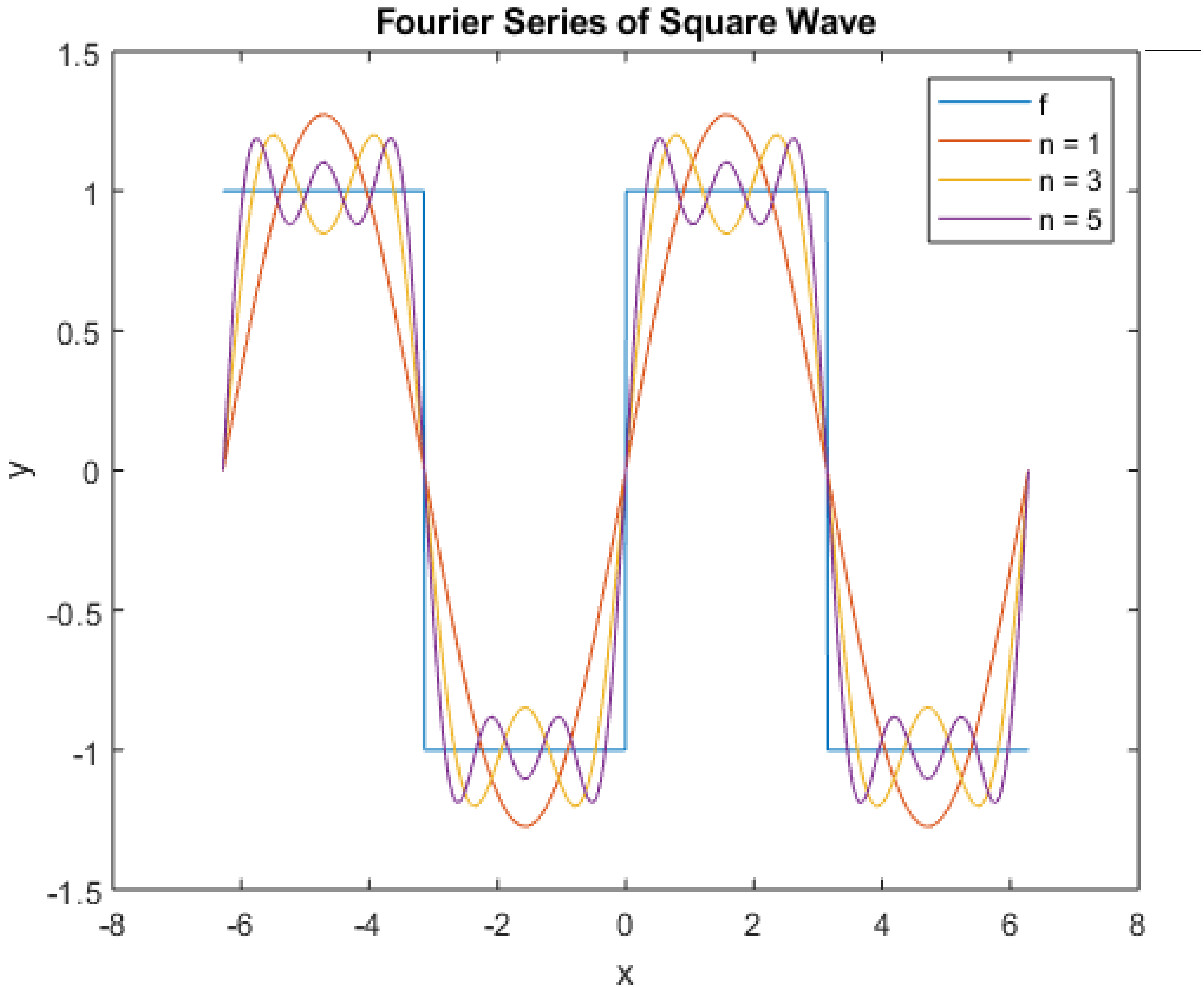
Review4 Fourier Series

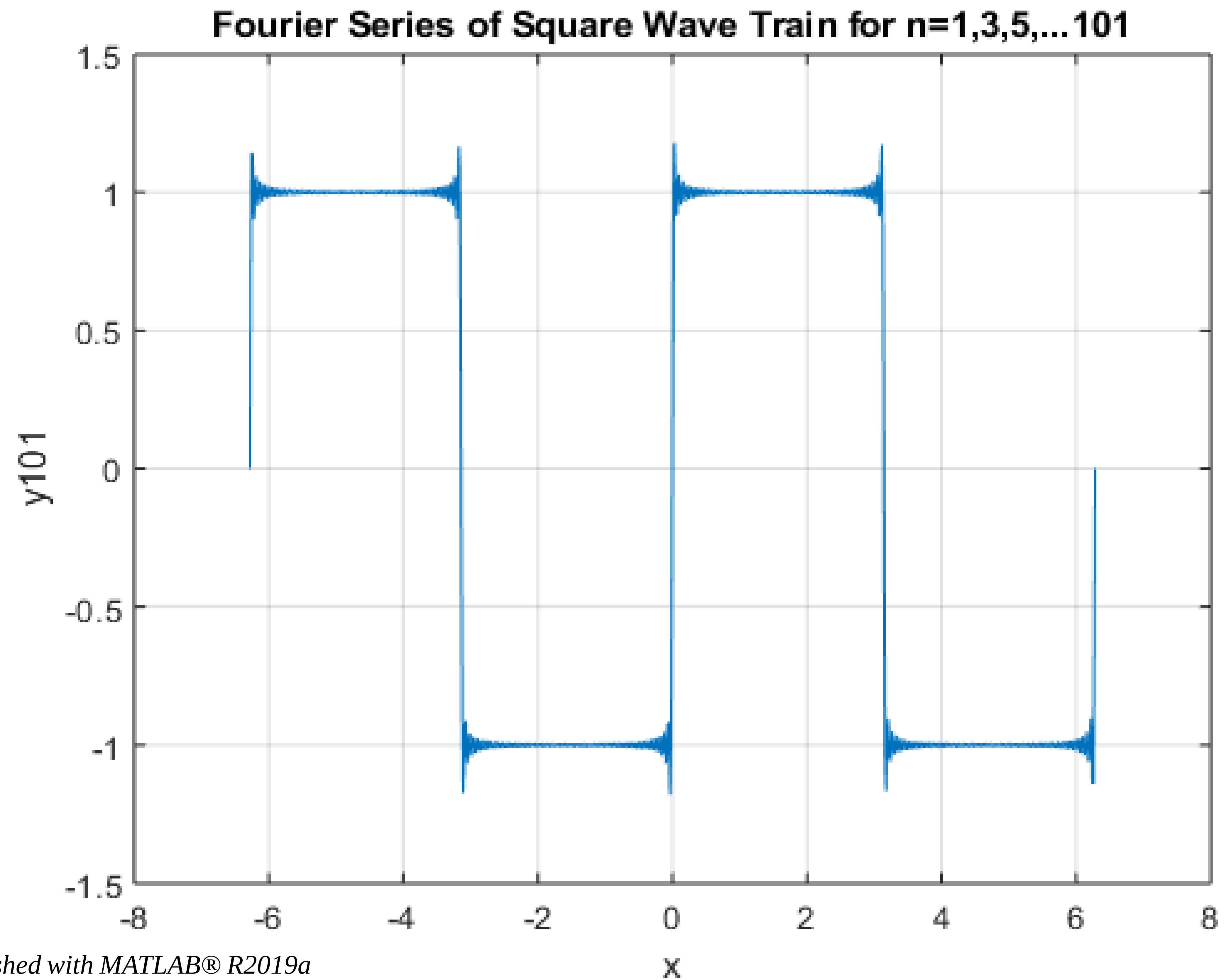
Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.



Fourier Series of Square Wave
f
n = 1
n = 3
n = 5
1.5
1
0.5
0
-0.5
-1
-1.5
y
-8
-6
-4
-2
0
2
4
6
8
x



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CODE from M-file FourierSeriestest1.m
A square wave A (T=2*pi)

Show 3 terms and then 51 odd terms to N=101

```
clear, clf
close
nmax = 5; %  Iterations for the Fourier series
x = linspace(-2*pi,2*pi,1000); % Domain of plot  1000 points
% Plot of function f(x)- the square wave from -2pi to 2pi
for i = 1:length(x)
if x(i) <= -pi | (x(i) > 0 & x(i) <= pi)
f(i) = 1;
end
if (x(i) > -pi & x(i) <= 0) | x(i) > pi
f(i) = -1;
end

end
figure(1),plot(x,f),title('Press a key for next Harmonic')
hold on
```

MATLAB_Fourier_SquareWave   ON Course Website

[-T/2, T/2] INTERVAL

On the interval $[-T/2, T/2]$, the limits of integration for the Fourier series can be changed from $[-\pi, \pi]$ by assigning to the integration variable t the value $2\pi t/T$. The period of the function is thus T.

