DSP-First, 2/e

TLH MODIFIED

LECTURE # CH2-3
Complex Exponentials
& Complex Numbers

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READING ASSIGNMENTS

- This Lecture:
  - Chapter 2, Sects. 2-3 to 2-5

- Appendix A: Complex Numbers

  - Complex Exponentials

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Dr. Van Veen Strikes Again

3. Complex Numbers Review (Wouldn't hurt to review) 10:22

Review of how to work with complex numbers in rectangular and polar coordinates.

https://www.youtube.com/watch?v=UAn9uah7puU&list=PLGI7M8vwfrFNO-gQ1xoJmN3bJy2-wp2J3

4. Complex Sinusoid Representations for Real Sinusoids 13:21

https://www.youtube.com/watch?v=Tm3gI6PQOYo&feature=youtu.be

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LECTURE OBJECTIVES

- Introduce more tools for manipulating complex numbers Euler Eq.
  - Conjugate
  - Multiplication & Division
  - Powers
  - N-th Roots of unity

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

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Complex Exponentials

January 22, 2019

The exponential function $e^{\lambda t}$ is an *eigenfunction* of various operations, such as differentiation, integration, and time shifting. Thus, when these operations are performed on $e^{\lambda t}$, the result is a constant times the exponential function.

Since for any constant λ,

$$\frac{d}{dt}(e^{\lambda t}) = \lambda(e^{\lambda t}), \tag{1}$$

the exponential is an eigenfunction of the differentiation operator.

How about an example:

Let $z = re^{j\omega t}$, So that

$$\frac{d}{dt}(e^{j\omega t}) = j\omega(e^{j\omega t}) \tag{2}$$

Notice the shift by j which is 90 degrees or $e^{j\pi/2}$.

If the time shift operator is defined as $\mathcal{T}_s[f(t)] = f(t - t_0)$, then

$$\mathcal{T}_s[e^{j\omega t}] = e^{j\omega(t-t_0)} = e^{-j\omega t_0}\, e^{j\omega t}. \tag{3}$$

Since the first term in the product is constant, setting $\lambda = e^{-\alpha t_0}$ results in the eigenvalue equation

$$\mathcal{T}_s[e^{\alpha t}] = \lambda e^{\alpha t}. \tag{4}$$

These relationships and the result for the integral of $z = re^{j\omega t}$ are the reasons for using the exponential form of the sinusoids.

Another reason for using expontentials is that the exponentials are simple to multiply. Just multiply the magnitudes and add the angles.

LECTURE OBJECTIVES

- Phasors = Complex Amplitude
  - Complex Numbers **represent** Sinusoids
  - Take Real or Complex part

$$\boxed{A\cos(\omega t+\varphi)=\Re\{(Ae^{j\varphi})e^{j\omega t}\}}$$

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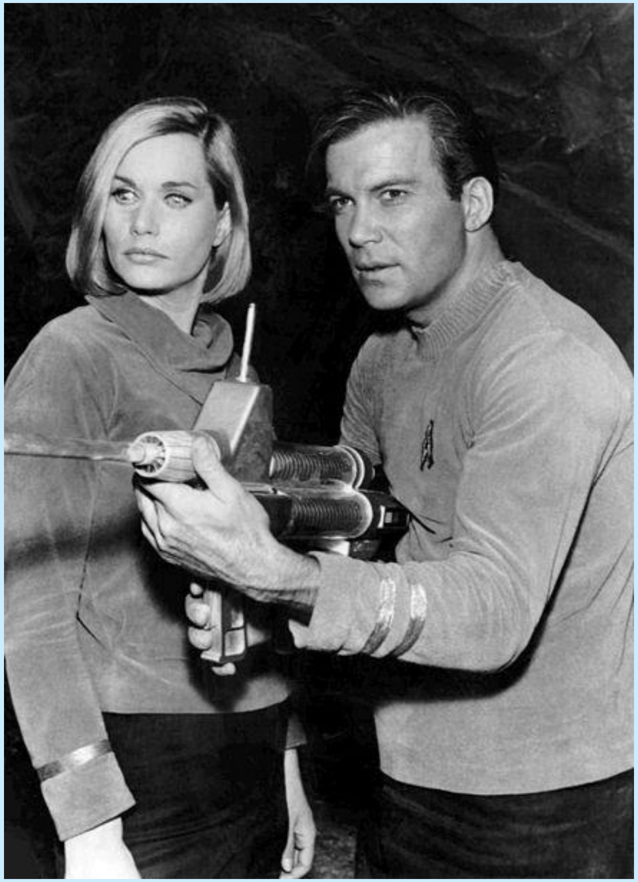
Phasors Not Phasers

Captain Kirk &
Sally Kellerman



STAR TREK

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WHY? What do we gain?

- Sinusoids are the basis of DSP,
  - but trig identities are very tedious

- Abstraction of complex numbers
  - Represent cosine functions
  - Can replace most trigonometry with **algebra**
- **Avoid (Most) all Trigonometric manipulations**

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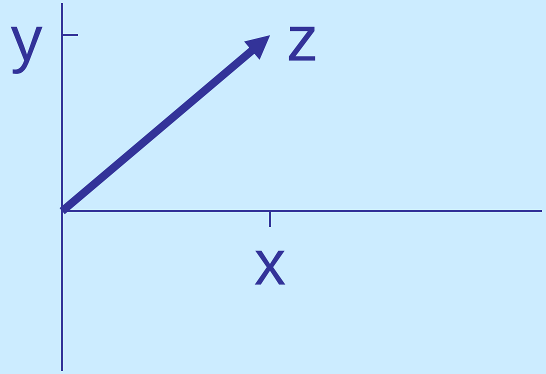
COMPLEX NUMBERS

- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$



Cartesian coordinate system

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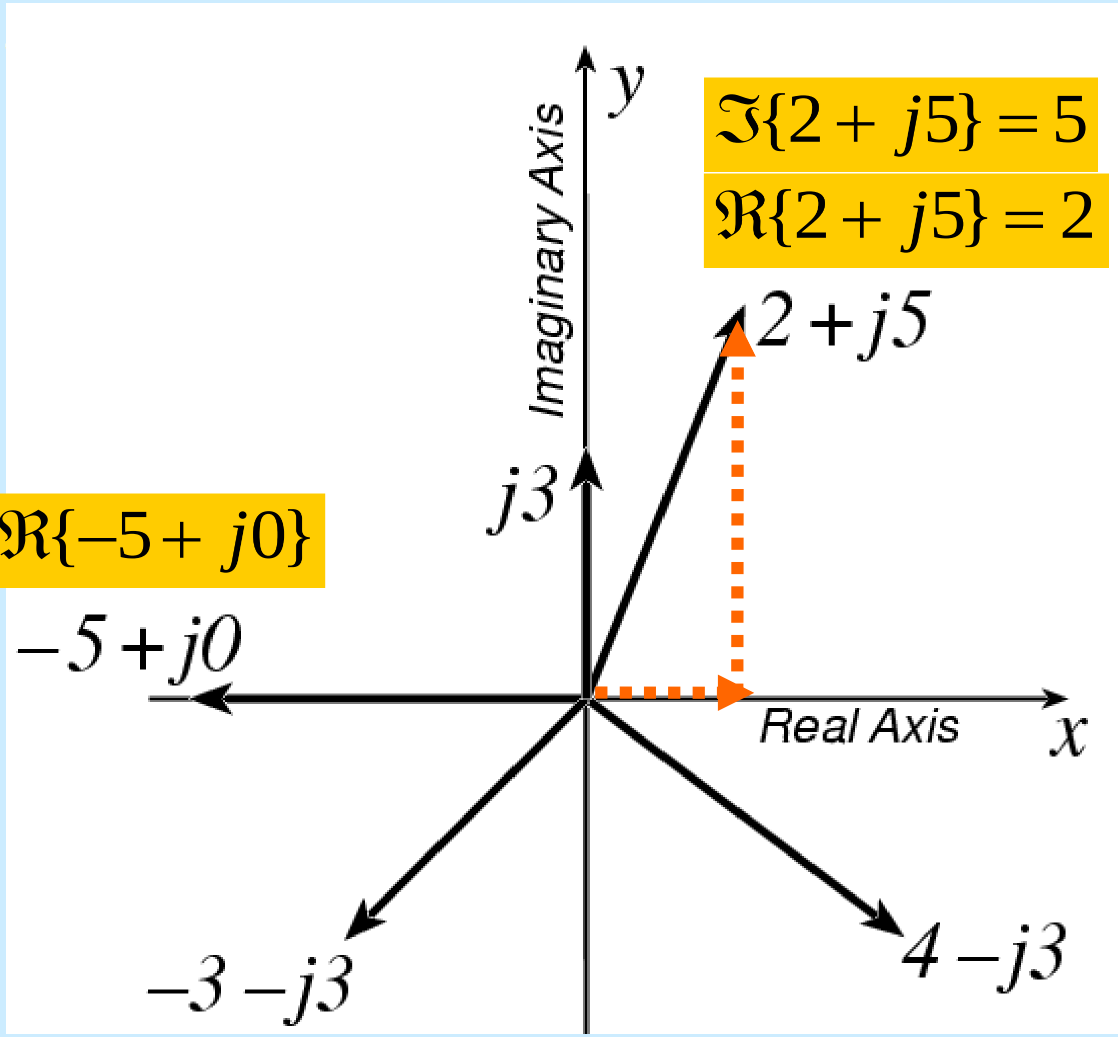
PLOT COMPLEX NUMBERS



$$\Im\{2+j5\}=5$$

$$\Re\{2+j5\}=2$$

$$-5=\Re\{-5+j0\}$$

Real part:

$$x=\Re\{z\}$$

Imaginary part:

