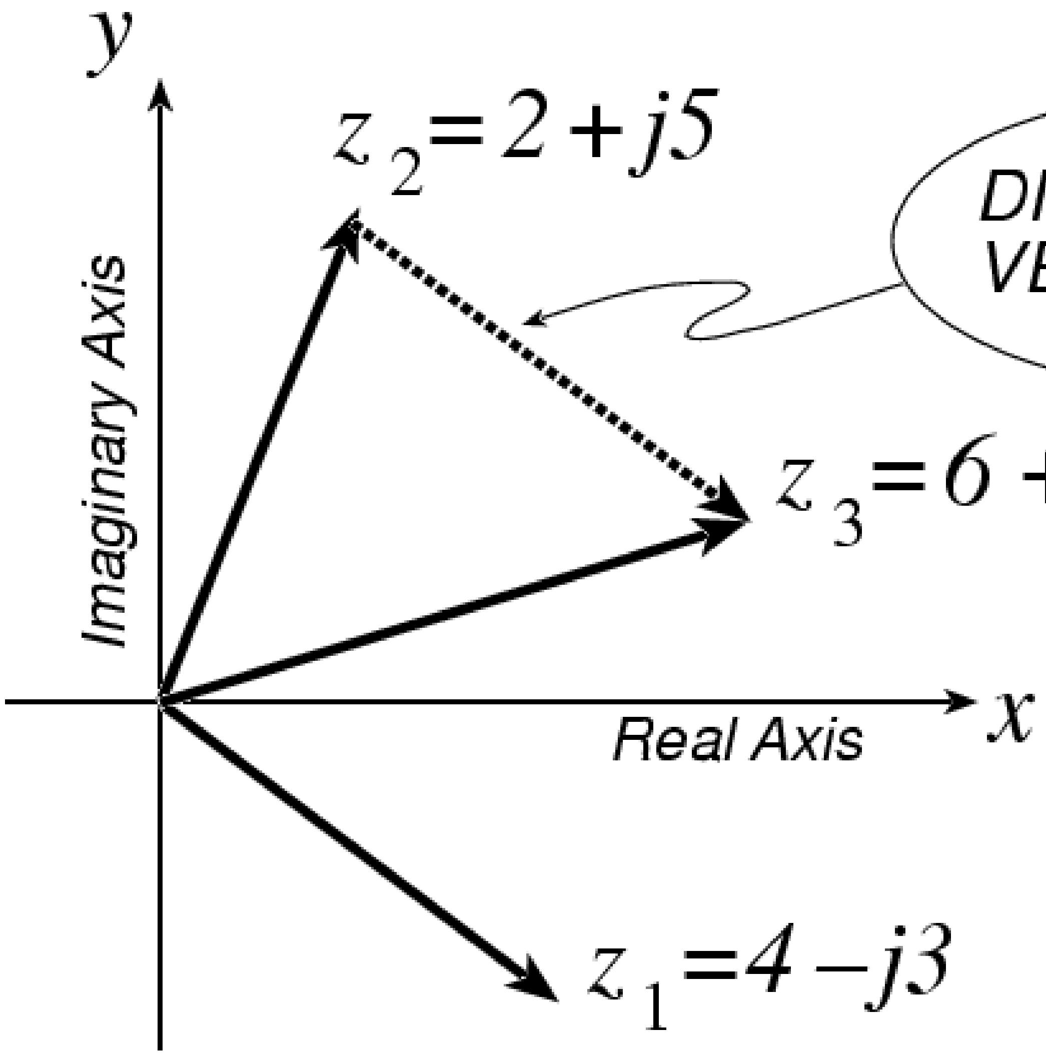
Real part:

$$x=\Re\{z\}$$

Imaginary part:

