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# DSP First, 2/e

Lecture Ch3

Fourier Series Analysis

SEE COURSE WEBSITE FourierCh8_TLH

# LECTURE OBJECTIVES

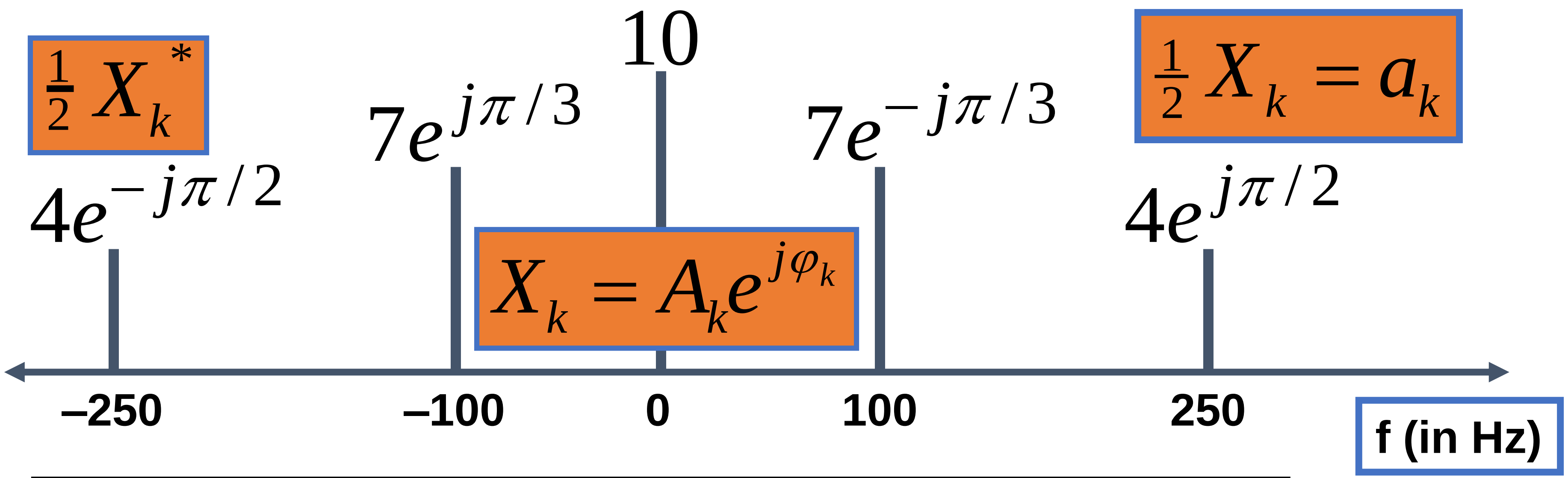
- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\mathbf{x(t+T_0) = x(t)}$
  - Draw spectrum from the Fourier Series coefficients

# SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq




$$x(t) = \boxed{a_0} + \sum_{k=1}^{N} \left\{ \boxed{a_k} e^{j2\pi f_k t} + \boxed{a_k^*} e^{-j2\pi f_k t} \right\}$$