Assuming that $f(t)$ is continuous on the interval $-T/2 \leq t \leq T/2$, the coefficients a_n and b_n can be computed by the formulas

Trig Form

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt,$$

2xAverage over a Period

$$\begin{aligned} a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2n\pi t}{T}\right)dt, \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2n\pi t}{T}\right)dt, \end{aligned} \tag{8.20}$$

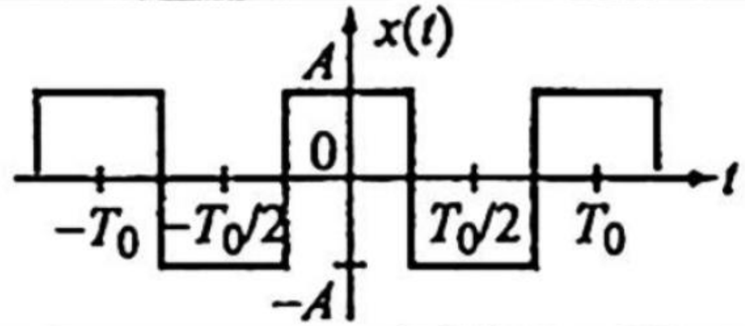
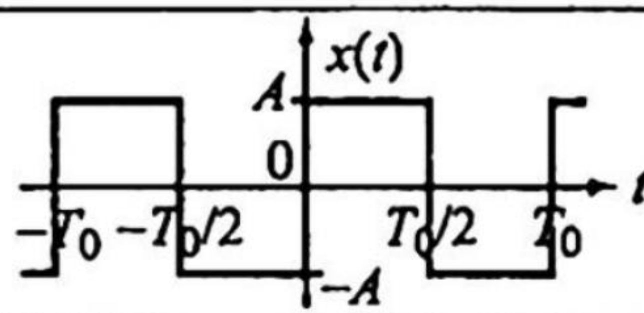
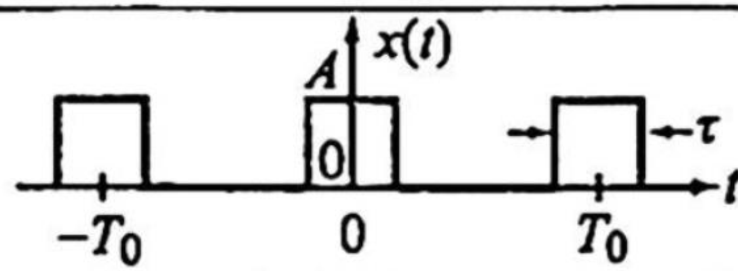
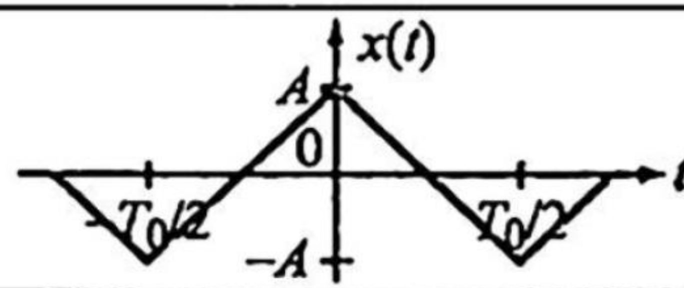
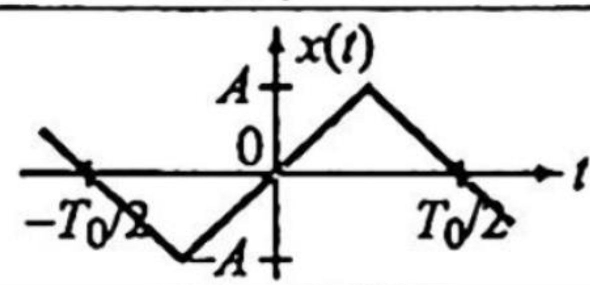
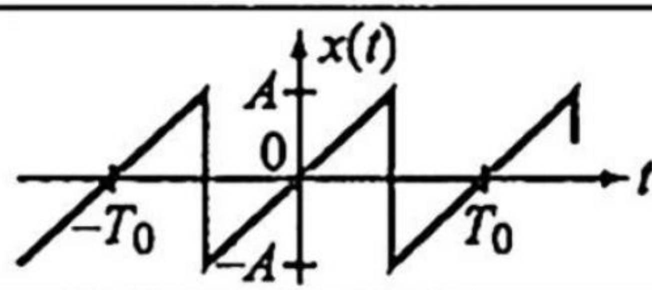
where $n = 1, 2, \ldots$ is any positive integer.