$$y=\Im\{z\}$$

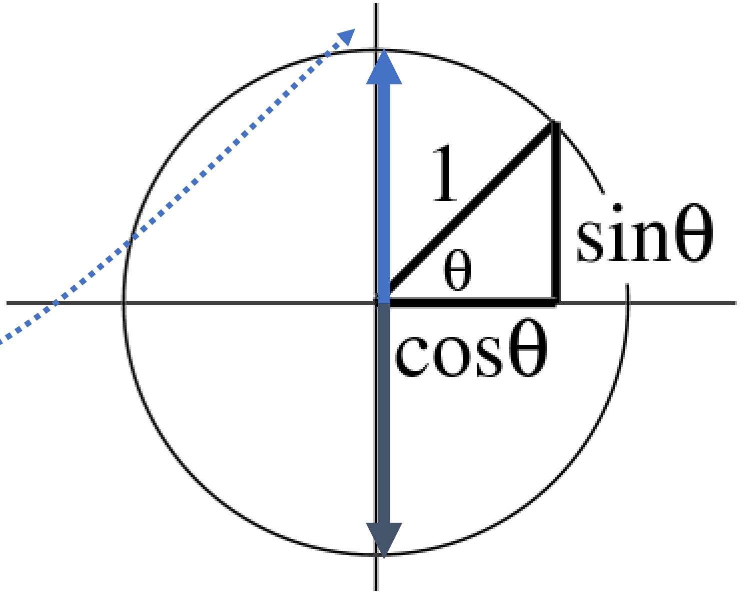
COMPLEX ADDITION = VECTOR Addition



$$
\begin{aligned}
z_3 &= z_1 + z_2 \\
&= (4 - j3) + (2 + j5) \\
&= (4 + 2) + j(-3 + 5) \\
&= 6 + j2
\end{aligned}
$$

*** POLAR FORM ***

- **Vector** Form
  - **Length** =1
  - **Angle** = θ
- Common Values
  - j has angle of 0.5π
  - -1 has angle of π
  - $-$ j has angle of 1.5π
  - also, angle of $-$j **could** be $-0.5\pi = 1.5\pi - 2\pi$
  - because the PHASE is **AMBIGUOUS**



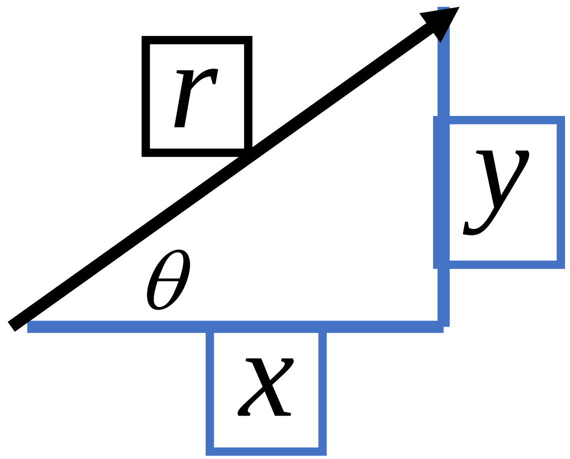
POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$

Most calculators do Polar-Rectangular



$$x = r\cos\theta$$

$$y = r\sin\theta$$

Need a notation for POLAR FORM

MATLAB:

P = atan2(Y,X) returns the four-quadrant inverse tangent ($\tan^{-1}$) of Y and X, which must be real.

ATAN2   WORKS IN ALL 4 QUADRANTS

```
>> a1=-1        %  a1 =    -1
>> b1=-1        % b1 =    -1  %  3rd Quadrant values
>> y3 = atan2(b1,a1)    % y3 =   -2.3562 rad
>> y3deg=rad2deg(y3)   % y3deg =   -135 degrees
```

ATAN

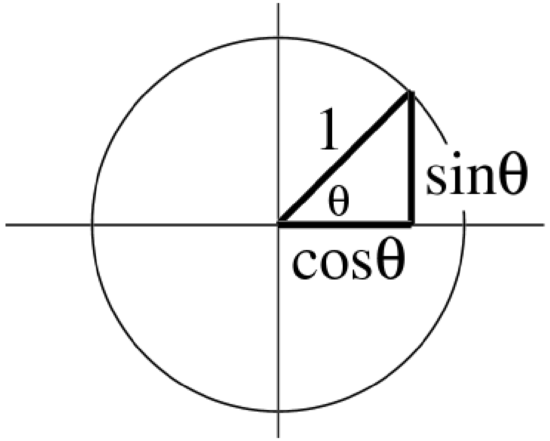
```
>> y4=atan(b1/a1)          % y4 =     0.7854 rad
```

WRONG 😂

```
>> y4deg=rad2deg(y4)    % y4deg =    45 deg
```

Euler’s FORMULA

- **Complex Exponential**
  - [Real part is cosine]
  - Imaginary part is sine
  - Magnitude is one