# Harmonic Signal->Periodic

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}}$$

*Sums of **Harmonic** complex exponentials are **Periodic** signals*

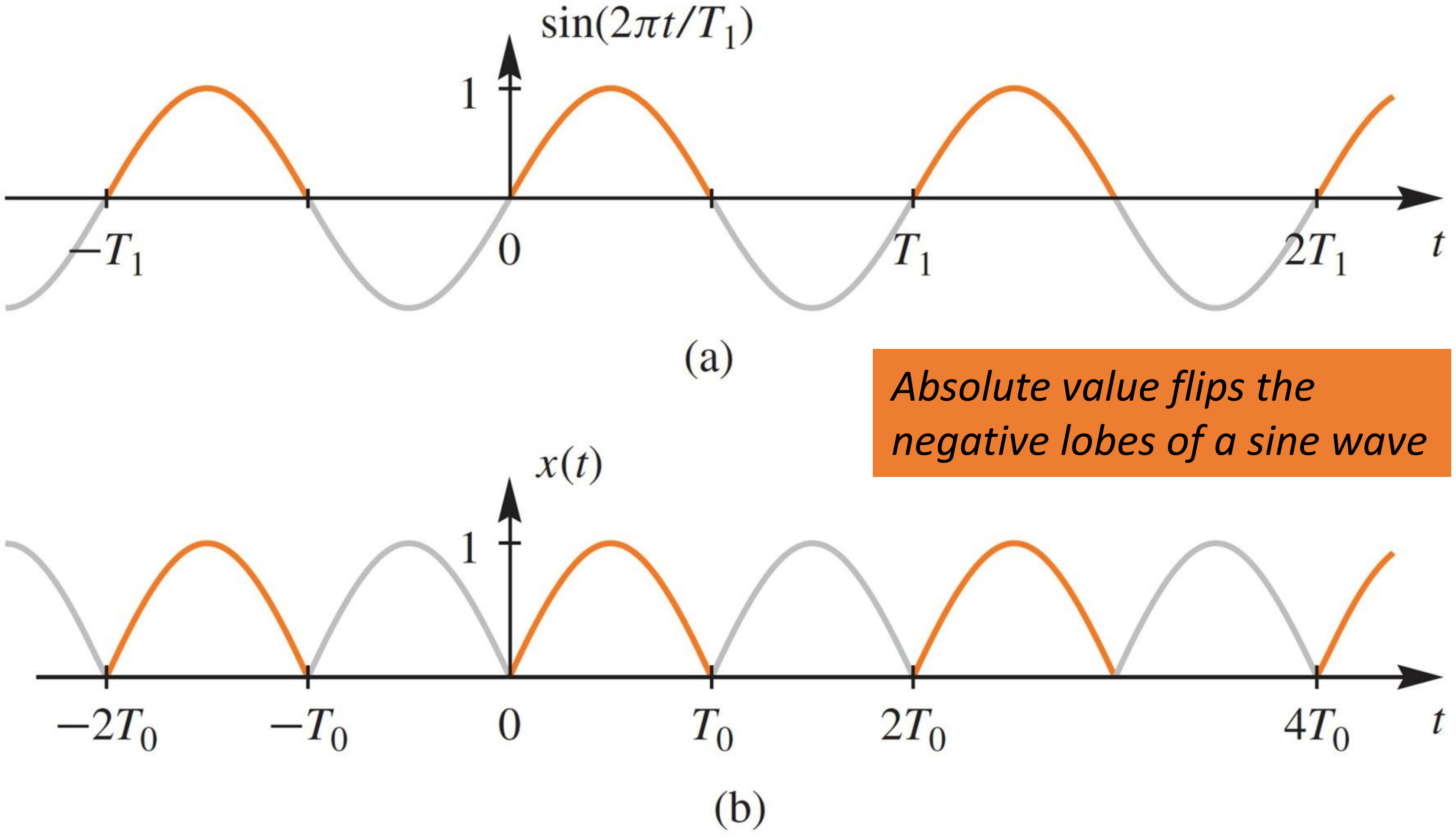
PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$\boxed{2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}}$$

# Full-Wave Rectified Sine

$$x(t) = \left|\sin(2\pi t / T_1)\right| \qquad \text{Period is } T_0 = \tfrac{1}{2} T_1$$

- Frequency
- Doubles




*Absolute value flips the negative lobes of a sine wave*

# Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

**Full-Wave Rectified Sine**

$$x(t) = \left|\sin(2\pi t/T_1)\right|$$

$$Period: T_0 = \tfrac{1}{2}T_1$$

$$\Rightarrow x(t) = \left|\sin(\pi t/T_0)\right|$$

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} \sin(\tfrac{\pi}{T_0}t)e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k-1)t}dt - \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k+1)t}dt$$

$$= \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

# Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

$$= \frac{1}{2\pi(2k-1)}\left(e^{-j(\pi/T_0)(2k-1)T_0} - 1\right) - \frac{1}{2\pi(2k+1)}\left(e^{-j(\pi/T_0)(2k+1)T_0} - 1\right)$$

$$= \frac{1}{\pi(2k-1)}\left(e^{-j\pi(2k-1)} - 1\right) - \frac{1}{\pi(2k+1)}\left(e^{-j\pi(2k+1)} - 1\right)$$

$$= \left(\frac{2k+1-(2k-1)}{\pi(4k^2-1)}\right)\left(-(-1)^{2k} - 1\right) = \frac{-2}{\pi(4k^2-1)}$$