The Fourier series on the interval $[-T/2, T/2]$ is thus

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2n\pi t}{T}\right) + b_n\sin\left(\frac{2n\pi t}{T}\right)\right]. \tag{8.21}$$

Note: $\frac{a\_0}{2}$

Average over Period

	Waveform	Fourier Series
1. Square Wave		$x(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$
2. Time-Shifted Square Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{4A}{n\pi} \sin\left(\frac{2n\pi t}{T_0}\right)$
3. Pulse Train		$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left(\frac{n\pi\tau}{T_0}\right) \cos\left(\frac{2n\pi t}{T_0}\right)$
4. Triangular Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2} \cos\left(\frac{2n\pi t}{T_0}\right)$
5. Shifted Triangular Wave		$x(t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{2n\pi t}{T_0}\right)$
6. Sawtooth		$x(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2A}{n\pi} \sin\left(\frac{2n\pi t}{T_0}\right)$

EXAMPLE 8.4 ***Fourier series square wave example***

A square wave of amplitude A and period T shown in Figure 8.4 can be defined as

$$f(t) = \begin{cases} A, & 0 < t < \dfrac{T}{2}, \\ -A, & -\dfrac{T}{2} < t < 0, \end{cases}$$

with $f(t) = f(t+T)$, since the function is periodic.

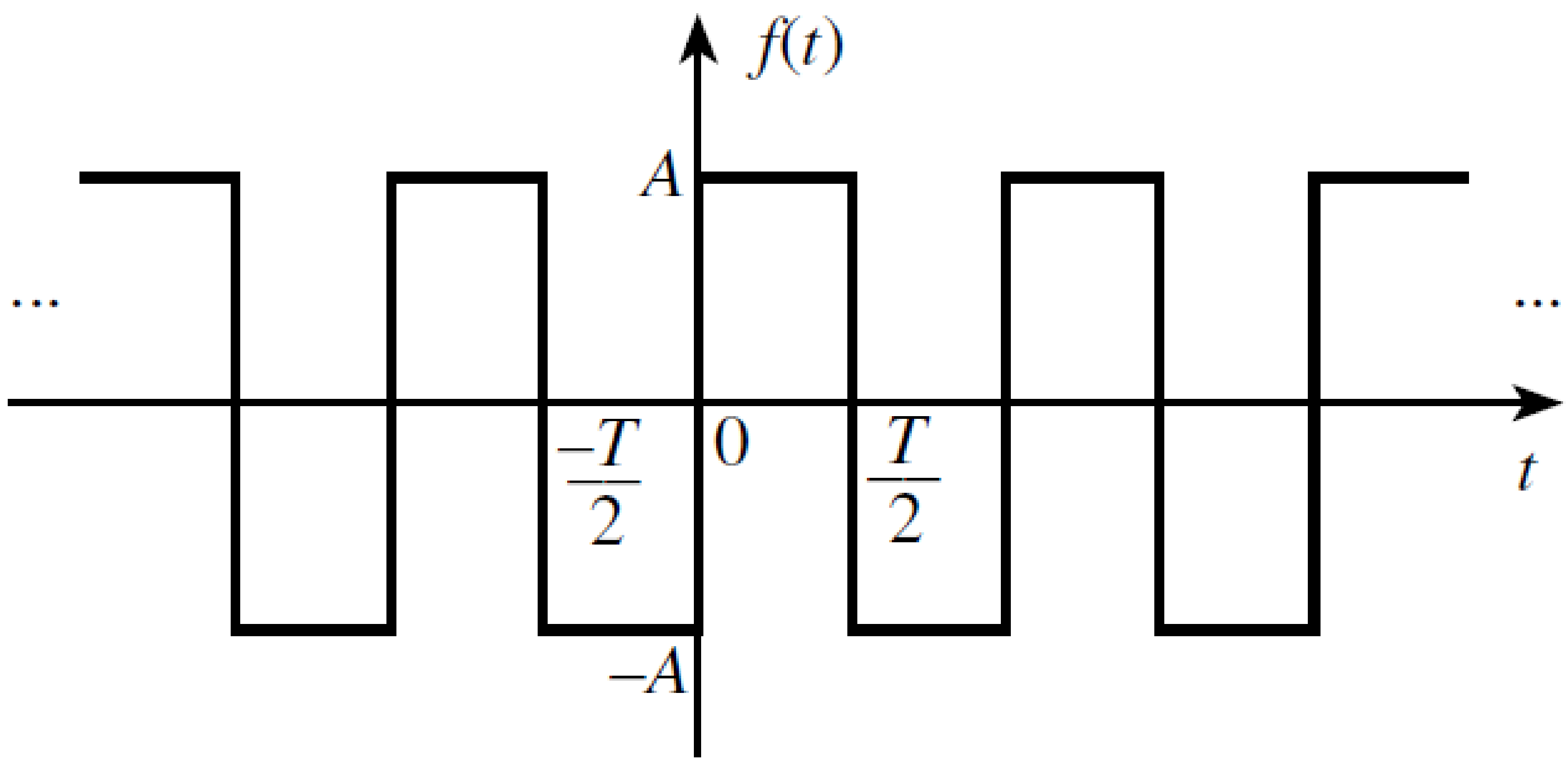


FIGURE 8.4 *Square wave of Example 8.4*

The first observation is that $f(t)$ is odd, which yields the result that $a_0 = 0$ and $a_i = 0$ for every coefficient of the cosine terms. Letting $\omega_0 = 2\pi/T$, the coefficients b_n are