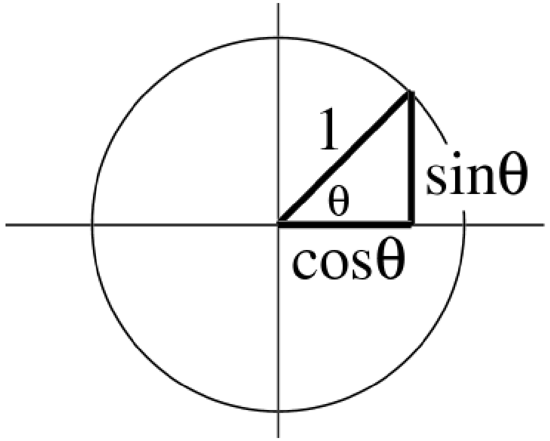


$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

Cosine = Real Part

- Complex Exponential
  - **Real part is cosine**
  - Imaginary part is sine



$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

$$\boxed{\Re\{re^{j\theta}\} = r\cos(\theta)}$$

Common Values of exp(jθ)

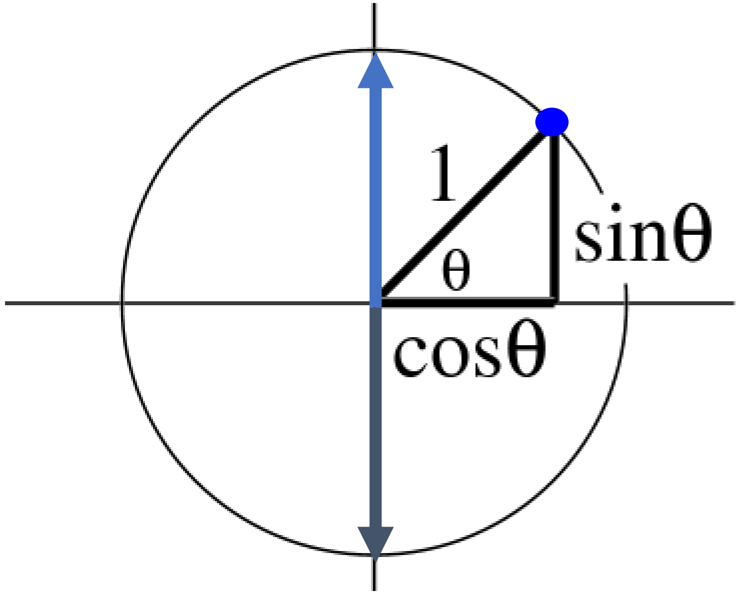
- Changing the angle

$$\theta = 0 \rightarrow 1 = 1 + j0 = e^{j0} = e^{j2n\pi}$$

$$\theta = \pi \rightarrow -1 = -1 + j0 = e^{j\pi} = e^{j(2n+1)\pi}$$

$$\boxed{\theta = \pi/2 \rightarrow j = e^{j\pi/2} = e^{j(2n+1/2)\pi}}$$

$$\boxed{\theta = 3\pi/2 \rightarrow -j = e^{j3\pi/2} = e^{-j\pi/2} = e^{j(2n-1/2)\pi}}$$



$$1 \pm j = \sqrt{2}e^{\pm j\pi/4} \qquad a^2 + b^2 = c^2$$

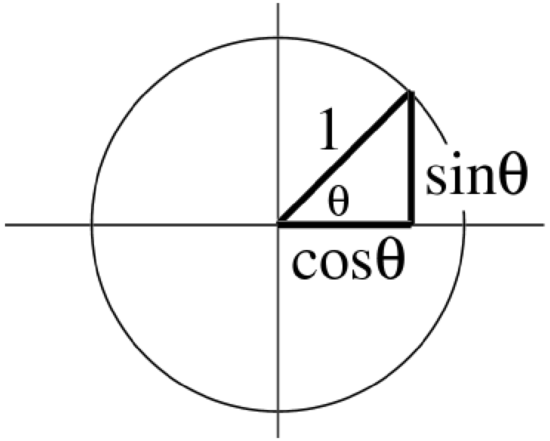
$$\theta = \arctan(\frac{b}{a})$$

IF $a \geq 0$

COMPLEX EXPONENTIAL

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

- Interpret this as a **Rotating Vector**
  - $\boldsymbol{\theta = \omega t}$
  - Angle changes vs. time
  - ex: $\omega=20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