$$y=\Im\{z\}$$

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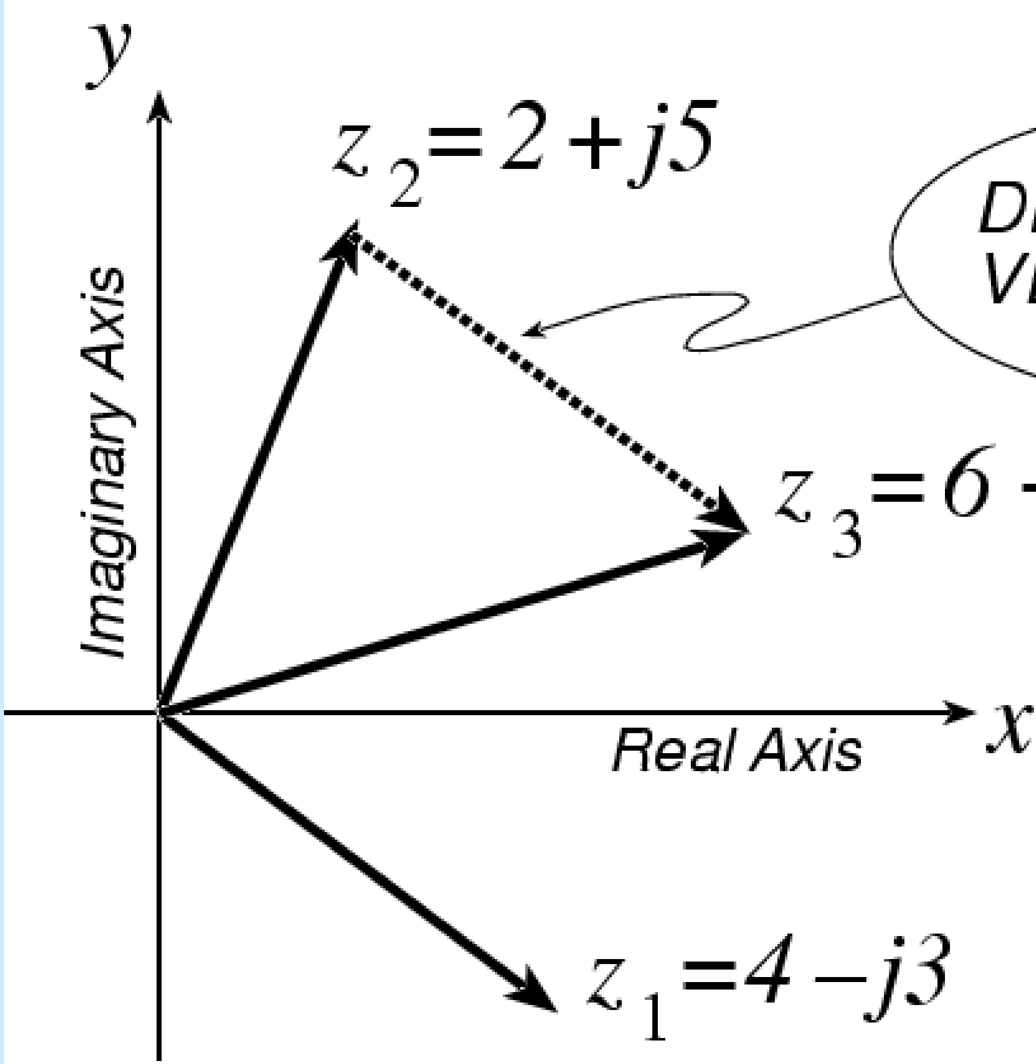
COMPLEX ADDITION = VECTOR Addition



$$
\begin{aligned}
z_3 &= z_1 + z_2 \\
&= (4 - j3) + (2 + j5) \\
&= (4 + 2) + j(-3 + 5) \\
&= 6 + j2
\end{aligned}
$$

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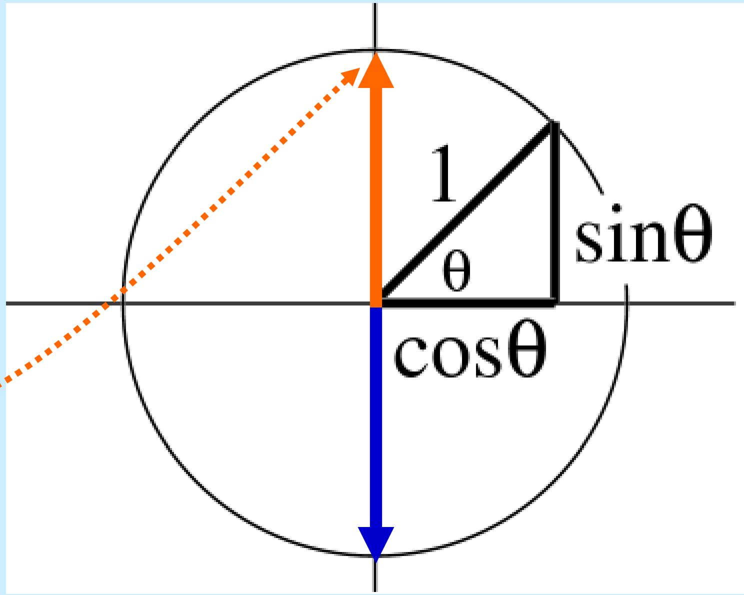
*** POLAR FORM ***

- <u>**Vector**</u> Form
  - **Length** =1
  - **Angle** = θ
- Common Values
  - **j** has angle of 0.5π
  - -1 has angle of π
  - **– j** has angle of 1.5π
  - also, angle of **–j could** be $-0.5\pi = 1.5\pi - 2\pi$
  - because the PHASE is **AMBIGUOUS**



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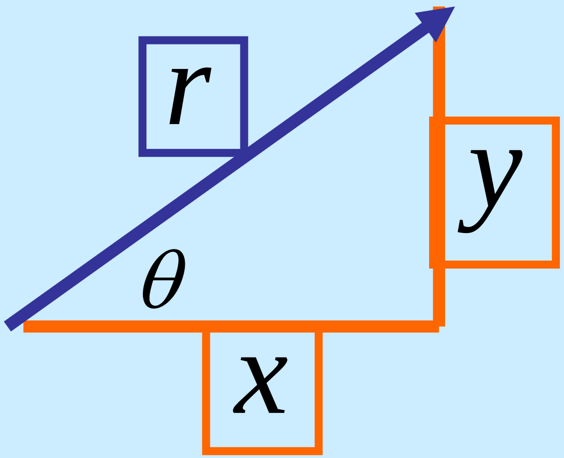
POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)



$$r^2 = x^2 + y^2$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Most calculators do Polar-Rectangular

