**3315 REVIEW 7 CHAPTER 6**

# DSP First, 2/e

Frequency Response

of FIR Filters

Ch6 Presentation 2

**Overview:** In chapter 6 the frequency response function for FIR filters is introduced.

**When a pure sinusoid passes through a linear time-invariant filter, the output is a sinusoid at the same frequency, but its magnitude and phase might be changed.**

In this chapter, we derive the frequency response formulas for several common FIR filters. Plots of the magnitude and phase versus frequency summarize how the filter treats sinusoidal inputs over the entire range of possible input frequencies.

Finally, the concept of filtering is introduced. Since all signals can be decomposed into sinusoidal components, the frequency response function characterizes frequency regions called stop bands and pass bands, where the FIR filter will reject signal components or pass them nearly undistorted.

# SINUSOIDAL FIDELITY – ONE TEST TO SEE IF SYSTEM IS LTI.

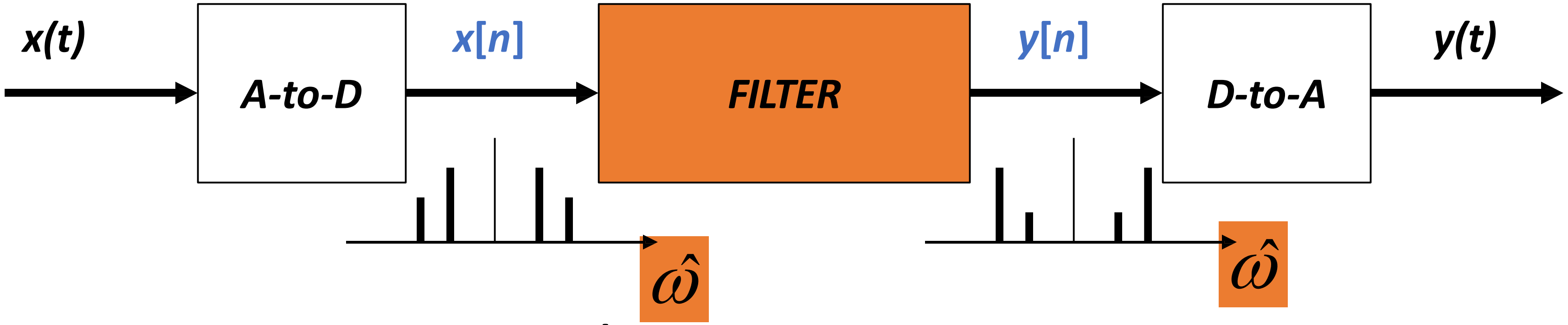
- SINUSOIDAL INPUT SIGNAL
  - DETERMINE the FIR FILTER OUTPUT
- FREQUENCY RESPONSE of FIR
  - PLOTTING vs. Frequency
  - MAGNITUDE vs. Freq
  - PHASE vs. Freq

**MAG**

**PHASE**

$$H(e^{j\hat{\omega}}) = \left|H(e^{j\hat{\omega}})\right| e^{j\angle H(e^{j\hat{\omega}})}$$

# DIGITAL “FILTERING”




- CONCENTRATE on the **SPECTRUM**
- SINUSOIDAL INPUT
  - INPUT x[n] = SUM of SINUSOIDS
  - Then, OUTPUT y[n] = SUM of SAME SINUSOIDS IF LTI.

# SINUSOIDAL RESPONSE TO LTI SYSTEMS

- INPUT: x[n] = SINUSOID
- OUTPUT: y[n] will also be a SINUSOID
  - **Different Amplitude and Phase**
  - **SAME** Frequency
- AMPLITUDE & PHASE CHANGE
  - Called the **FREQUENCY RESPONSE**

# COMPLEX EXPONENTIAL

$$x[n] = Ae^{j\varphi}e^{j\hat{\omega}n} \qquad -\infty < n < \infty$$

**x[n] is the input signal—a complex exponential**

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

**FIR DIFFERENCE EQUATION**

# COMPLEX EXP OUTPUT

- Use the FIR “Difference Equation” Input sinusoid

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k A e^{j\varphi} e^{j\hat{\omega}(n-k)}$$

$$= \left( \sum_{k=0}^{M} b_k e^{j\hat{\omega}(-k)} \right) A e^{j\varphi} e^{j\hat{\omega}n}$$

