ALGORITHM FOR FOURIER SERIES

1. DETERMINE THE PERIOD T seconds

   a) FUNDAMENTAL FREQUENCY $f_0 = \frac{1}{T}$ Hertz

   b) RADIAN FREQUENCY $\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$

2. CONSIDER THE GENERAL SERIES K+H Eq 3.4

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)\right]$$

3. ARE THERE SPECIAL CONDITIONS

   $f(t)$ EVEN ? YES, $b_k = 0$ for all k

   $f(t)$ ODD ? YES, $a_k = 0$ for all k

4. PEFORM THE INTEGRATIONS

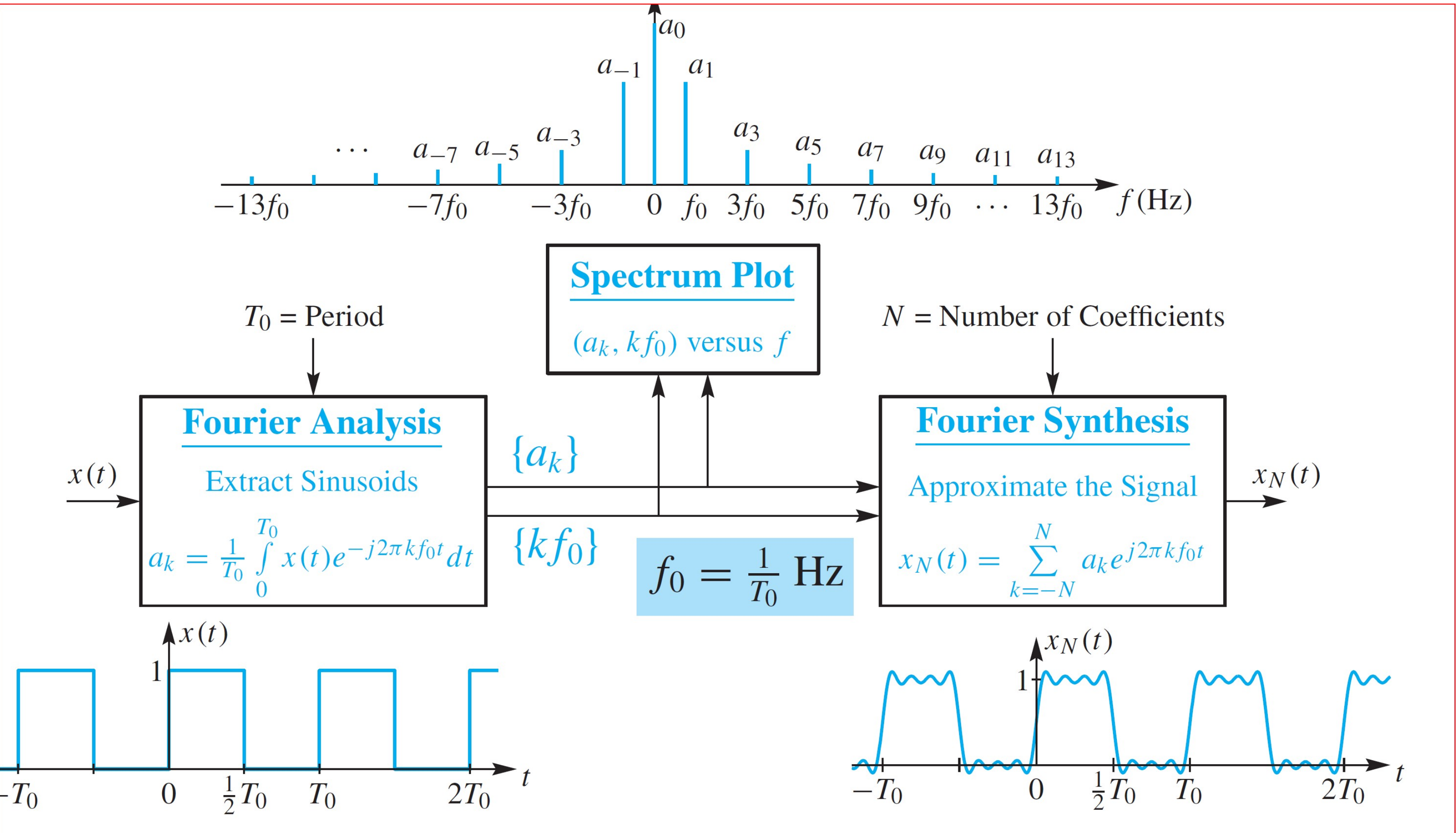
$$a_0 = \frac{1}{T}\int_0^T f(t)\,dt \; ; \quad a_k = \frac{2}{T}\int_0^T f(t)\cos(k\omega_0 t)\,dt$$

$$b_k = \frac{2}{T}\int_0^T f(t)\sin(k\omega_0 t)\,dt \qquad k = 1, 2, 3, \ldots$$