$$b_n = 2\left(\frac{2}{T}\right)\int_0^{T/2} A\sin(n\omega_0 t)\,dt.$$

The result is

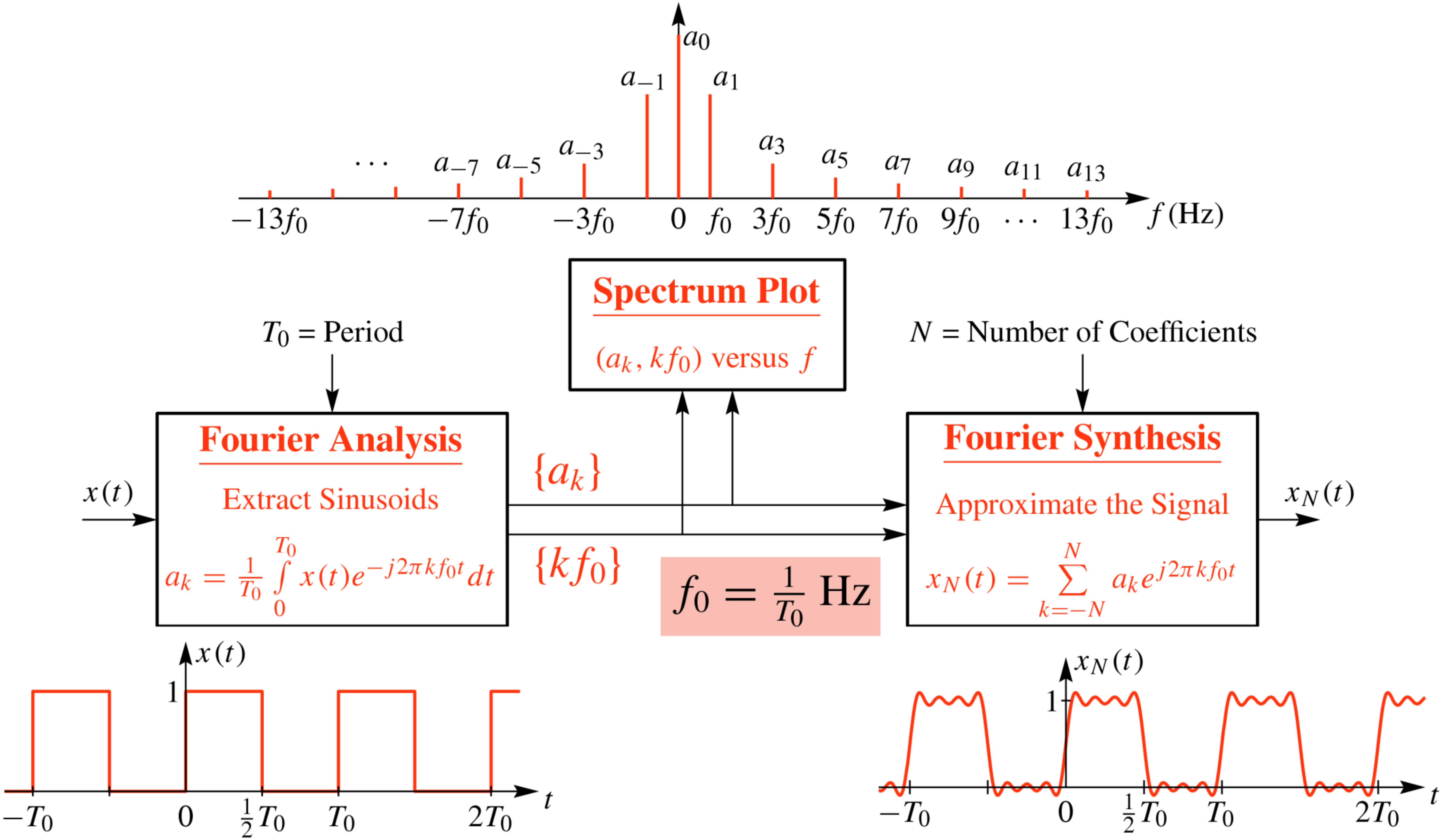
$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

where $(2n-1)$ is introduced to assure that only odd terms are included in the summation. The sine waves that make up the Fourier series for the odd square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right],$$

Importance in signal processing:

A Fourier transform of a signal tells you **what frequencies are present in your signal and in what proportions**.