$$= H(\hat{\omega}) A e^{j\varphi} e^{j\hat{\omega}n}$$

H IS THE TRANSFER FUNCTION

# EXAMPLE 6.1

REMEMBER - PI < $\hat{\omega} \le$ PI

$$\{b_k\} = \{1,\ 2,\ 1\}$$

$$H(e^{j\hat{\omega}}) = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

$$= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}})$$

$$= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega})$$

**EXPLOIT SYMMETRY**

Since $(2 + 2\cos\hat{\omega}) \geq 0$

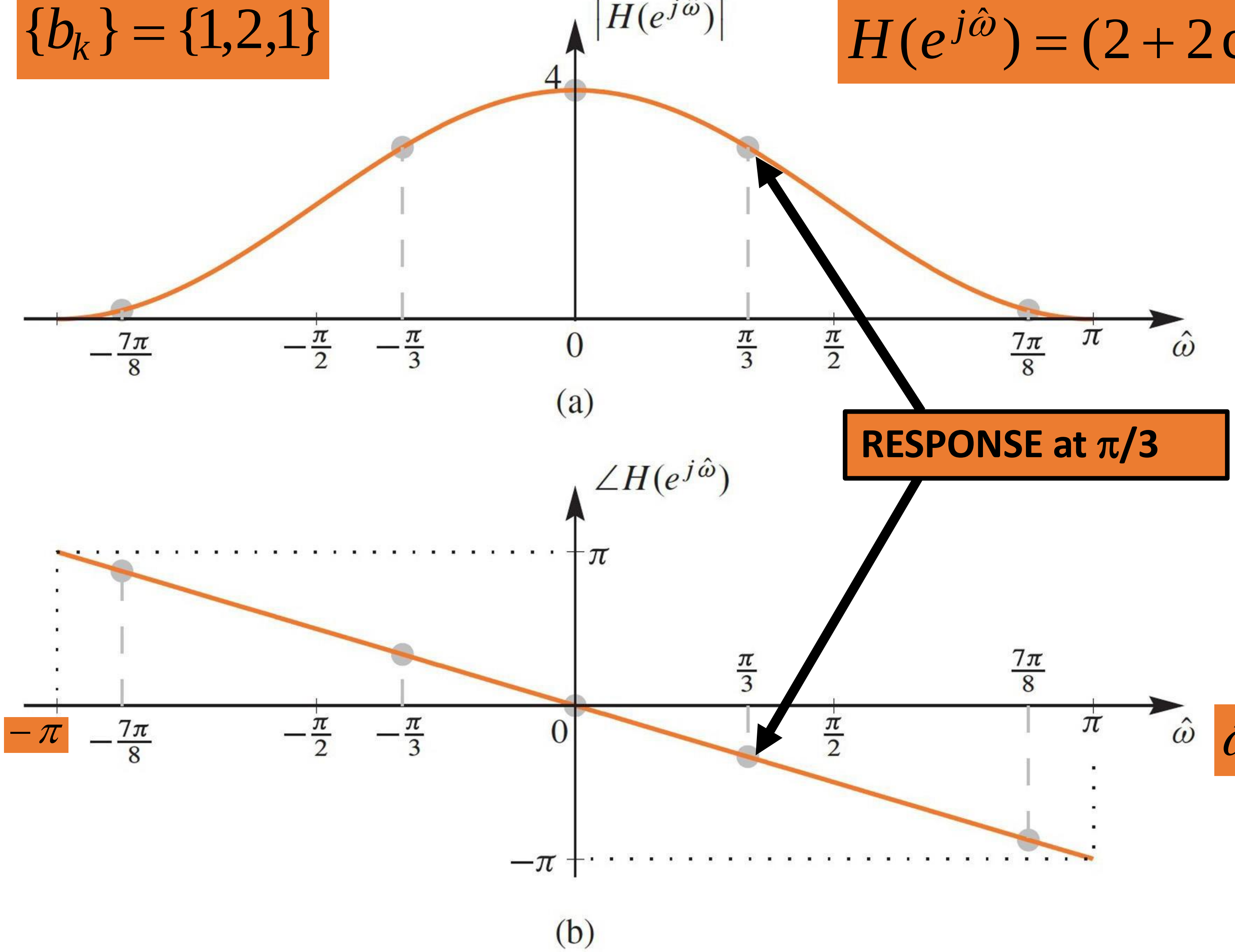
Magnitude is $\left|H(e^{j\hat{\omega}})\right| = (2 + 2\cos\hat{\omega})$

and Phase is $\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$

# PLOT of FREQ RESPONSE

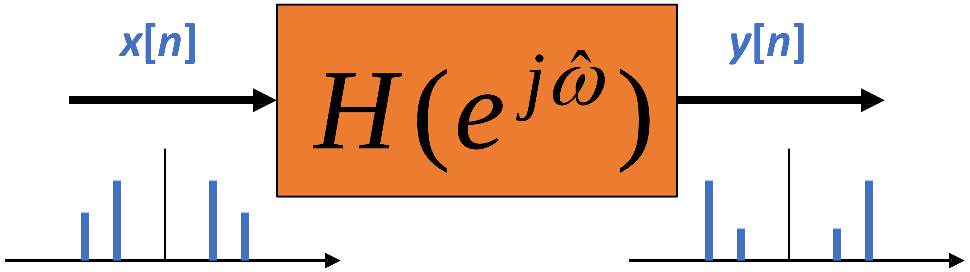
$\{b_k\} = \{1,2,1\}$

$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$



# EXAMPLE 6.2

Find $y[n]$ when $H(e^{j\hat{\omega}})$ is known and $x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$




$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

# EXAMPLE 6.2 (answer)

$$\text{Find } y[n] \text{ when } x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$$

$$\text{Evaluate } H(e^{j\hat{\omega}}) \text{ at } \hat{\omega} = \pi/3$$

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$H(e^{j\hat{\omega}}) = 3e^{-j\pi/3} \qquad @\,\hat{\omega} = \pi/3$$

$$y[n] = \left(3e^{-j\pi/3}\right) \times 2e^{j\pi/4}e^{j(\pi/3)n} = 6e^{-j\pi/12}e^{j(\pi/3)n}$$

# Moving Average Filtering

- The moving average filter occurs frequency enough that we should consider finding a general expression for the frequency response
- The difference equation for an $L$-point averager is

$$y[n] = \frac{1}{L}\sum_{k=0}^{L-1} x[n-k]$$

6.32 ~~(6.30)~~

- The frequency response is

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\hat{\omega}k}$$

~~(6.31)~~ 6.25/L

- It can be shown that

$$\sum_{k=0}^{L-1} \alpha^k = \begin{cases} \dfrac{1-\alpha^L}{1-\alpha}, \alpha \neq 1 \\ L, \qquad \alpha = 1 \text{ (why?)} \end{cases} \qquad 6.24 \qquad \cancel{(6.32)}$$

- ~~Apply (6.32) to (6.31)~~ by setting $\alpha = e^{-j\hat{\omega}}$

$$H(e^{j\hat{\omega}}) = \frac{1}{L}\left(\frac{1-e^{-j\hat{\omega}L}}{1-e^{-j\hat{\omega}}}\right)$$

$$= \frac{1}{L}\left(\frac{e^{-j\hat{\omega}L/2}(e^{j\hat{\omega}L/2}-e^{-j\hat{\omega}L/2})}{e^{-j\hat{\omega}/2}(e^{j\hat{\omega}/2}-e^{-j\hat{\omega}/2})}\right) \qquad \cancel{(6.33)}$$

$$= \left(\frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}\right)e^{-j\hat{\omega}(L-1)/2}$$