# Fourier Coefficients: $a_k$

- $a_k$ is a function of k
  - Complex Amplitude for k-th Harmonic
  - NOTE: $\frac{1}{k^2}$ $for\ large\ k$
  - Does not depend on the period, $T_0$
  - DC value is

$$\boxed{a_k = \frac{-2}{\pi(4k^2-1)}}$$

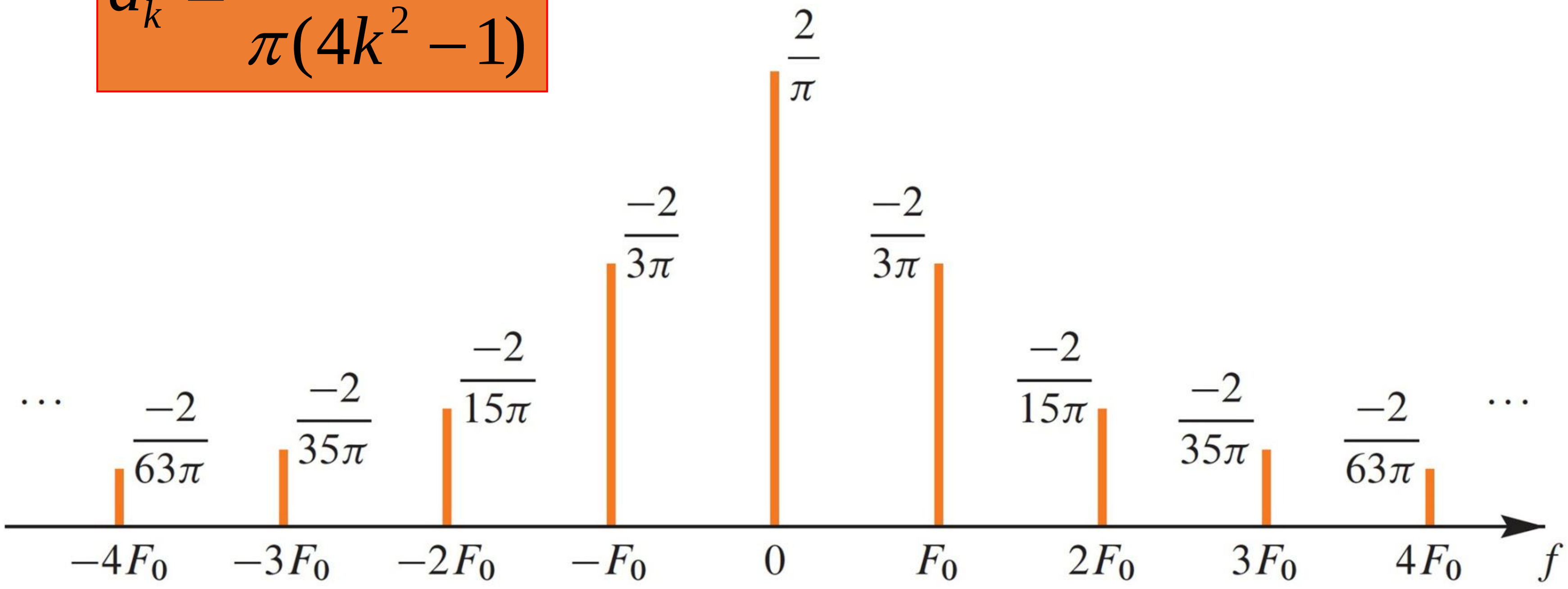
$$\boxed{a_0 = 2/\pi = 0.6336}$$

# Spectrum from Fourier Series

Plot $a_k$ for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$$F_0 = 1/T_0 \quad \text{and} \quad \omega_0 = 2\pi F_0$$



# Fourier Trig Series

☐ EXAMPLE 8.4 ***Fourier series square wave example***

A square wave of amplitude $A$ and period $T$ shown in Figure 8.4 can be defined as

$$f(t) = \begin{cases} A, & 0 < t < \dfrac{T}{2}, \\ -A, & -\dfrac{T}{2} < t < 0, \end{cases}$$

with $f(t) = f(t + T)$, since the function is periodic.