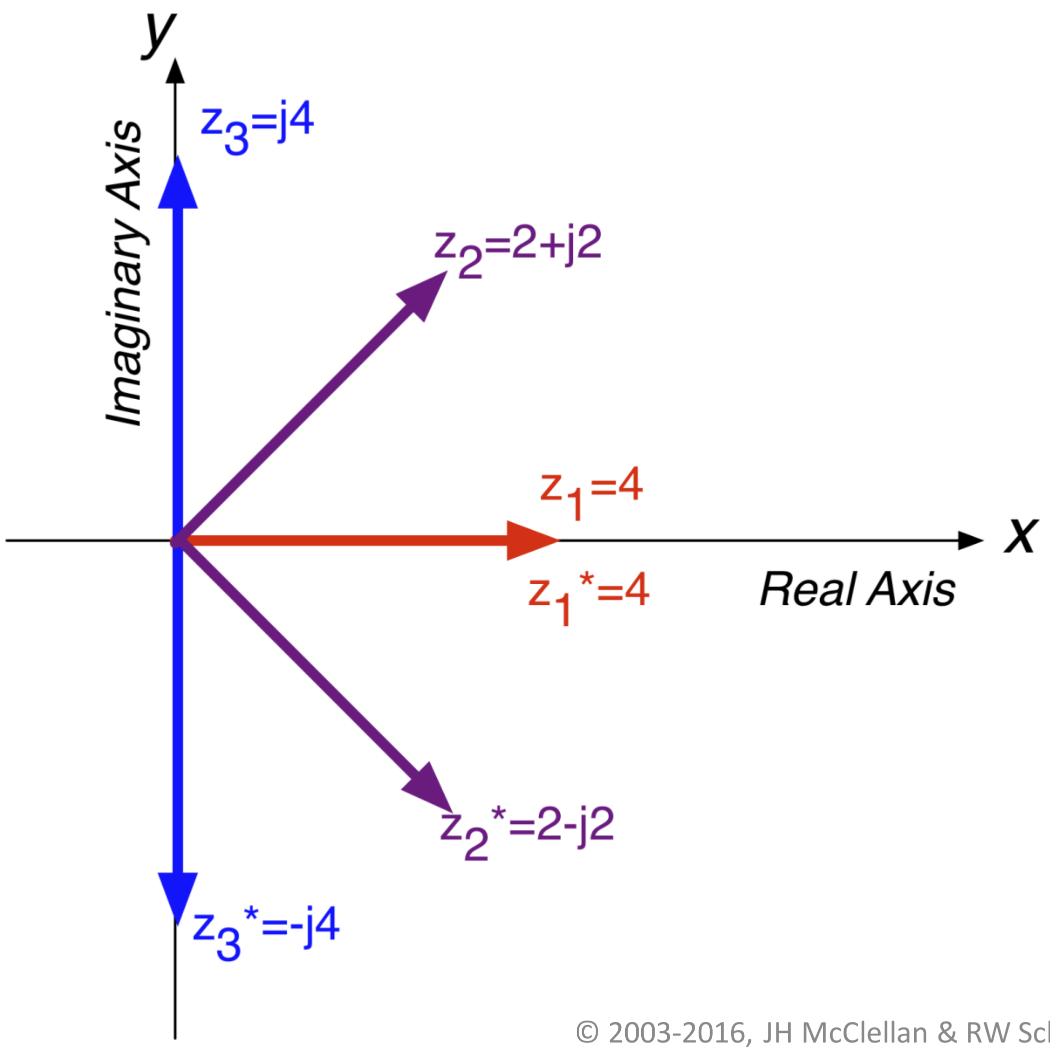
$$A\cos(\omega t + \varphi) = \Re\{Ae^{j(\omega t+\varphi)}\}$$
$$= \Re\{Ae^{j\varphi}e^{j\omega t}\}$$

COMPLEX CONJUGATE (z*)

- Useful concept: change the sign of **all j's**
- **RECTANGULAR**: If z = x + ***j*** y, then the complex conjugate is z* = x – ***j*** y
- **POLAR**: Magnitude is the same but angle has sign change

$$\boxed{z = re^{j\theta} \Rightarrow z^* = re^{-j\theta}}$$

COMPLEX CONJUGATION



- Flips vector about the real axis!