**6.25**

- Notice that the phase response is composed of a linear term $e^{-j\hat{\omega}(L-1)/2}$ and $\pm\pi$ due to the sign changes of $\sin(\hat{\omega}L/2)/[L\sin(\hat{\omega}/2)]$
- In MATLAB

$$D_L(e^{j\hat{\omega}}) = \texttt{diric}(\hat{\omega}, L) = \frac{\sin(\hat{\omega}L/2)}{L\sin(\hat{\omega}/2)}$$

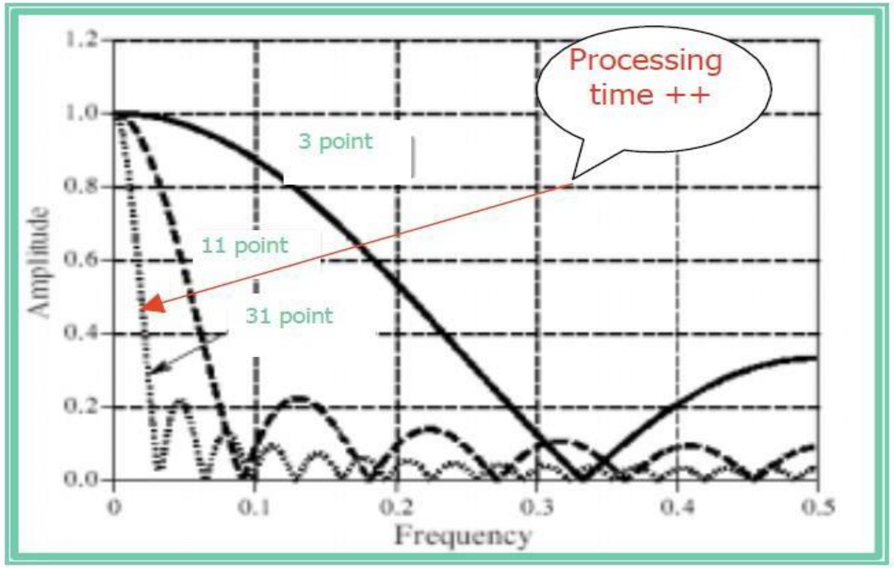
can be used for analyzing moving average filters

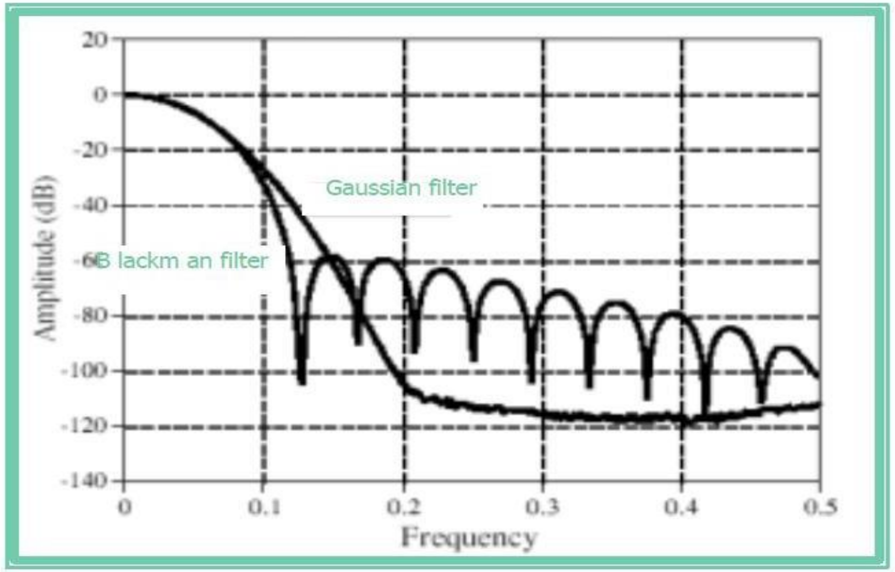
**Peter Gustav Lejeune Dirichlet**

February 1805 – 5 May 1859

# Embedded DSP: Moving Average Filters

- The frequency response is mathematically described by the Fourier Transform of the rectangular pulse.
- H[f]=sin(Pi f M) / M sin(Pi f)
- The roll-off is very slow, the stopband attenuation is very weak !
- The moving average filter is a good smoothing filter but a bad low-pass-filter !



## Example: Lowpass Averager

- Consider a 5-point moving average filter wrapped up between a C-to-D and D-to-C system
- We assume a sampling rate of 1000 Hz and an input composed of two sinusoids

$$x(t) = \cos[2\pi(100)t] + 3\cos[2\pi(300)t]$$

- Find the system frequency response in terms of the analog frequency variable $f$, and find the steady-state output $y(t)$
- We will use `freqz()` to obtain the frequency response

```
>> w = -pi:pi/100:pi;
>> H = freqz(ones(1,5)/5,1,w);
>> subplot(211)
```

help freqz

freqz Frequency response of digital filter

[H,W] = freqz(B,A,N) returns the N-point complex frequency response vector H and the N-point frequency vector W in radians/sample of the filter:

```
             jw               -jw              -jmw
      jw  B(e)    b(1) + b(2)e + .... + b(m+1)e
   H(e) = ---- = ------------------------------------
             jw               -jw              -jnw
          A(e)    a(1) + a(2)e + .... + a(n+1)e

given numerator and denominator coefficients in vectors B and A.
```