Need a notation for POLAR FORM

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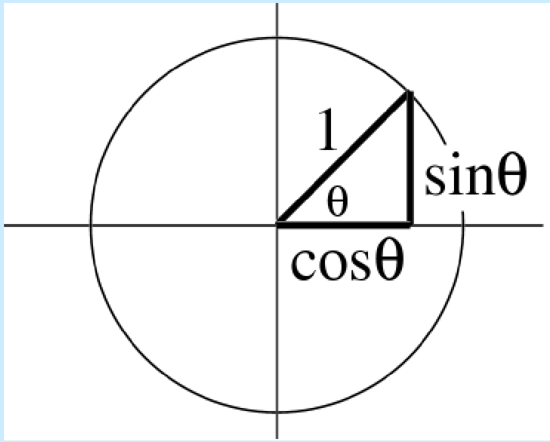
Euler's FORMULA

- Complex Exponential
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

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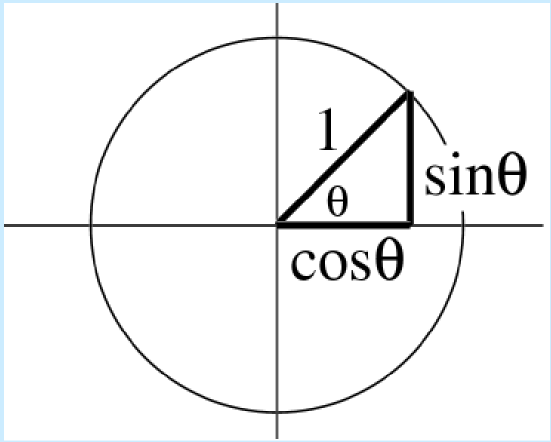
Cosine = Real Part

- Complex Exponential
  - **Real part is cosine**
  - Imaginary part is sine



$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

$$\Re\{re^{j\theta}\} = r\cos(\theta)$$

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Common Values of exp(jθ)

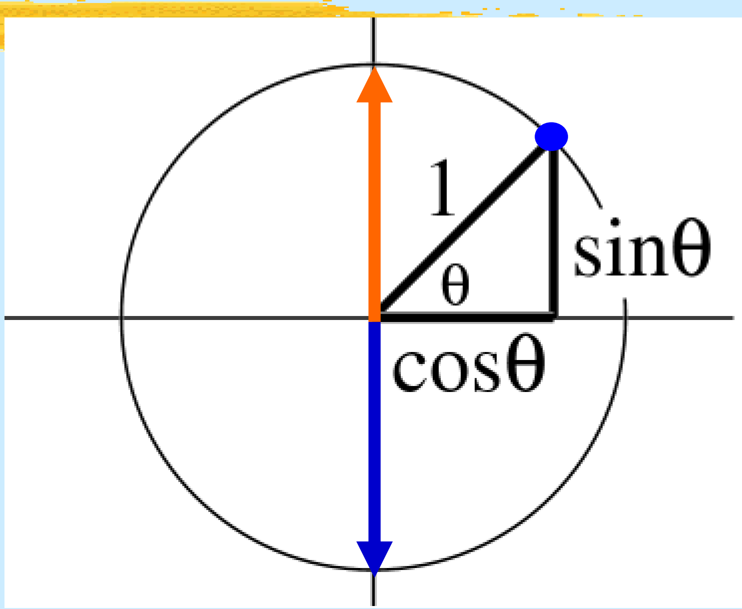
- Changing the angle

$$\theta = 0 \rightarrow 1 = 1 + j0 = e^{j0} = e^{j2n\pi}$$

$$\theta = \pi \rightarrow -1 = -1 + j0 = e^{j\pi} = e^{j(2n+1)\pi}$$

$$\boxed{\theta = \pi/2 \rightarrow j = e^{j\pi/2} = e^{j(2n+1/2)\pi}}$$

$$\boxed{\theta = 3\pi/2 \rightarrow -j = e^{j3\pi/2} = e^{-j\pi/2} = e^{j(2n-1/2)\pi}}$$



$$1 \pm j = \sqrt{2}e^{\pm j\pi/4}$$

$$a^2 + b^2 = c^2$$

$$\theta = \arctan\left(\frac{b}{a}\right)$$

IF $a \geq 0$

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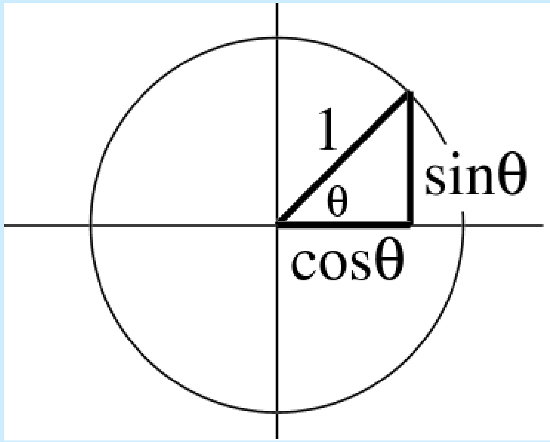
COMPLEX EXPONENTIAL

$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

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Cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re\{Ae^{j(\omega t+\varphi)}\}$$
$$= \Re\{Ae^{j\varphi}e^{j\omega t}\}$$

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COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re\{Ae^{j\varphi}e^{j\omega t}\}$$

Sinusoid = REAL PART of complex exp: $z(t)=(Ae^{j\phi})e^{j\omega t}$

$$x(t) = \Re\{Xe^{j\omega t}\} = \Re\{z(t)\}$$

Complex AMPLITUDE = X, which is a constant

$$X = Ae^{j\varphi} \qquad \text{when } z(t) = Xe^{j\omega t}$$

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POP QUIZ: Complex Amp

- Find the COMPLEX AMPLITUDE for:

$$\boxed{x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)}$$

- Use EULER's FORMULA:

$$x(t) = \Re\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\}$$

$$= \Re\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\}$$

$$\boxed{X = \sqrt{3}e^{j0.5\pi}}$$

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POP QUIZ-2: Complex Amp

- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is: $\boxed{X = \sqrt{3} + j3}$
- Convert ***X*** to **POLAR**:

$$x(t) = \Re\{(\sqrt{3} + j3)e^{j(120\pi t)}\}$$

$$= \Re\{\sqrt{12}e^{j\pi/3}e^{j120\pi t}\}$$

$$\boxed{\Rightarrow x(t) = \sqrt{12}\cos(120\pi t + \pi/3)}$$

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Note atan $(3/\sqrt{3})$ –is atan$(\sqrt{3}) = \pi/3$