USES OF CONJUGATION

- Conjugates useful for many calculations
- Real part:

$$\frac{z+z^*}{2}=\frac{(x+jy)+(x-jy)}{2}=x=\Re\{z\}$$

- Imaginary part:

$$\frac{z-z^*}{2j}=\frac{j2y}{2j}=y=\Im\{z\}$$

Inverse Euler Relations

- Cosine is real part of exp, sine is imaginary part
- Real part:

$$\frac{z+z^*}{2} = \Re\{z\}$$

$$z = e^{j\theta}, \Rightarrow \Re\{e^{j\theta}\} = \frac{e^{j\theta}+e^{-j\theta}}{2} = \cos(\theta)$$

- Imaginary part:

$$\frac{z-z^*}{2j} = y = \Im\{z\}$$

$$z = e^{j\theta}, \Rightarrow \Im\{e^{j\theta}\} = \frac{e^{j\theta}-e^{-j\theta}}{2j} = \sin(\theta)$$

Mag & Magnitude Squared

- Magnitude Squared (polar form):

$$zz^* = (re^{j\theta})(re^{-j\theta}) = r^2 = |z|^2$$

- Magnitude Squared (Cartesian form):

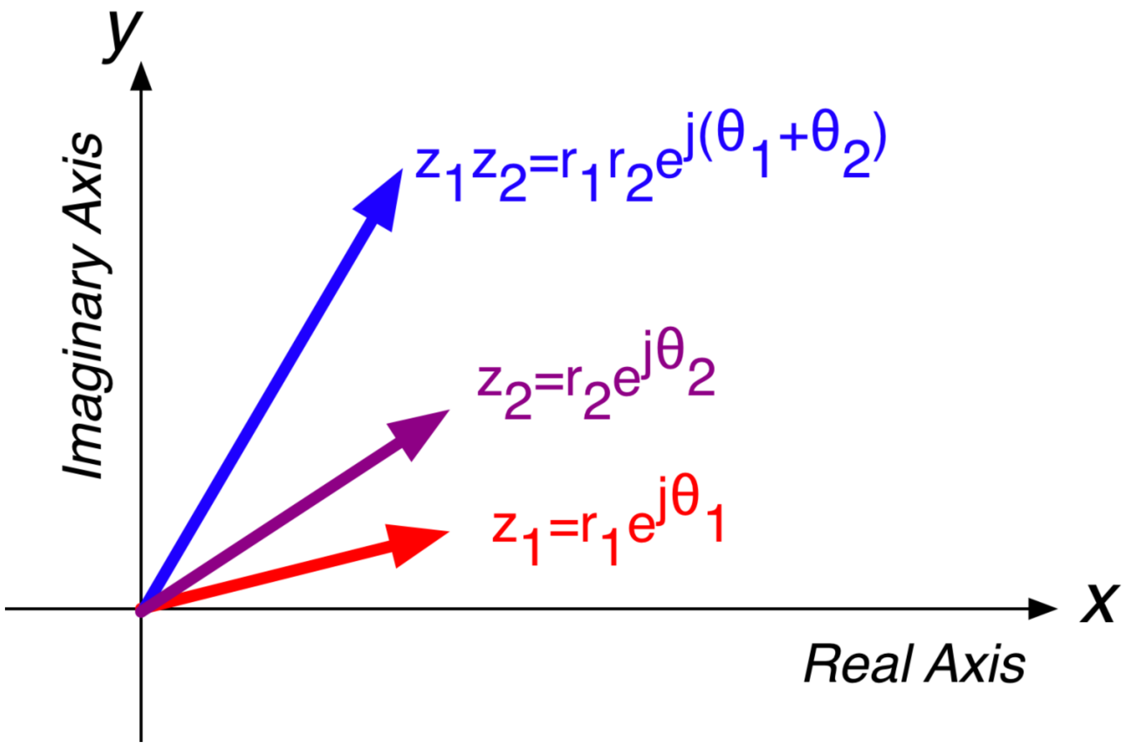
$$zz^* = (x + jy) \times (x - jy) = x^2 - j^2y^2 = x^2 + y^2$$