**FOR FIR FILTER THE DENOMINATOR IS 1**

**SOLVE THE PROBLEM BEFORE MATLAB – IF POSSIBLE!**
**AT LEAST KNOW THE RANGES INVOLVED**

$f_s = 1000\ Hz$ so $f_{max} = 500\ Hz$

$x(t) = \cos\{2\pi(100t)\} + 3\cos[2\pi\ 300t]$

$\hat{\omega} = \omega T_s = \frac{\omega}{f_s}$ $\quad \hat{\omega}_{100} = 2\pi \frac{100}{1000} = 0.2\pi$ $\quad \hat{\omega}_{300} = 0.6\pi$

Range of $\hat{\omega}$ $\quad -\frac{500}{1000} \times 2\pi < \hat{\omega} < \frac{500}{1000} 2\pi$

$-\pi < \hat{\omega} < \pi$

Digital PLOT $\quad -\pi < \hat{\omega} < \pi$

Analog PLOT $\quad -500\ Hz < f < 500\ Hz$

EXAMPLE 5PT MOVING AVERAGE FIR

$$y[n] = \sum_{k=0}^{4} b_k\, x[n-k] = \frac{1}{5}\left\{ x[n] + x[n-1] + x[n-2] + x[n-3] + x[n-4] \right\}$$

$b_k = \frac{1}{5} \quad k = 0, 1, \ldots 4$

EXPECT MAGNITUDE OF FREQ RESPONSE

$$\left|H(e^{j\hat{\omega}})\right| = \left|D_L(\hat{\omega})\right| = \left|\frac{\sin(\hat{\omega} L/2)}{\sin(\hat{\omega}/2)}\right| \quad 6.27 \text{ Pg } 215$$

HERE $L=5$ SO

$$\left|H_5(e^{j\hat{\omega}})\right| = \left|\frac{\sin(\hat{\omega}\, 5/2)}{\sin(\hat{\omega}/5)}\right|$$

zeros of $H_5$ where $\hat{\omega} = k \cdot 2\pi/L = k\, 2\pi/5$

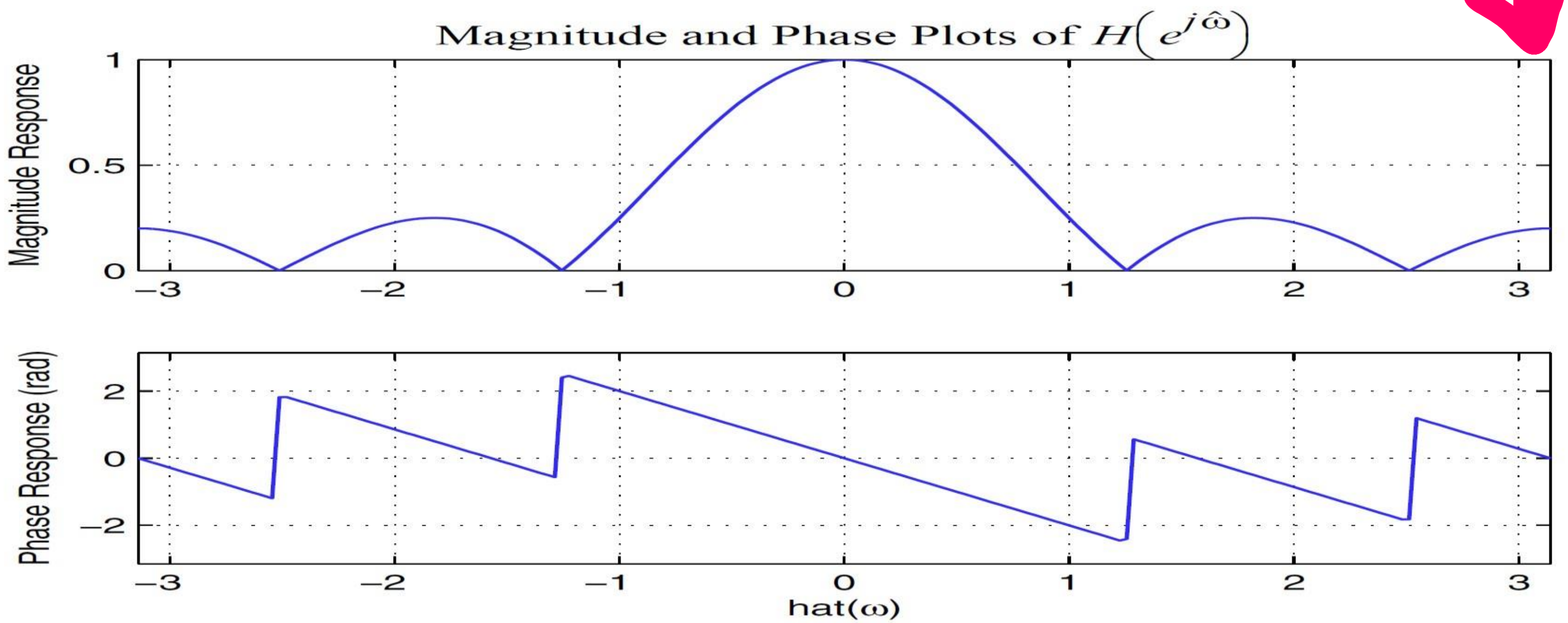
$2\pi/5 \quad 4\pi/5$

EXPECT 5 EXTREMA from $-\pi$ to $\pi$

```
>> plot(w,abs(H))
>> axis([-pi pi 0 1]); grid
>> ylabel('Magnitude Response')
>> subplot(212)
>> plot(w,angle(H))
>> axis([-pi pi -pi pi]); grid
>> ylabel('Phase Response (rad)')
>> xlabel('hat(\omega)')
```