Remember 60 degree angle

Cos = 1/2 Sin = $\sqrt{3}/2$

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COMPLEX CONJUGATE (z*)

- Useful concept: change the sign of **all j's**
- **RECTANGULAR**: If $z = x + j\,y$, then the complex conjugate is $z^* = x - j\,y$
- **POLAR**: Magnitude is the same but angle has sign change

$$\boxed{z = re^{j\theta} \Rightarrow z^* = re^{-j\theta}}$$

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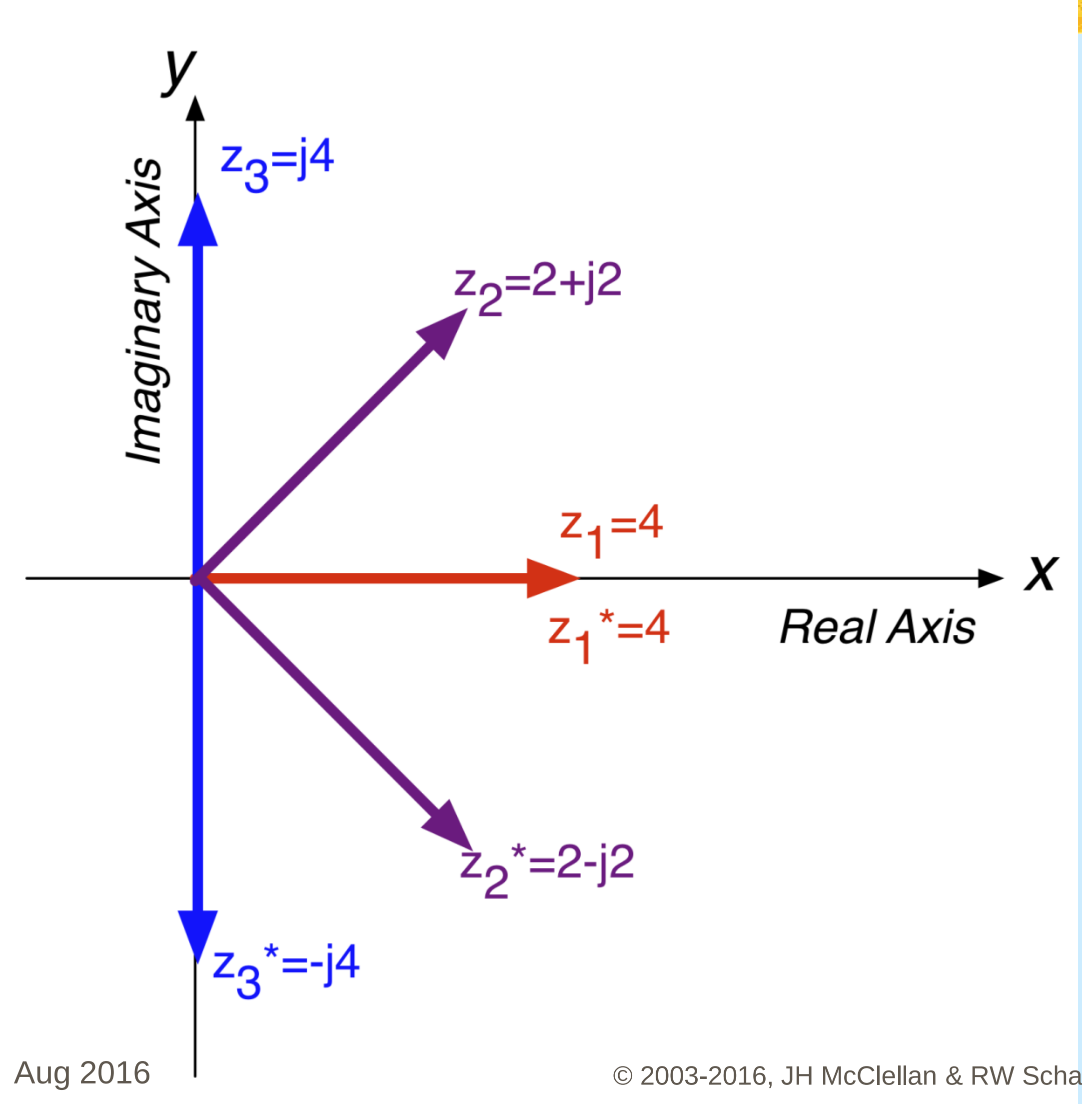
COMPLEX CONJUGATION



- Flips vector about the real axis!