- Magnitude of complex exponential is one:

$$|e^{j\theta}|^2 = \cos^2(\theta) + \sin^2(\theta) = 1$$

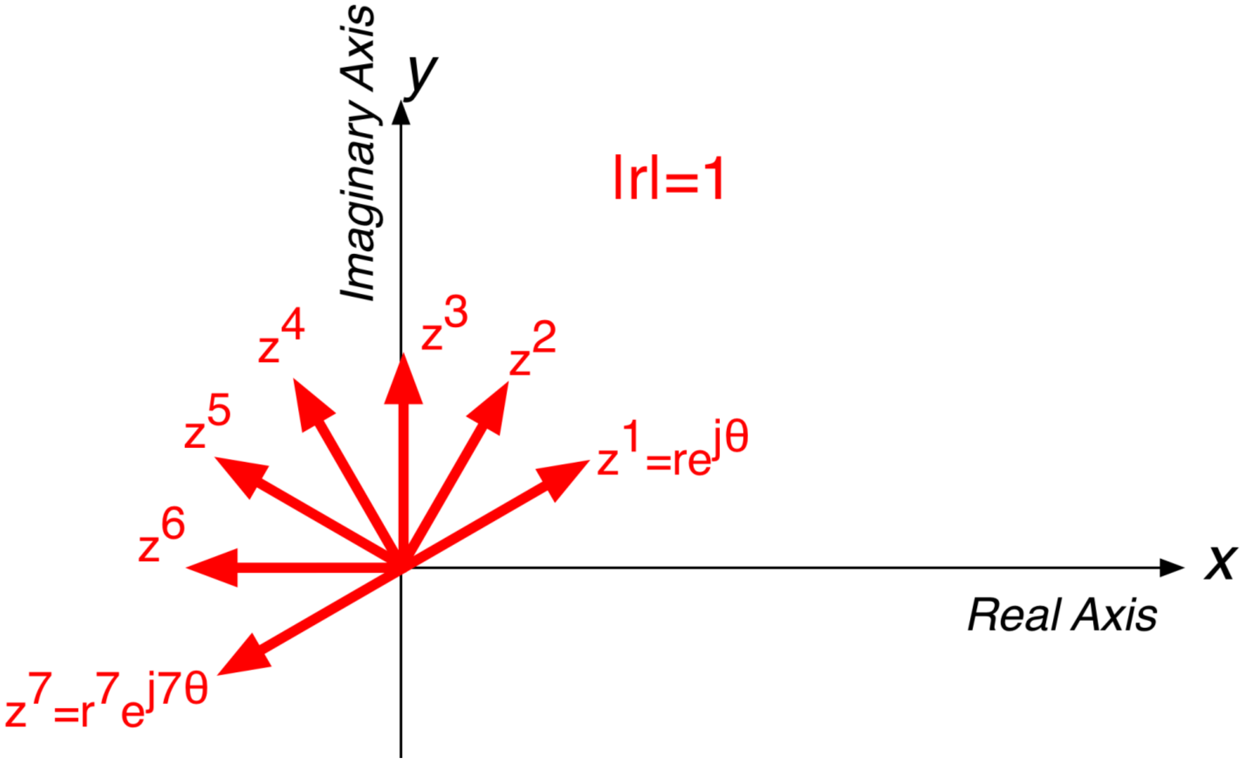
COMPLEX MULTIPLY = VECTOR ROTATION

- Multiplication/division scales and rotates vectors