5. WRITE THE SERIES FROM (2) above

OR $$f(t) = \sum_{-\infty}^{\infty} \alpha_n e^{jn\omega_0 t}$$

$$\alpha_n = \frac{1}{T}\int_T f(t) e^{-jn\omega_0 t}$$



a_0
a_{-1} a_1
$\cdots$ a_{-7} a_{-5} a_{-3} a_3 a_5 a_7 a_9 a_{11} a_{13}
$-13f_0$ $-7f_0$ $-3f_0$ 0 f_0 $3f_0$ $5f_0$ $7f_0$ $9f_0$ $\cdots$ $13f_0$ f (Hz)
Spectrum Plot
(a_k, kf_0) versus f
T_0 = Period
N = Number of Coefficients
Fourier Analysis
Extract Sinusoids
$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j2\pi kf_0 t}dt$
$x(t)$
$\{a_k\}$
$\{kf_0\}$
$f_0 = \frac{1}{T_0}$ Hz
Fourier Synthesis
Approximate the Signal
$x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi kf_0 t}$
$x_N(t)$
$x(t)$
1
$-T_0$ 0 $\frac{1}{2}T_0$ T_0 $2T_0$ t
$x_N(t)$
1
$-T_0$ 0 $\frac{1}{2}T_0$ T_0 $2T_0$ t

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FOURIER SERIES SUMMARY For a periodic signal $f(t)$ with period T, the various forms of the Fourier series and the power associated with the signal are shown in Table 8.2.

TABLE 8.2 *Fourier Series Representation*

Series	*Power*
Sine and cosine series:	
$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2n\pi f_0 t) + b_n \sin(2n\pi f_0 t)]$	$\left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$
where $a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(2n\pi f_0 t)\,dt$, $b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(2n\pi f_0 t)\,dt$	
Shifted cosine:	
$\frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(2\pi n f_0 t + \theta_n)$	$\left(\frac{a_0}{2}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} c_n^2$
where $c_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$	
Complex series:	
$\sum_{n=-\infty}^{\infty} \alpha_n e^{i2\pi n f_0 t}$	$\sum_{n=-\infty}^{\infty} \lvert\alpha_n\rvert^2$
where $\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-i2n\pi f_0 t}\,dt$	

Comparing the power relations in the table, the coefficients of the shifted cosine series are related to those for the sine and cosine series as

$$c_n^2 = a_n^2 + b_n^2,$$

with $c_0 = a_0/2$. The complex series coefficients are related to the coefficients of the trigonometric series as

$$\alpha_n^2 = \frac{1}{2}(a_n^2 + b_n^2)$$

with $\alpha_0 = a_0/2$.

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VARY PERIOD T_0 AS $T \to \infty$ SERIES → TRANSFORM