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USES OF CONJUGATION

- Conjugates useful for many calculations
- Real part:

$$\frac{z+z^*}{2} = \frac{(x+jy)+(x-jy)}{2} = x = \Re\{z\}$$

- Imaginary part:

$$\frac{z-z^*}{2j} = \frac{j2y}{2j} = y = \Im\{z\}$$

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Inverse Euler Relations

- Cosine is real part of exp, sine is imaginary part
- Real part:

$$\frac{z + z^*}{2} = \Re\{z\}$$

$$z = e^{j\theta}, \Rightarrow \Re\{e^{j\theta}\} = \frac{e^{j\theta} + e^{-j\theta}}{2} = \cos(\theta)$$

- Imaginary part:

$$\frac{z - z^*}{2j} = y = \Im\{z\}$$

$$z = e^{j\theta}, \Rightarrow \Im\{e^{j\theta}\} = \frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin(\theta)$$

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Mag & Magnitude Squared

- Magnitude Squared (polar form):

$$zz^* = (re^{j\theta})(re^{-j\theta}) = r^2 = |z|^2$$

- Magnitude Squared (Cartesian form):

$$zz^* = (x + jy) \times (x - jy) = x^2 - j^2 y^2 = x^2 + y^2$$

- Magnitude of complex exponential is one:

$$|e^{j\theta}|^2 = \cos^2(\theta) + \sin^2(\theta) = 1$$

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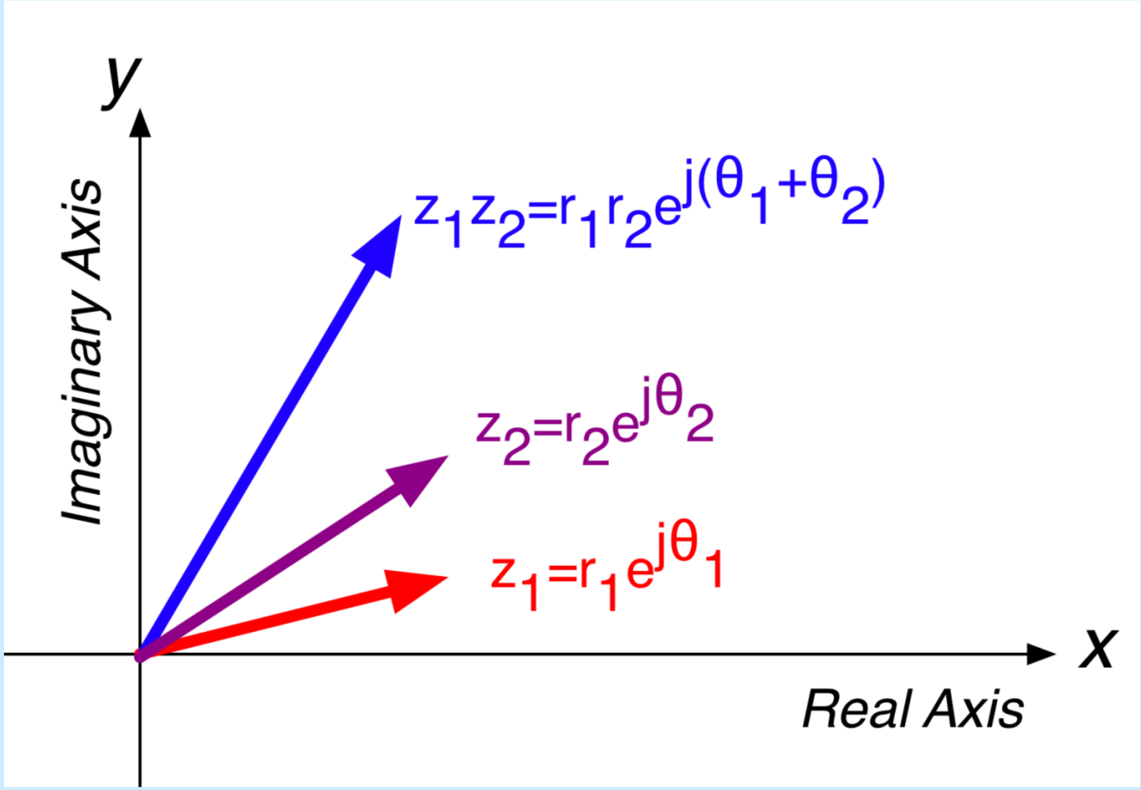
COMPLEX MULTIPLY = VECTOR ROTATION

- Multiplication/division scales and rotates vectors



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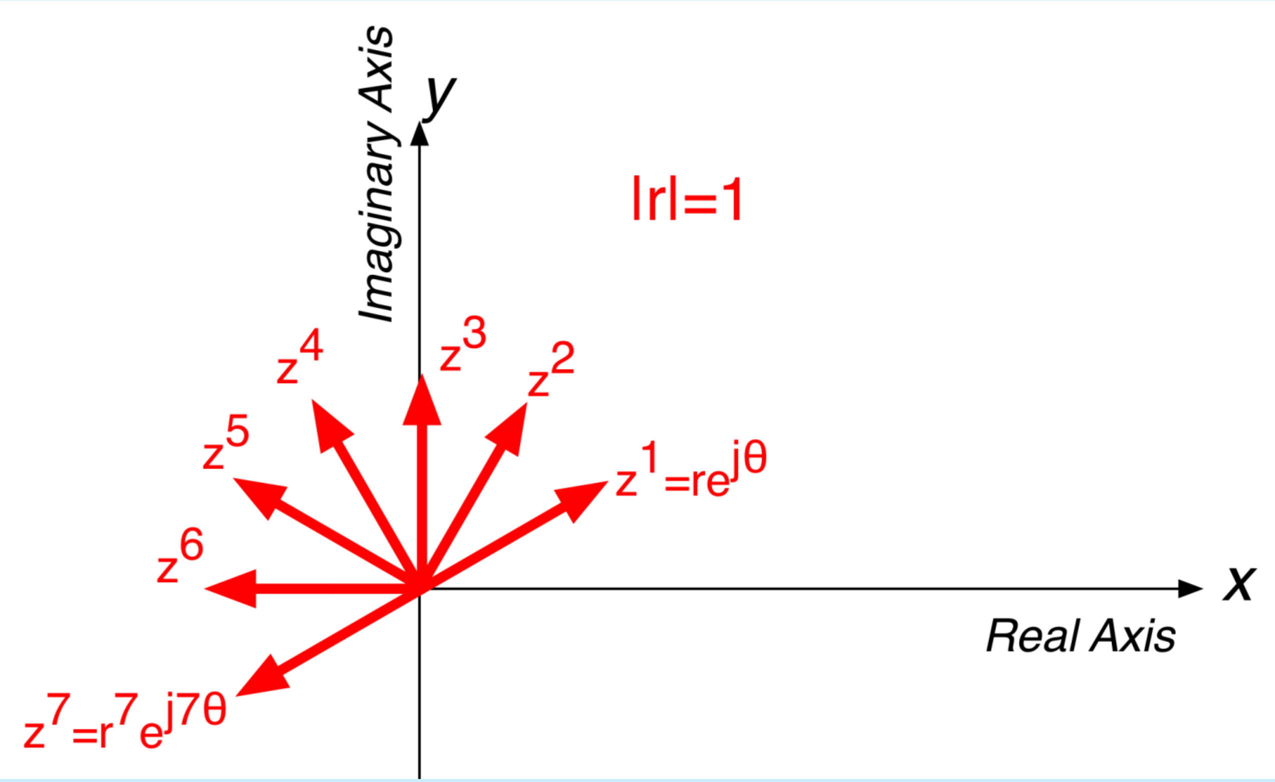
POWERS