POWERS

- Raising to a power N rotates vector by Nθ and scales vector length by r^N



MORE POWERS

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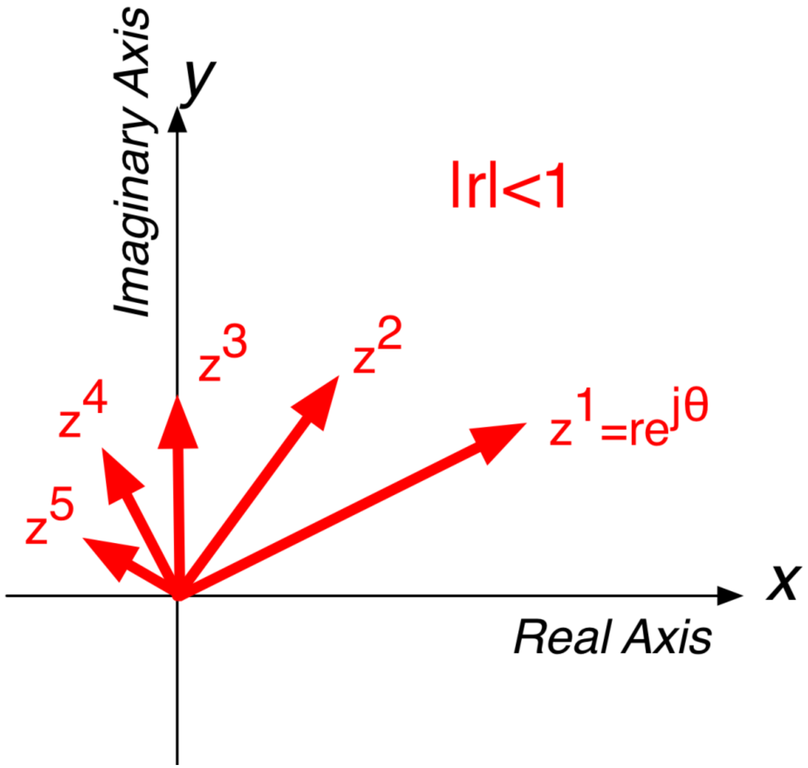
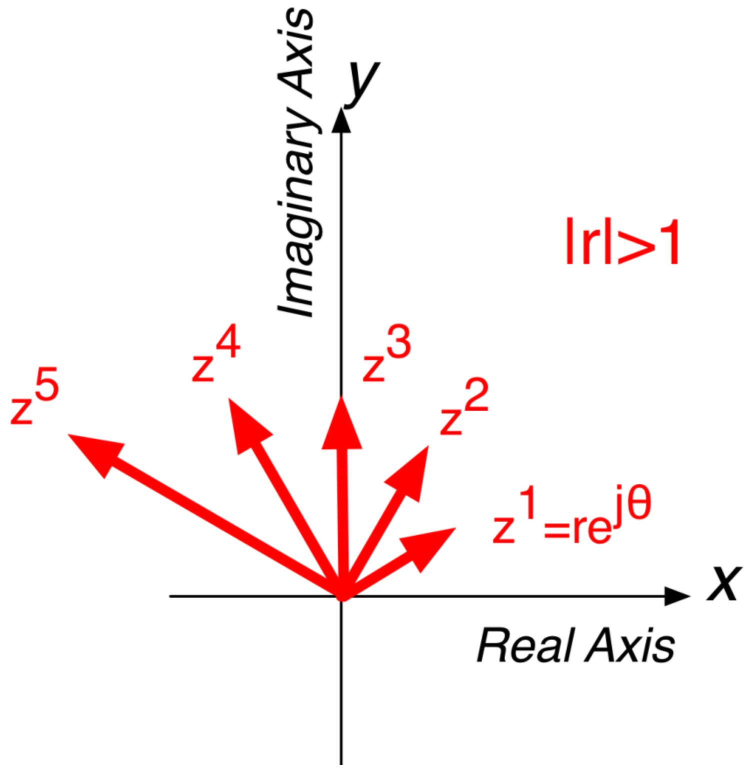