Harmonic Signal->Periodic

Complex form – Note Limits

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k \underline{F_0 t}}$$

*Sums of **Harmonic** complex exponentials are **Periodic** signals*

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

Compute a_k

Work with the Fourier Series Integral

ANALYSIS via Fourier Series

For **PERIODIC** signals: $x(t+T_0) = x(t)$

Draw spectrum from the Fourier Series coeffs

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

Fourier Series Integral

NOTE: a_0 here is the Average of x(t) over a Period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Freq. $F_0 = 1/T_0$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

- THIS IS THE AVERAGE!

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \qquad \text{(DC component)}$$

☐ EXAMPLE 8.5 ***Complex Series Square Wave Example***

Consider the odd square wave of Example 8.4 and the complex Fourier coefficients

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{0} (-A)e^{-in\omega_0 t}\, dt + \frac{1}{T}\int_{0}^{T/2} (A)e^{-in\omega_0 t}\, dt, \tag{8.29}$$

which leads to the series

$$f(t) = \frac{2A}{i\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}, \tag{8.30}$$

as defined in Equation 8.23.

This form contains complex coefficients, but the series can be written in terms of sine waves by combining the corresponding terms for positive and negative arguments. To determine the coefficients, the amount of difficulty is about the same for the trigonometric series and the complex series. However, the complex series perhaps has an advantage when the magnitude of the coefficients are of interest.

Each coefficient has the form

$$\alpha_n = \frac{2A}{in\pi} = \frac{2A}{n\pi}e^{-i\pi/2}, \qquad n = \pm 1, \pm 3, \ldots,$$

and the coefficients for even values, $n = 0, \pm 2, \ldots$, are zero. Notice that the

The trigonometric series is derived from the complex series by expanding the complex series of Equation 8.30 as

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t} \\ &= \cdots - \frac{2A}{3\pi i} e^{-i3\omega_0 t} - \frac{2A}{\pi i} e^{-i\omega_0 t} + \frac{2A}{\pi i} e^{i\omega_0 t} + \frac{2A}{3\pi i} e^{i3\omega_0 t} + \cdots \end{aligned}$$

and recognizing the sum of negative and positive terms for each n as $2\sin(n\omega_0 t)$. The trigonometric series becomes

$$f(t) = \frac{4A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right) = \frac{4A}{\pi}\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

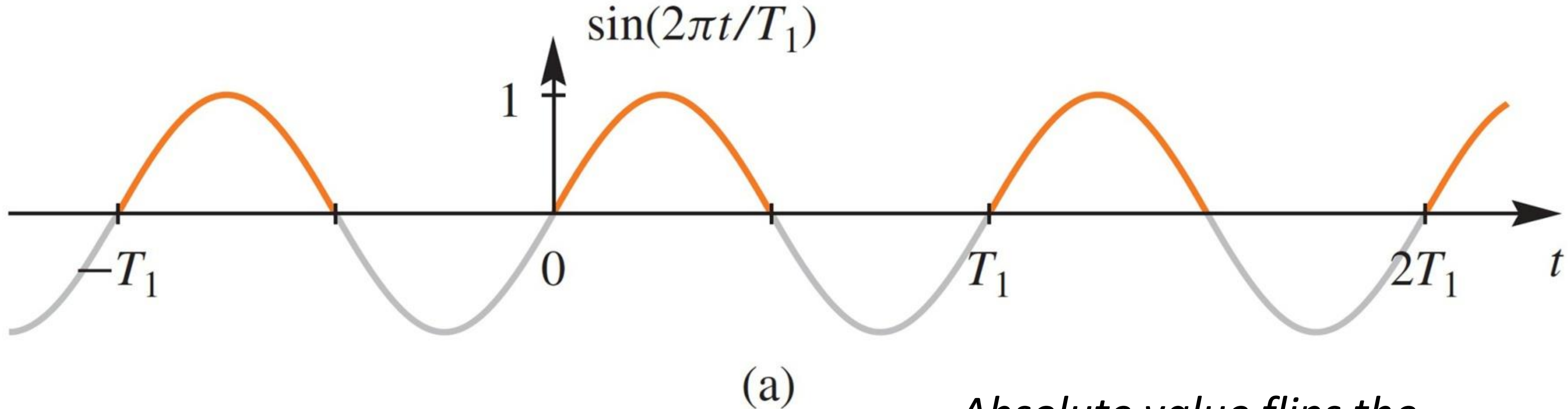
which is the result of Example 8.4.