- Raising to a power N rotates vector by Nθ and scales vector length by r^N



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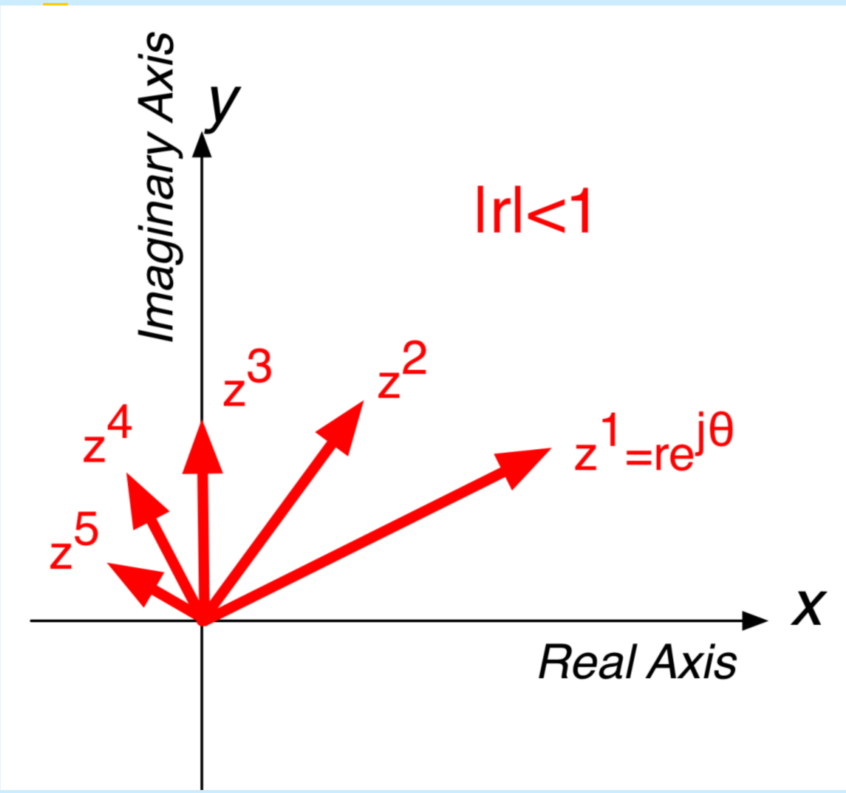
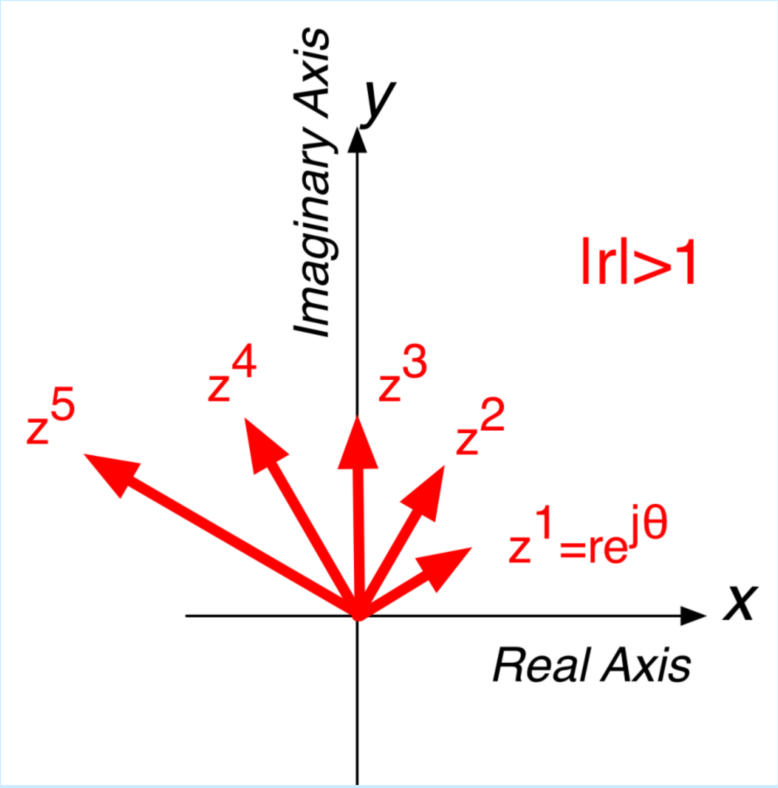
MORE POWERS