TOWARD THE TRANSFORM

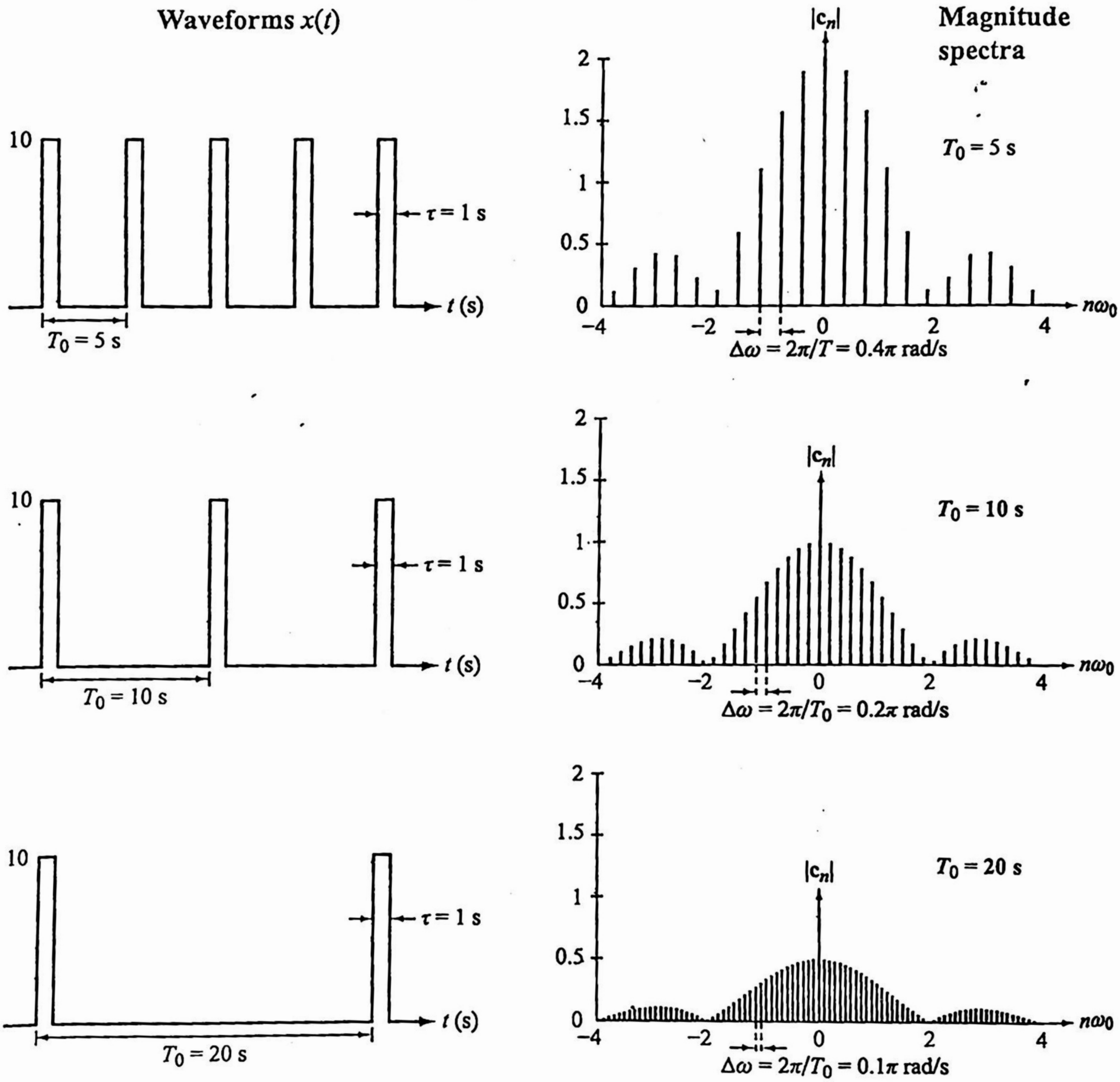
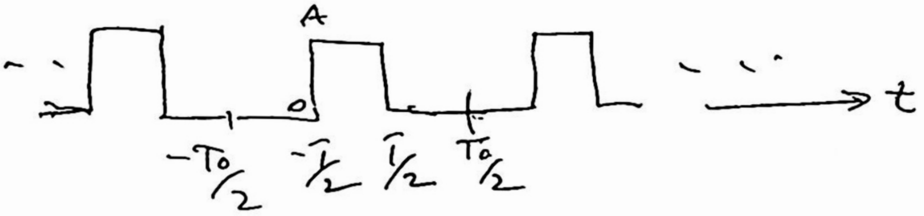


Figure 5-12: Line spectra for pulse trains with $T_0/\tau = 5$, 10, and 20.

EXAMPLE 3.2 EVEN PULSE TRAIN



$T = T_0$

$\omega_0 = \frac{2\pi}{T_0}$

FOURIER SERIES IS

I. $x(t) = \frac{A\tau}{T} + \frac{2A}{\pi} \sum_1 \frac{1}{k} \sin\left(\frac{k\pi\tau}{T_0}\right) \cos\left(\frac{2k\pi t}{T_0}\right)$

EXAMPLE $T_0 = 2$ sec $\tau = 1$ sec $A = 1$

$x(t) = \frac{1}{2} + \frac{2}{\pi} \sum_1^{\infty} \frac{1}{k} \sin\left(\frac{k\pi}{2}\right) \cos(k\pi t)$ Eq 3 p103

NOTE $\sin(k\pi/2) = 0$ for Even k! Eq 3.1 p104

II TO CONVERT TO EXPONENTIAL

$C_0 = \frac{1}{2}$ $C_k = \frac{1}{2} a_k = \frac{1}{\pi k} \sin\left(\frac{k\pi}{2}\right)$ Eq 3.20 P108

$x(t) = \frac{1}{2} + \frac{1}{\pi} \sum_{-\infty}^{\infty} \frac{1}{k} \sin\left(\frac{k\pi}{2}\right) e^{jk\pi t}$

EVEN VALUES OF k yield 0! $(e^{jk\omega_0 t})$

THIS SOLVES ALL EVEN PULSES

WITH $\tau < T_0$ USING RESULT I!

