Solution of Differential Equations and Filters

Contents

I.Fourier Series Solution of Differential Equation

Fourier Series DE Example

Consider the simple circuit of Figure 8.8 consisting of a resistor R and capacitor C. The input voltage is designated $f(t)$ and the output voltage across the capacitor is $y(t)$.

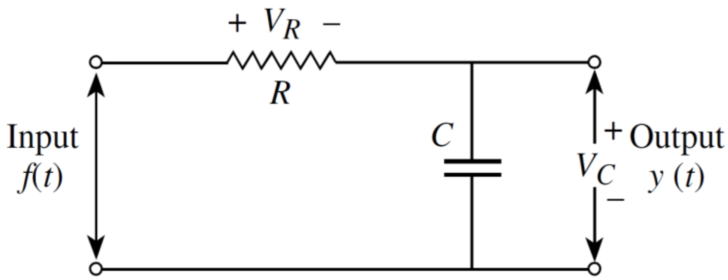


FIGURE 8.8 *RC circuit*

Sherlock Holmes once said – Having seen a dripping faucet, one could envision Niagara Falls.

I.1 Suppose the input can be written as a Fourier Series:

Differentiation of the Fourier series, in effect, multiplies the original series by n and thus increases the magnitude of the coefficients. Using the complex series, the derivative is

$$\frac{df(t)}{dt} = \frac{d}{dt}\sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} (in\omega_0)\alpha_n e^{in\omega_0 t}. \quad \textbf{(8.44)}$$

Consider the nth-order differential equation with constant coefficients subject to a harmonic series of sinusoids input as the forcing function. The differential equation is thus

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = \sum_{n=1}^{N} A_n e^{in\omega_0 t}. \quad \textbf{(8.45)}$$

Following the discussion in Chapter 5, if the frequencies $n\omega_0$ are not those of the characteristic equation, the assumed solution is

$$y(t) = \sum_{n=1}^{N} \alpha_n e^{in\omega_0 t}, \quad \textbf{(8.46)}$$

using the method of undetermined coefficients. Substituting the kth term in the differential equation leads to the relationship

$$\alpha_k[(i\omega_k)^n + a_{n-1}(i\omega_k)^{n-1} + \cdots + a_1(i\omega_k) + a_0] = A_k.$$

Letting $H^{-1}(ik\omega_0)$ designate the term in brackets, the solution for the kth coefficient of the solution is

$$\alpha_k = H(ik\omega_0)A_k,$$

and the complete particular solution is

$$y(t) = \sum_{n=1}^{N} H(in\omega_0)\, A_n e^{in\omega_0 t}. \quad \textbf{(8.47)}$$

I.2 Thus, the result from a LTI differential equation is another Fourier Series multiplied by the transfer function H.

To derive the differential equation for the circuit, apply Kirchhoff's voltage law, with the result

$$f(t) = V_R(t) + V_C(t).$$

Using Ohm's law, $V_R = Ri(t)$, where $i(t)$ is the current through the circuit. Since the current is proportional to the change in voltage across the capacitor,

$$i(t) = C\frac{dV_C(t)}{dt},$$

and Kirchhoff's law can be written

$$f(t) = RC\frac{dV_C(t)}{dt} + V_C(t).$$

Letting $y(t) = V_C(t)$ leads to the resulting equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}f(t).$$

If the input voltage can be written as

$$f(t) = \sum_{n=1}^{N} A_n e^{in\omega_0 t},$$

the solution according to Equation 8.47 is

$$y(t) = \sum_{n=1}^{N} H(in\omega_0) \times A_n e^{in\omega_0 t},$$

and the function $H(in\omega_0)$ is

$$H(in\omega_0) = \frac{1}{1 + in\omega_0 RC}.$$

I.3 The Solution has the same frequencies as the original series but the Magnitude and Phase will change.

$$\begin{aligned} |H(in\omega_0)| &= \frac{1}{\sqrt{1 + (n\omega_0 RC)^2}}, \\ \arg[H(in\omega_0)] &= -\tan^{-1}(n\omega_0 RC). \end{aligned} \qquad \textbf{(8.48)}$$

We could plot the Frequency Response in Amplitude and Phase.