Imaginary Axis
y
|r|<1
z3
z2
z4
z1=rejθ
z5
x
Real Axis


Imaginary Axis
y
|r|>1
z4
z3
z5
z2
z1=rejθ
x
Real Axis

ROOTS OF UNITY

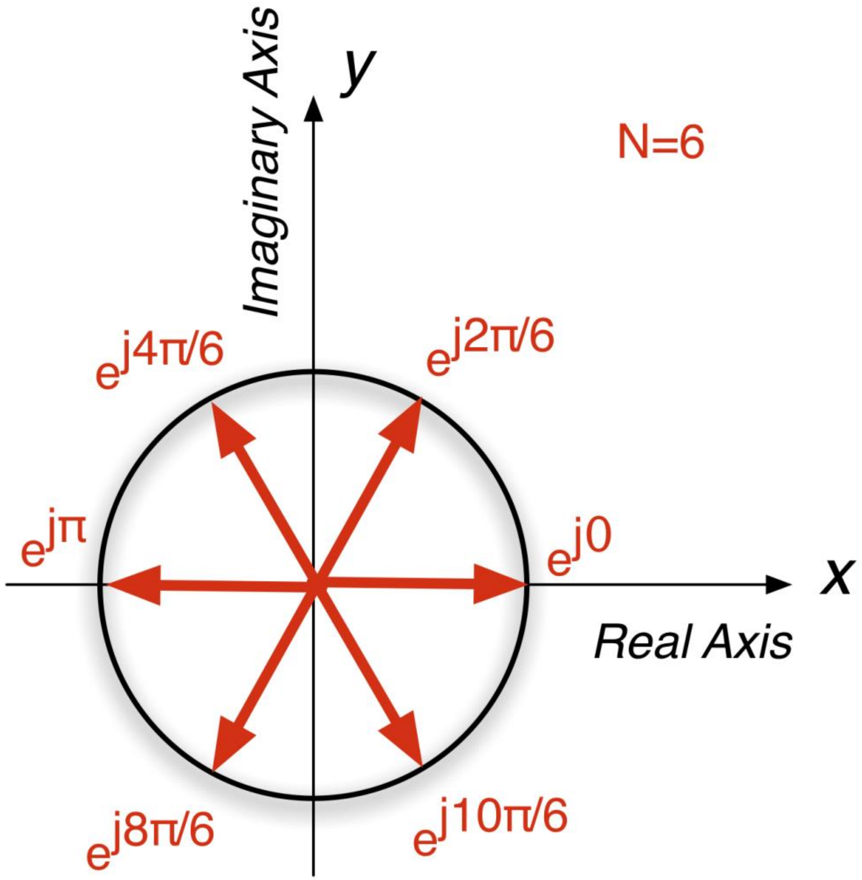
- We often have to solve $z^N=1$
- How many solutions?

$$z^N = r^N e^{jN\theta} = 1 = e^{j2\pi k}$$

$$\Rightarrow r = 1, \quad N\theta = 2\pi k \Rightarrow \theta = \frac{2\pi k}{N}$$

$$z = e^{j2\pi k/N}, \;\; k = 0,1,2,\ldots N-1$$

ROOTS OF UNITY for N=6



- Solutions to $z^N=1$ are N equally spaced vectors on the unit circle!
- What happens if we take the sum of all of them?

Sum the Roots of Unity

- Looks like the answer is zero (for N=6)

$$\sum_{k=0}^{N-1} e^{j2\pi k/N} = 0 ?$$

- Write as geometric sum

$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} \quad \text{then let} \quad r = e^{j2\pi/N}$$

$$\text{Numerator} \quad 1-r^N = 1-(e^{j2\pi/N})^N = 1-e^{j2\pi} = 0$$

Integrate Complex Exp

- Needed later to describe periodic signals in terms of sinusoids (Fourier Series)
  - Especially over one period

$$\int_a^b e^{j\theta}\,d\theta = \left.\frac{e^{j\theta}}{j}\right|_a^b = \frac{e^{jb}-e^{ja}}{j}$$

$$\int_0^T e^{j2\pi t/T}\,dt = \frac{e^{j2\pi T/T}-e^{j0}}{j} = \frac{1-1}{j} = 0$$

BOTTOM LINE

- **CARTESIAN**: Addition/subtraction is most efficient in Cartesian form
- **POLAR**: good for multiplication/division
- STEPS:
  - Identify arithmetic operation
  - Convert to easy form
  - Calculate
  - Convert back to original form

Review Appendix A and TLH Ch2 on Course Website

Harman Chapter 2 Pages 55-60 Complex Numbers
and
MATLAB Complex Numbers

TABLE 2.4 *MATLAB commands for complex numbers*

Command	*Format*
`z=x+yi, z=x+yj`	Complex number
`z=r*exp(i*theta)`	Polar form
abs	Magnitude $\lvert z \rvert = \sqrt{x^2 + y^2}$
angle	Angle in radians $(-\pi, \pi)$; $\theta = \tan^{-1}(y/x)$
conj	Complex conjugate $x - yi$
imag	Complex imaginary part y
real	Complex real part x
Plotting:	
compass	Draws complex numbers as arrows on polar plot
feather	Draws complex numbers as arrows on linear plot