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Table 5-4: Fourier series expressions for a select set of periodic waveforms. U+Y p 204

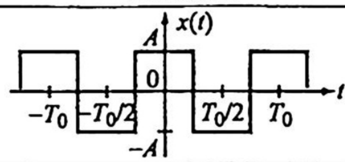
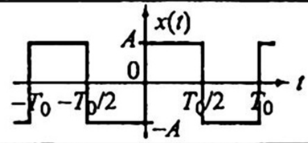
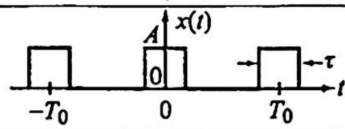
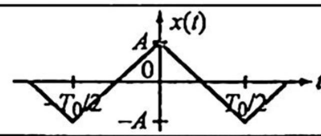
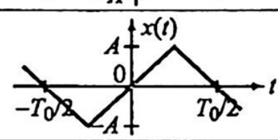
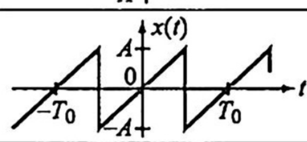
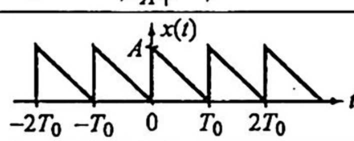
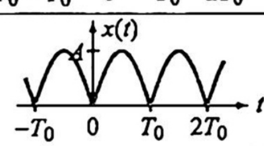
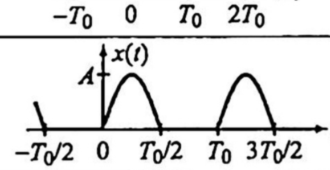
	Waveform	Fourier Series
1. Square Wave		$x(t)=\sum_{n=1}^{\infty}\frac{4A}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{2n\pi t}{T_0}\right)$
2. Time-Shifted Square Wave		$x(t)=\sum_{\substack{n=1\\ n=\text{odd}}}^{\infty}\frac{4A}{n\pi}\sin\left(\frac{2n\pi t}{T_0}\right)$
3. Pulse Train		$x(t)=\frac{A\tau}{T_0}+\sum_{n=1}^{\infty}\frac{2A}{n\pi}\sin\left(\frac{n\pi\tau}{T_0}\right)\cos\left(\frac{2n\pi t}{T_0}\right)$
4. Triangular Wave		$x(t)=\sum_{\substack{n=1\\ n=\text{odd}}}^{\infty}\frac{8A}{n^2\pi^2}\cos\left(\frac{2n\pi t}{T_0}\right)$
5. Shifted Triangular Wave		$x(t)=\sum_{\substack{n=1\\ n=\text{odd}}}^{\infty}\frac{8A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{2n\pi t}{T_0}\right)$
6. Sawtooth		$x(t)=\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2A}{n\pi}\sin\left(\frac{2n\pi t}{T_0}\right)$
7. Backward Sawtooth		$x(t)=\frac{A}{2}+\sum_{n=1}^{\infty}\frac{A}{n\pi}\sin\left(\frac{2n\pi t}{T_0}\right)$
8. Full-Wave Rectified Sinusoid		$x(t)=\frac{2A}{\pi}+\sum_{n=1}^{\infty}\frac{4A}{\pi(1-4n^2)}\cos\left(\frac{2n\pi t}{T_0}\right)$
9. Half-Wave Rectified Sinusoid		$x(t)=\frac{A}{\pi}+\frac{A}{2}\sin\left(\frac{2\pi t}{T_0}\right)+\sum_{\substack{n=2\\ n=\text{even}}}^{\infty}\frac{2A}{\pi(1-n^2)}\cos\left(\frac{2n\pi t}{T_0}\right)$

Table 5-3: Fourier series representations for a real-valued periodic function $x(t)$. U+Y p 200

Cosine/Sine	Amplitude/Phase	Complex Exponential
$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$	$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t + \phi_n)$	$x(t) = \sum_{n=-\infty}^{\infty} \mathbf{x}_n e^{jn\omega_0 t}$
$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t)\, dt$	$c_n e^{j\phi_n} = a_n - jb_n$	$\mathbf{x}_n = \lvert\mathbf{x}_n\rvert e^{j\phi_n};\ \mathbf{x}_{-n} = \mathbf{x}_n^*;\ \phi_{-n} = -\phi_n$
$a_n = \frac{2}{T_0} \int_0^{T_0} x(t) \cos n\omega_0 t\, dt$	$c_n = \sqrt{a_n^2 + b_n^2}$	$\lvert\mathbf{x}_n\rvert = c_n/2;\ x_0 = c_0$
$b_n = \frac{2}{T_0} \int_0^{T_0} x(t) \sin n\omega_0 t\, dt$	$\phi_n = \begin{cases} -\tan^{-1}(b_n/a_n), & a_n > 0 \\ \pi - \tan^{-1}(b_n/a_n), & a_n < 0 \end{cases}$	$\mathbf{x}_n = \frac{1}{T_0} \int_0^{T_0} x(t)\, e^{-jn\omega_0 t}\, dt$
$a_0 = c_0 = x_0;\ a_n = c_n \cos\phi_n;\ b_n = -c_n \sin\phi_n;\ \mathbf{x}_n = \frac{1}{2}(a_n - jb_n)$		