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ROOTS OF UNITY

- We often have to solve $z^N = 1$
- How many solutions?

$$z^N = r^N e^{jN\theta} = 1 = e^{j2\pi k}$$

$$\Rightarrow r = 1, \quad N\theta = 2\pi k \Rightarrow \theta = \frac{2\pi k}{N}$$

$$z = e^{j2\pi k/N}, \; k = 0, 1, 2, \ldots N-1$$

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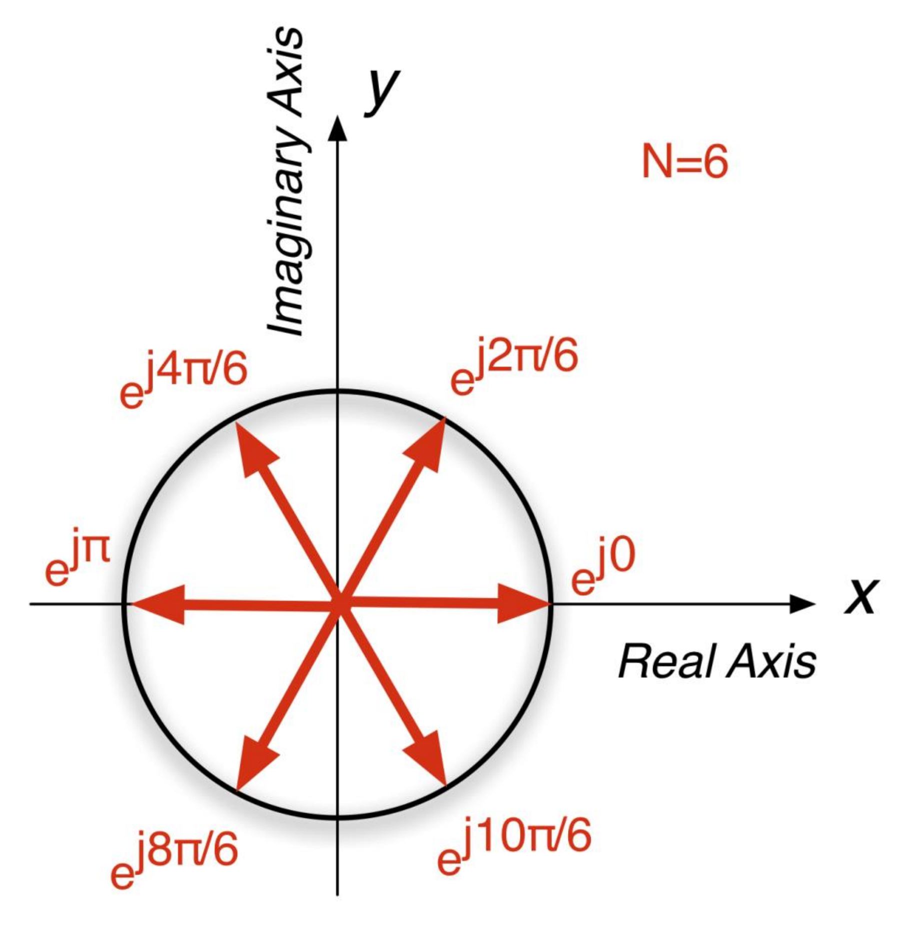
ROOTS OF UNITY for N=6



- Solutions to $z^N=1$ are N equally spaced vectors on the unit circle!
- What happens if we take the sum of all of them?

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Sum the Roots of Unity

- Looks like the answer is zero (for N=6)

$$\sum_{k=0}^{N-1} e^{j2\pi k/N} = 0?$$

- Write as geometric sum

$$\sum_{k=0}^{N-1} r^k = \frac{1-r^N}{1-r} \quad \text{then let } r = e^{j2\pi/N}$$

$$\text{Numerator} \quad 1-r^N = 1-(e^{j2\pi/N})^N = 1-e^{j2\pi} = 0$$

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Integrate Complex Exp

- Needed later to describe periodic signals in terms of sinusoids (Fourier Series)
  - Especially over one period

$$\int_a^b e^{j\theta} d\theta = \left.\frac{e^{j\theta}}{j}\right|_a^b = \frac{e^{jb} - e^{ja}}{j}$$

$$\int_0^T e^{j2\pi t/T} dt = \frac{e^{j2\pi T/T} - e^{j0}}{j} = \frac{1-1}{j} = 0$$

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BOTTOM LINE

- **CARTESIAN**: Addition/subtraction is most efficient in Cartesian form
- **POLAR**: good for multiplication/division
- STEPS:
  - Identify arithmetic operation
  - Convert to easy form
  - Calculate
  - Convert back to original form

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Review Appendix A and TLH Ch2 on Course Website

Harman Chapter 2 Pages 55-60 Complex Numbers
and
MATLAB Complex Numbers

TABLE 2.4 *MATLAB commands for complex numbers*

Command	*Format*
`z=x+yi, z=x+yj`	Complex number
`z=r*exp(i*theta)`	Polar form
abs	Magnitude $\lvert z \rvert = \sqrt{x^2 + y^2}$
angle	Angle in radians $(-\pi, \pi)$; $\theta = \tan^{-1}(y/x)$
conj	Complex conjugate $x - yi$
imag	Complex imaginary part y
real	Complex real part x
Plotting:	
compass	Draws complex numbers as arrows on polar plot
feather	Draws complex numbers as arrows on linear plot