II Fourier Transform Solution of a Differential Equation

Applying the differentiation theorem to the differential equation,

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t) \qquad \textbf{(8.52)}$$

by forming the Fourier transform of both sides of the equation, we find that

$$[\,(i\omega)^n + a_{n-1}(i\omega)^{n-1} + \cdots + a_1(i\omega) + a_0\,]Y(i\omega) = F(i\omega).$$

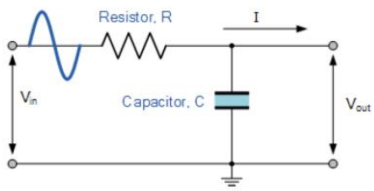
The solution for $y(t)$ could be found by solving the transformed equation as

$$Y(i\omega) = \frac{F(i\omega)}{[\,(i\omega)^n + a_{n-1}(i\omega)^{n-1} + \cdots + a_1(i\omega) + a_0\,]}$$

and taking the inverse Fourier transform. An example will demonstrate this application to the solution of differential equations.

III. Passive Low Pass Filter Impedance Method

https://www.electronics-tutorials.ws/filter/filter_2.html



We also know that the capacitive reactance of a capacitor in an AC circuit is given as:

$$X_C = \frac{1}{2\pi f C} \text{ in Ohm's}$$

Opposition to current flow in an AC circuit is called **impedance**, symbol Z and for a series circuit consisting of a single resistor in series with a single capacitor, the circuit impedance is calculated as:

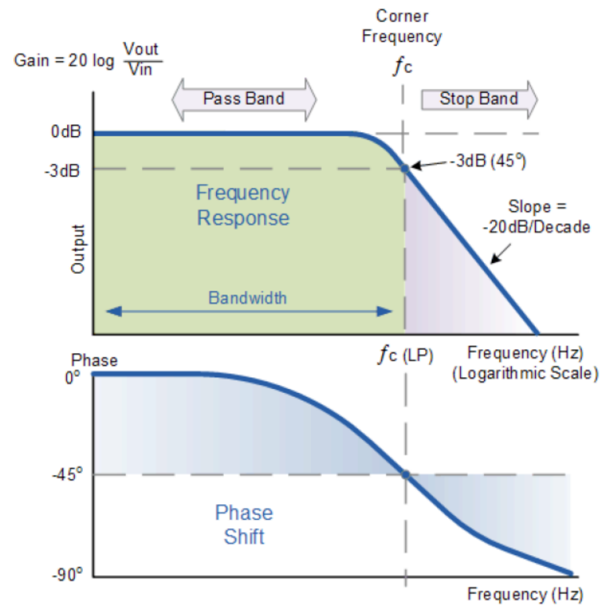
$$Z = \sqrt{R^2 + X_C^2}$$

RC Potential Divider Equation

$$V_{out} = V_{in} \times \frac{X_C}{\sqrt{R^2 + X_C^2}} = V_{in} \frac{X_C}{Z}$$

III.1 Frequency Response of a Low Pass Filter - Like RC filter.

Frequency Response of a 1st-order Low Pass Filter



Cutoff Frequency

The formula for cutoff frequency (corner frequency) is

$$f_c = \frac{1}{2\pi RC}$$

where R and C are the values of Resistance and Capacitance. For a simple RC low pass filter, cut-off (3dB point) is defined as when the resistance is the same magnitude as the capacitive reactance

III.2 RC values R= 4.7 k Ohms, C = 47 x 10^{-9} farads.

Voltage Output at a Frequency of 100Hz.

$$Xc = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 100 \times 47 \times 10^{-9}} = 33{,}863\Omega$$

$$V_{OUT} = V_{IN} \times \frac{Xc}{\sqrt{R^2 + X_C^2}} = 10 \times \frac{33863}{\sqrt{4700^2 + 33863^2}} = 9.9\text{v}$$

Voltage Output at a Frequency of 10,000Hz (10kHz).

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 10{,}000 \times 47 \times 10^{-9}} = 338.6\Omega$$

$$V_{OUT} = V_{IN} \times \frac{X_C}{\sqrt{R^2 + X_C^2}} = 10 \times \frac{338.6}{\sqrt{4700^2 + 338.6^2}} = 0.718\text{v}$$

IV. Laplace Transform Solution

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt. \tag{9.1}$$

The variable of integration t in Equation 9.1 is a dummy variable and may be replaced by any other symbol. In order for the quantity st to be dimensionless in the factor e^{-st}, if t represents time, then s has dimensions of frequency . In general, s is complex and is written

$$s = \sigma + i\omega$$

in which σ and ω are real.