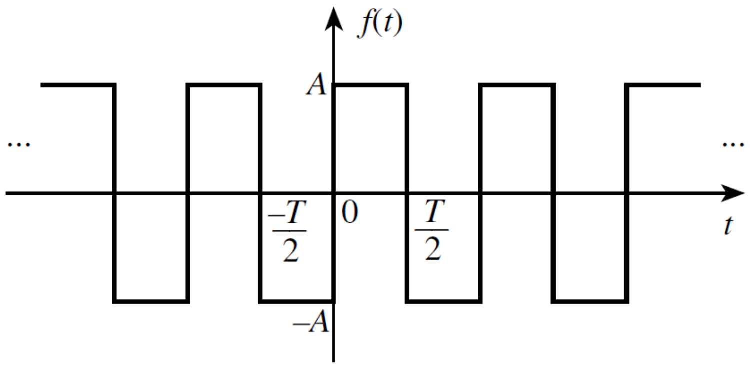



**FIGURE 8.4** *Square wave of Example 8.4*

The first observation is that $f(t)$ is odd, which yields the result that $a_0 = 0$ and $a_i = 0$ for every coefficient of the cosine terms. Letting $\omega_0 = 2\pi/T$, the coefficients $b_n$ are

$$b_n = 2\left(\frac{2}{T}\right)\int_0^{T/2} A\sin(n\omega_0 t)\,dt.$$

The result is

$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

where $(2n-1)$ is introduced to assure that only odd terms are included in the summation. The sine waves that make up the Fourier series for the odd square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right],$$

so the series consists not only of sine terms, as expected, but also odd harmonics appear. This is due to the rotational symmetry of the function since the wave shapes on alternate half-cycles are identical in shape but reversed in sign. Such waveforms are produced in certain types of rotating electrical machinery.

$\square$

□ **EXAMPLE 8.5** ***Complex Series Square Wave Example***

Consider the odd square wave of Example 8.4 and the complex Fourier coefficients

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{0}(-A)e^{-in\omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2}(A)e^{-in\omega_0 t}\,dt, \tag{8.29}$$

which leads to the series

$$f(t) = \frac{2A}{i\pi}\sum_{n=-\infty}^{\infty}\frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}, \tag{8.30}$$

as defined in Equation 8.23.

This form contains complex coefficients, but the series can be written in terms of sine waves by combining the corresponding terms for positive and negative arguments. To determine the coefficients, the amount of difficulty is about the same for the trigonometric series and the complex series. However, the complex series perhaps has an advantage when the magnitude of the coefficients are of interest.

Each coefficient has the form

$$\alpha_n = \frac{2A}{in\pi} = \frac{2A}{n\pi}e^{-i\pi/2}, \qquad n = \pm 1, \pm 3, \ldots,$$

and the coefficients for even values, $n = 0, \pm 2, \ldots$, are zero. Notice that the coefficients decrease as the index $n$ increases. The use of these coefficients to compute the *frequency spectrum* of $f(t)$ is considered later.

The trigonometric series is derived from the complex series by expanding the complex series of Equation 8.30 as

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty}\alpha_n e^{in\omega_0 t} \\ &= \cdots - \frac{2A}{3\pi i}e^{-i3\omega_0 t} - \frac{2A}{\pi i}e^{-i\omega_0 t} + \frac{2A}{\pi i}e^{i\omega_0 t} + \frac{2A}{3\pi i}e^{i3\omega_0 t} + \cdots \end{aligned}$$

and recognizing the sum of negative and positive terms for each $n$ as $2\sin(n\omega_0 t)$. The trigonometric series becomes

$$f(t) = \frac{4A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

which is the result of Example 8.4.

$\square$