Range –pi to pi

500 HZ

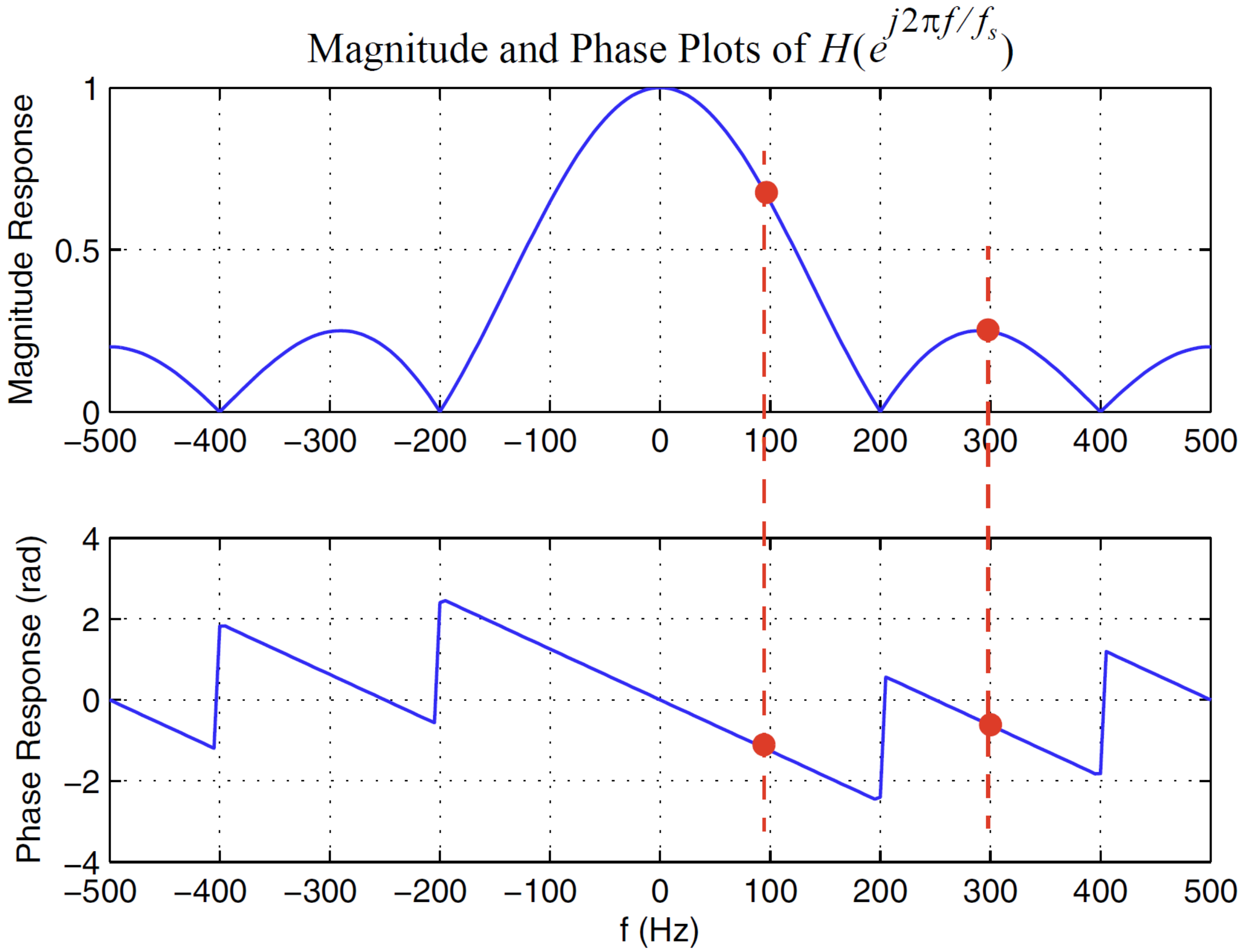



```
>> subplot(211)
>> plot(w*1000/(2*pi),abs(H))
>> grid
>> ylabel('Magnitude Response')
>> subplot(212)
>> plot(w*1000/(2*pi),angle(H))
>> grid
>> ylabel('Phase Response (rad)')
>> xlabel('f (Hz)')
```

- The output $y(t)$ will be of the same form as the input $x(t)$, except the sinusoids at 100 and 300 Hz need to have the filter frequency response applied
  - Note 100 Hz and 300 Hz < 1000/2 = 500 Hz (no aliasing)
- To properly apply the filter frequency response we need to convert the analog frequencies to the corresponding discrete-time frequencies

$$\begin{aligned} 100\text{Hz} &\to 2\pi \cdot \frac{100}{1000} = 2\pi \cdot 0.1 = 0.2\pi \\ 300\text{Hz} &\to 2\pi \cdot \frac{300}{1000} = 2\pi \cdot 0.3 = 0.6\pi \end{aligned} \tag{6.40}$$

*Filtering Sampled Continuous-Time Signals*




EACH POINT IS A TRANSFER VALUE SHOWING THE CHANGE IN MAGNATUDE AND PHASE AT EACH FREQUENCY.

With pi = 500 Hz, expect zeros at 2*pi/5 = 200 Hz and 4*pi/5 = 400 Hz.