IV.1 Laplace Transform Examples

Laplace Transform Examples

a. Consider the piecewise continuous function $f(t)$ defined as

$$f(t) = \begin{cases} 0, & t < 0, \\ Ae^{-at}, & t \geq 0. \end{cases}$$

The Laplace transform is

$$\begin{aligned} F(s) = \mathcal{L}[f(t)] &= \int_0^\infty Ae^{-at}e^{-st}\,dt \\ &= \int_0^\infty Ae^{-(s+a)t}\,dt \\ &= A\frac{e^{-(s+a)t}}{-(s+a)}\bigg|_0^\infty = \frac{A}{s+a} \end{aligned} \tag{9.2}$$

provided that $s + a > 0$.[1] Thus, we make the transform correspondence

$$Ae^{-at}\ (t \geq 0) \Leftrightarrow \frac{A}{s+a}. \tag{9.3}$$

TABLE 9.2 *Example Laplace transforms*

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$U(t)$	$\frac{1}{s}$	$\cos(\omega t)U(t)$	$\frac{s}{s^2+\omega^2}$
$tU(t)$	$\frac{1}{s^2}$	$\sin(\omega t)U(t)$	$\frac{\omega}{s^2+\omega^2}$
$t^nU(t)$	$\frac{n!}{s^{n+1}}$	$\exp(-at)\cos(\omega t)U(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
$\exp(-at)U(t)$	$\frac{1}{s+a}$	$\exp(-at)\sin(\omega t)U(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$

IV.2 MATLAB functions for Laplace

TABLE 9.5 *MATLAB commands for Laplace analysis*

Command	*Result*
Numerical operations:	
conv	Convolution and polynomial multiplication
polyder	Derivative of a polynomial
roots	Roots of a polynomial
residue	Partial fraction expansion
Symbolic Math Toolbox operations:	
laplace	Laplace transform
ilaplace	Inverse Laplace transform
Special Symbolic functions:	
Dirac	Dirac delta function
Heaviside	Unit step function

IV.3 Differential Equation Solution for Laplace

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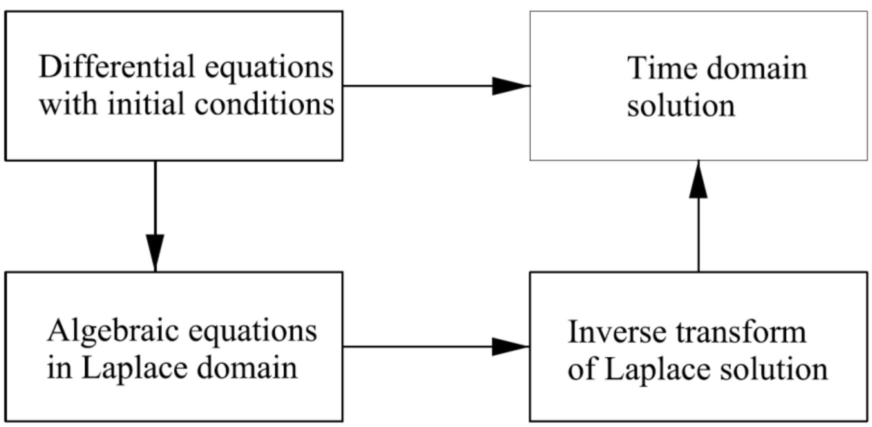

FIGURE 9.1 *Procedure for solving linear differential equations*

IV.4 Laplace and Fourier

The Laplace transform $F(s) = \int_0^\infty f(t)e^{-st}\,dt$ can be viewed as an extension of the Fourier transform in which the function $f(t)$ is represented by exponential functions of the form e^{st}, where $s = \sigma + i\omega$ is a complex frequency. Thus, the Fourier transform is a special case in which $s = +i\omega$ when $f(t)$ is zero for $t < 0$ and the Fourier integral exists.[8]

TABLE 9.7 *Laplace and Fourier transform pairs*

$f(t)$	$F(s)$	$F(i\omega)$
$\delta(t)$	1	1
$\exp(-at)U(t)$	$\frac{1}{s+a}$	$\frac{1}{i\omega+a}$
$t\exp(-at)U(t)$	$\frac{1}{(s+a)^2}$	$\frac{1}{(i\omega+a)^2}$

We conclude that when $f(t)$ is causal and absolutely integrable so that the Fourier transform exists, the relationship between the Laplace and Fourier transforms is

$$F(i\omega) = F(s)|_{s\to i\omega} \qquad F(s) = F(i\omega)|_{i\omega\to s}\,.$$

V. Difference Equations

A general form for an Nth-order linear difference equation with constant coefficients is

$$\begin{aligned} a_0\, y(n) \quad &+ \quad a_1\, y(n-1) + \cdots + a_{N-1}\, y(n-N+1) + a_N\, y(n-N) \\ &= \quad b_0\, x(n) + b_1\, x(n-1) + \cdots + b_M\, x(n-M), \end{aligned} \tag{10.4}$$

where $a_0, \ldots, a_N$ are constant with a_0 and a_N nonzero. The coefficients $b_0, \ldots, b_M$ are also constant with b_0 and b_M nonzero in the general case.