□

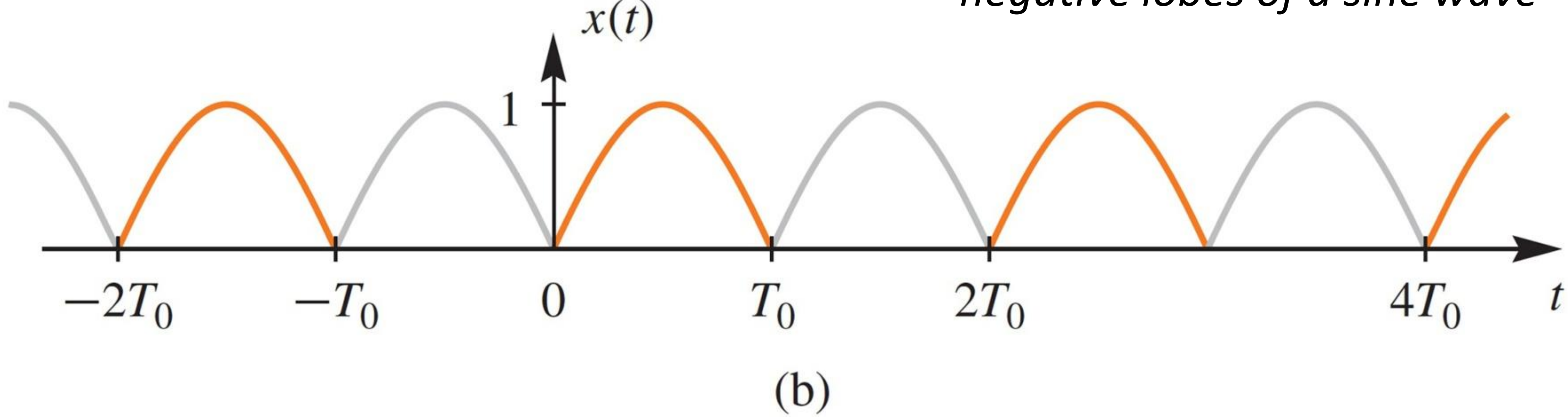
Full-Wave Rectified Sine

$$x(t) = \left|\sin(2\pi t / T_1)\right| \qquad \text{Period is } T_0 = \tfrac{1}{2} T_1$$

- Frequency
- Doubles



Absolute value flips the negative lobes of a sine wave



Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt}\,dt$$

Full-Wave Rectified Sine

$$x(t) = \left|\sin(2\pi t/T_1)\right|$$

$$Period: T_0 = \tfrac{1}{2}T_1$$

$$\Rightarrow x(t) = \left|\sin(\pi t/T_0)\right|$$

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} \sin(\tfrac{\pi}{T_0}t) e^{-j(2\pi/T_0)kt}\,dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}\,dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k-1)t}\,dt - \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k+1)t}\,dt$$

$$= \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

$$= \tfrac{1}{2\pi(2k-1)}\left(e^{-j(\pi/T_0)(2k-1)T_0} - 1\right) - \tfrac{1}{2\pi(2k+1)}\left(e^{-j(\pi/T_0)(2k+1)T_0} - 1\right)$$

$$= \tfrac{1}{\pi(2k-1)}\left(e^{-j\pi(2k-1)} - 1\right) - \tfrac{1}{\pi(2k+1)}\left(e^{-j\pi(2k+1)} - 1\right)$$

$$= \left(\tfrac{2k+1-(2k-1)}{\pi(4k^2-1)}\right)\left(-(-1)^{2k} - 1\right) = \frac{-2}{\pi(4k^2-1)}$$