- The frequency response for $L = 5$ in the general moving average filter is

$$H(e^{j\hat{\omega}}) = \frac{\sin(2.5\hat{\omega})}{5\sin(\hat{\omega}/2)}e^{-j2\hat{\omega}} \qquad (6.41)$$

- The frequency response at these two frequencies is

$$\begin{aligned} H(e^{j0.2\pi}) &= \frac{\sin(2.5(0.2\pi))}{5\sin((0.2\pi)/2)}e^{-j2(0.2\pi)} = 0.6472e^{-j0.4\pi} \\ H(e^{j0.6\pi}) &= \frac{\sin(2.5(0.6\pi))}{5\sin((0.6\pi)/2)}e^{-j2(0.6\pi)} = \underbrace{-0.2472e^{-j1.2\pi}}_{0.2472e^{-j0.2\pi}} \end{aligned} \qquad (6.42)$$

```
>> diric(0.2*pi,5)

ans = 0.6472

>> diric(0.6*pi,5)

ans = -0.2472 % also = 0.2472 at angle +/- pi
```

- The filter output $y[n]$ is

$$y[n] = 0.6472\cos[0.2\pi n - 0.4\pi] + 0.7416\cos[0.2\pi n - 1.2\pi + \pi] \quad (6.43)$$

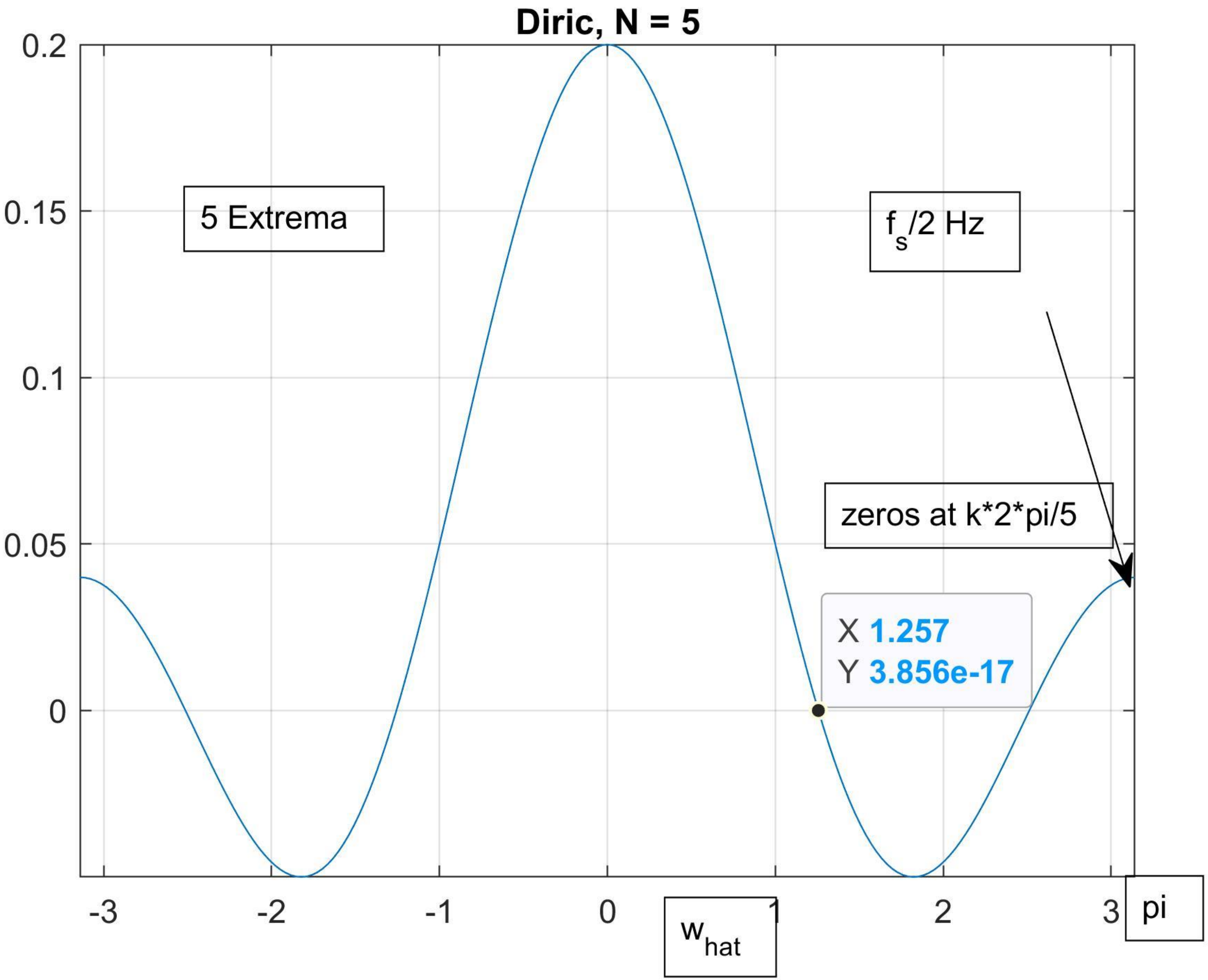
and the D-to-C output is

$$y(t) = 0.6472\cos[2\pi(100)t - 0.4\pi] + 0.7416\cos[2\pi(300)t - 0.2\pi] \quad (6.44)$$

Back to Analog

DSPF Chapter 6

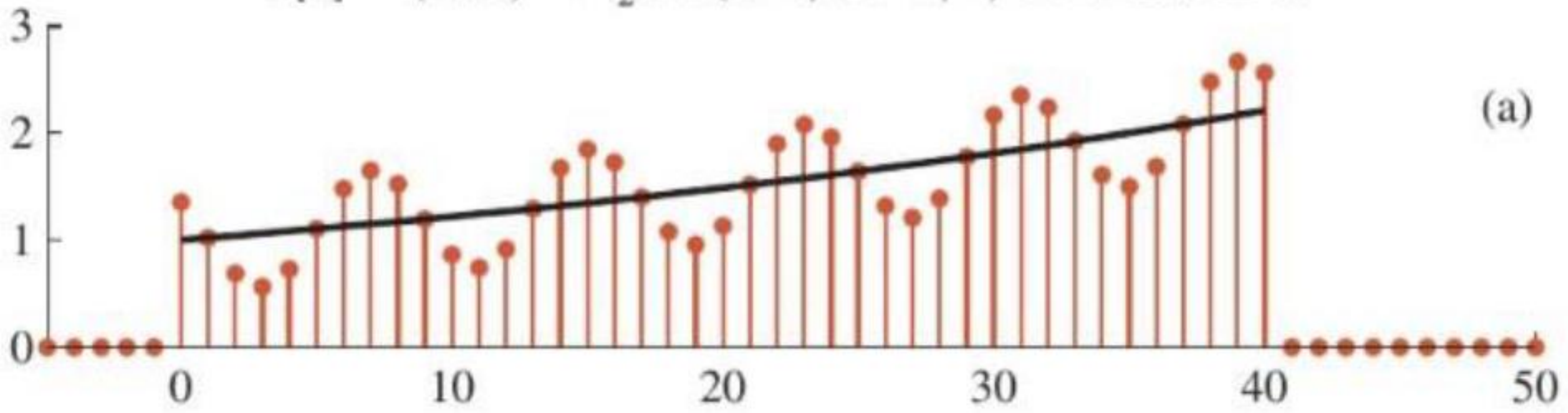
```
%%   Plot the Dirichlet function/N over the range 0
to pi,
% for N = 5; 5-point moving average. See DSPF Page
215
clc, clear all, clf
w=-pi:pi/100:pi;
w_zeros= [2*pi/5,4*pi/5]      %  1.2566     2.5133
figure(1)
plot(w,(1/5)*diric(w,5));
title('Diric, N = 5'),grid; axis tight;
```



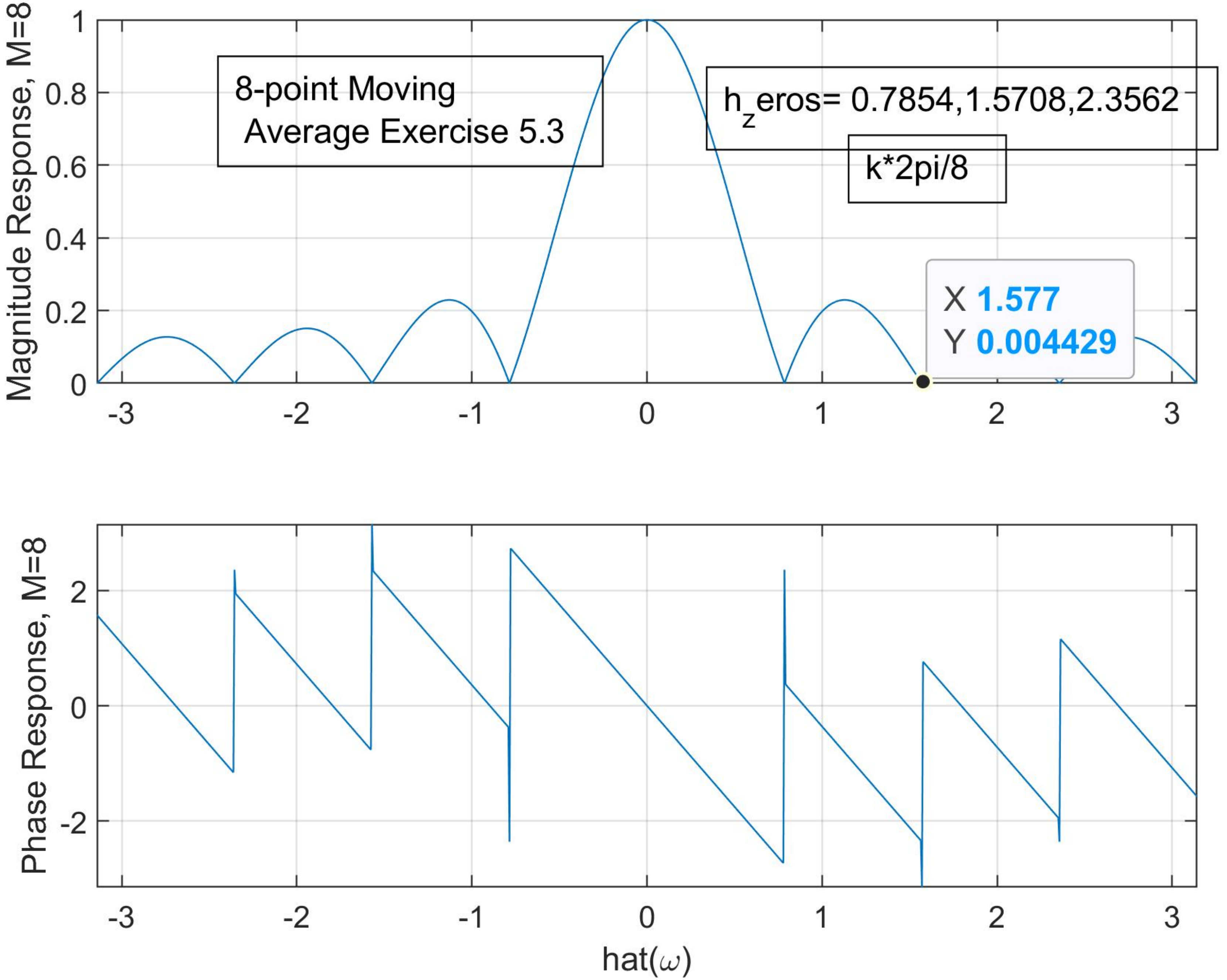
Diric, N = 5
5 Extrema
f_s/2 Hz
zeros at k*2*pi/5
X 1.257
Y 3.856e-17
0.2
0.15
0.1
0.05
0
-3
-2
-1
0
1
2
3
w_hat
pi

## SOLUTION to EXERCISE 5.3: 

$$x[n] = (1.02)^n + \tfrac{1}{2}\cos(2\pi n/8 + \pi/4) \text{ for } 0 \le n \le 40$$