Fourier Coefficients: a_k

a_k is a function of k

Complex Amplitude for k-th Harmonic

NOTE: $\frac{1}{k^2}$ $for\ large\ k$

Does not depend on the period, T_0

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

DC value is

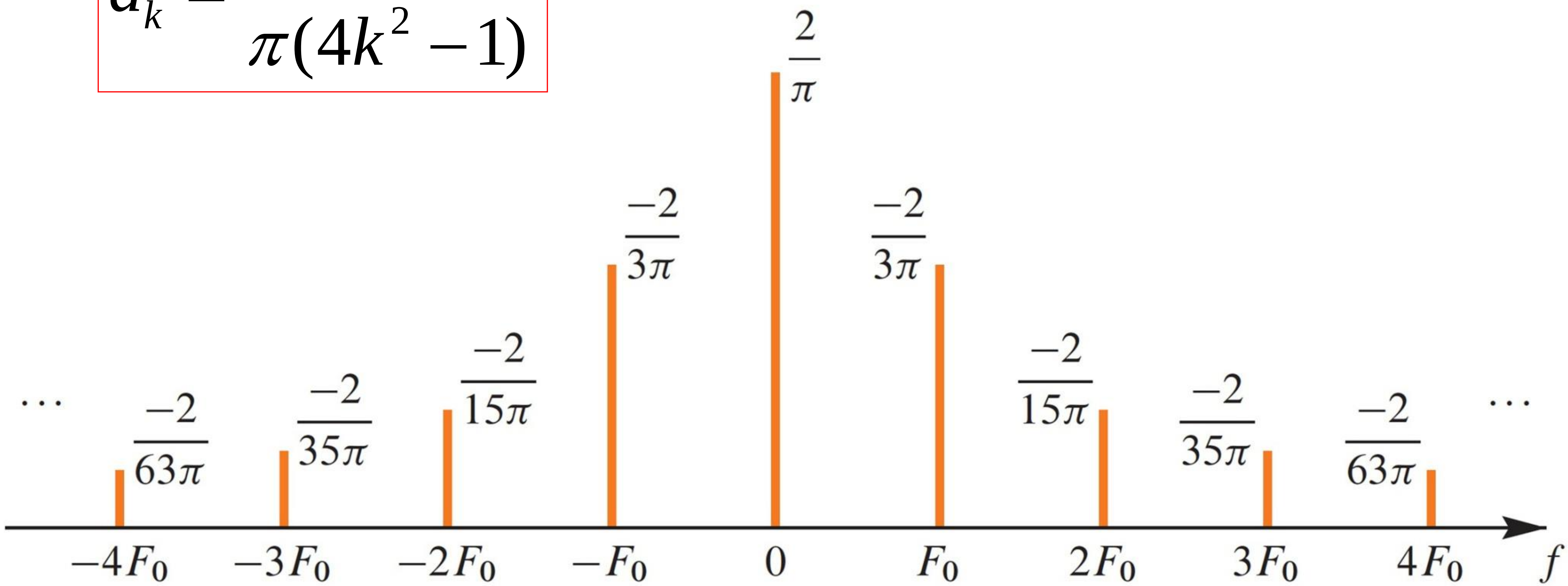
$$a_0 = 2/\pi = 0.6336$$

Spectrum from Fourier Series

Plot a_k for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$F_0 = 1/T_0$ and $\omega_0 = 2\pi F_0$



In an electric power system, a **harmonic** of a distorted (non-sinusoidal) periodic voltage or current is a sinusoidal voltage or current whose frequency is an integer multiple of the fundamental frequency of the distorted voltage or current, which is usually the fundamental frequency of the system, produced by the action of non-linear loads such as rectifiers, discharge lighting, or saturated magnetic devices.

Harmonic frequencies in the power grid are a frequent cause of power quality problems. Harmonics in power systems result in increased heating in the equipment and conductors, misfiring in variable speed drives, and torque pulsations in motors and generators.

Harmonics dc, 1 to 50, up to 9th harmonic for 400 Hz

Sampling system	
Resolution	16 bit analog to digital converter on 8 channels
Maximum sampling speed	200 kS/s on each channel simultaneously
RMS sampling	5000 samples on 10/12 cycles according to IEC61000-4-30
PLL synchronization	4096 samples on 10/12 cycles according to IEC61000-4-7
Nominal frequency	434-II and 435-II: 50 Hz and 60 Hz 437-II: 50 Hz, 60 Hz and 400 Hz

Reconstruct From Finite Number of Harmonic Components

Full-Wave Rectified Sinusoid

$$x(t) = \left|\sin(\pi t / T_0)\right|$$

$$T_0 = 10\,\text{ms}$$

$$\Rightarrow F_0 = 100\,\text{Hz}$$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$$a_0 = 2/\pi = 0.6336$$

$$x_N(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k\, e^{j2\pi k F_0 t} + a_k^*\, e^{-j2\pi k F_0 t} \right\}$$

How close is $x_N(t)$ to $x(t) = \left|\sin(\pi t / T_0)\right|$?

Reconstruct From Finite Number of Spectrum Components

Full-Wave Rectified Sinusoid

$$x(t) = |\sin(\pi t / T_0)|$$

$T_0 = 10\,\text{ms}$

$\Rightarrow F_0 = 100\,\text{Hz}$

$$a_0 = 2/\pi = 0.6336$$

$$a_k = \frac{-2}{\pi(4k^2-1)}$$