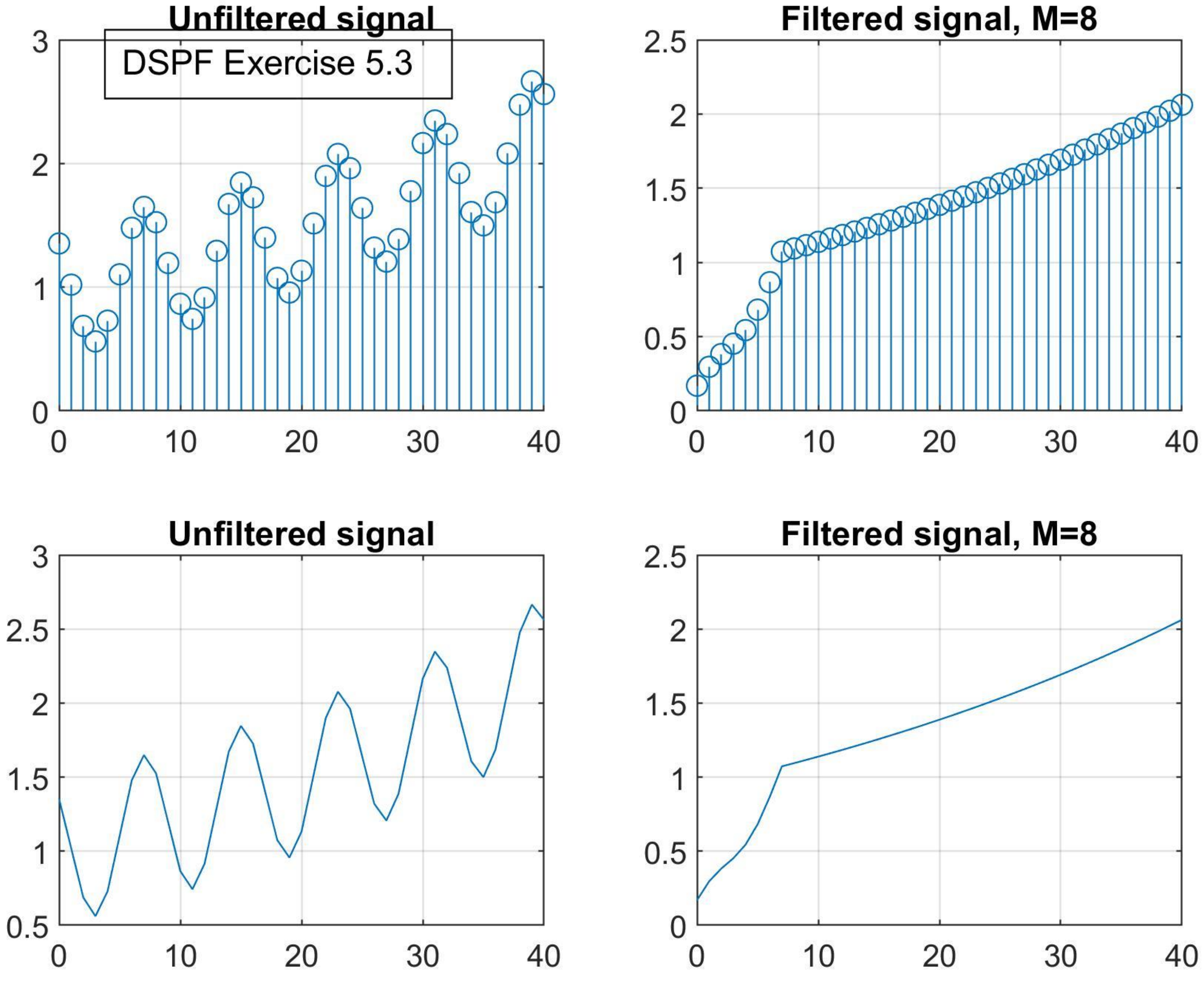



**KILL THE COSINE – (Need a zero in H at 2*pi/8 =0.7854)**



Magnitude Response, M=8
8-point Moving
Average Exercise 5.3
h_zeros= 0.7854,1.5708,2.3562
k*2pi/8
X 1.577
Y 0.004429
1
0.8
0.6
0.4
0.2
0
-3
-2
-1
0
1
2
3
Phase Response, M=8
2
0
-2
hat(ω)

```
Exercise 5_3  8-point moving average
%  Null out the cosine in a signal w_hat= k*2*pi/8
clc, clear all, clf
M=8
%windowSize = 8;
b=(1/M)*ones(1,M);    % b_i = 0.125 = (1/8)
a = 1;
%
% Create function
n=[0:1:40];    % 41 Points in n
x= (1.02).^n + 0.5*cos(2*pi*n/8 + pi/4);
%  w_hat = 2*pi/8   So use 8-point average with
%    zeros at k*2*pi/8  k= 1,2,3
y = filter(b,a,x);
figure(1)
subplot(2,2,1),stem(n,x),grid, title('Unfiltered signal')
subplot(2,2,2),stem(n,y),grid, title('Filtered signal, M=8')
%
subplot(2,2,3),plot(n,x),grid,title('Unfiltered signal')
subplot(2,2,4),plot(n,y),grid, title('Filtered signal, M=8')
```



Unfiltered signal
DSPF Exercise 5.3
Filtered signal, M=8
Unfiltered signal
Filtered signal, M=8

# *Convolution AND Frequency Domain*

Convolving two waveforms in the **time domain** means that you are **multiplying** their spectra (i.e. frequency content) in the frequency domain. By "multiplying" the spectra we mean that any frequency that is strong in **both** signals will be very strong in the convolved signal, and conversely any frequency that is weak in either input signal will be weak in the output signal.

**THIS IS ONE OF THE MAIN REASONS TO WORK IN THE FREQUENCY DOMAIN**

| Spatial Domain | | Frequency Domain |
|---|---|---|
| | CONVOLVE / MULTIPLY | |
| $g = f * h$ | $\longleftrightarrow$ | $G = FH$ |
| $g = fh$ | $\longleftrightarrow$ | $G = F * H$ |