In summation notation, Equation 10.4 becomes

$$\sum_{i=0}^{N} a_i y(n-i) = \sum_{i=0}^{M} b_i x(n-i). \tag{10.5}$$

A general form for an Nth-order linear difference equation with constant coefficients is

$$\begin{aligned} a_0\, y(n) \quad &+ \quad a_1\, y(n-1) + \cdots + a_{N-1}\, y(n-N+1) + a_N\, y(n-N) \\ &= \quad b_0\, x(n) + b_1\, x(n-1) + \cdots + b_M\, x(n-M), \end{aligned} \tag{10.4}$$

Note that the solution y(n) depends on both INPUTS x and OUTPUTS y from previous outputs. This is a general form. If the a’s are zero, this reduces to FIR form.

V1. Differential Equation to Difference Equation

One method of associating a differential equation with a difference equation approximates the derivatives with differences. For example, letting $\hat{y}(t)$ be a continuous function, the first derivative can be approximated as

$$\left.\frac{d\hat{y}(t)}{dt}\right|_{t=nT} \approx \frac{\hat{y}(nT+T)-\hat{y}(nT)}{T} \tag{10.23}$$

as we have seen previously in Chapter 6 using the Euler approximation. The second derivative can be approximated as

$$\begin{aligned}\left.\frac{d^2\hat{y}(t)}{dt^2}\right|_{t=nT} &\approx \frac{\dfrac{d\hat{y}(nT+T)}{dt}-\dfrac{d\hat{y}(nT)}{dt}}{T} \\ &= \frac{\hat{y}(nT+2T)-2\hat{y}(nT+T)+\hat{y}(nT)}{T^2}.\end{aligned} \tag{10.24}$$

From these approximations, a difference equation can be formed from a given differential equation.

V2. Low-Pass Digital Filter

LOWPASS FILTERS

We begin by introducing a particular type of filter called a *lowpass* or *smoothing* filter. This filter is used to remove noise from a signal by limiting the output variations of the filtered signal, and thus produce an output that is smoother than the input signal. After giving an example of a smoothing filter, we present the frequency response of filters described by difference equations.

☐ **EXAMPLE 10.7** ***First-Order Smoothing Filter***

Consider the discrete-time signal represented by the sequence

$$x(0), x(1), x(2), \cdots,$$

which we assume is a signal corrupted by random noise. One method of reducing the unwanted fluctuations is to compute a smoothed version of the signal according to the rule

$$y(n) = a\, y(n-1) + (1-a)\, x(n) \tag{10.29}$$

where $0 < a < 1$ and $y(-1) = 0$.[2] At each value n, the output $y(n)$ is formed as a weighted average of the new input $x(n)$ and the output $y(n-1)$ at the preceding time instant $n-1$. If the constant a is almost zero, the output is almost equal to the input. The closer a is to 1, the more the preceding output is weighted and the "smoother" is the output.

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[1] Some authors, particularly those describing signals and linear systems, use the notation $f[n]$ with square brackets to designate discrete values of $\hat{f}(t)$.

[2] It is assumed that the output $y(n)$ cannot begin before the input $x(0)$. Systems that do not respond before being stimulated are called causal.

Example 10.7

```
% SMOOTHER.M A smoothing filter defined as
%    y(n) = a*y(n-1) +(1-a)*x(n) , y(-1)=0
%    x(n) is input signal, y(n) is smoothed output
%
% Test signal is sin(w*t) with random noise
%  INPUT:  Weighing factor a
%  OUTPUT: Plot of x and y
%
clear, clf
w=2*pi/5;
t = linspace(0,10,100);     % Time steps
s = sin(w*t);               % Noiseless signal
% Add random noise
len=size(t);
na = 0.1;                   % Noise amplitude
noise = na*(rand(len)-.5);  % (-.05 to +.05)
x = s + noise;
%
% Weighing factor
a = input('Weighing factor a= ')
%
y(1)=(1-a)*x(1);
for I=2:100
 y(I) = a*y(I-1) + (1-a)*x(I);  % Digital Filter
end
plot(t,x,'--',t,y,'-')
xlabel('Time'), ylabel('Signals')
title(['Effect of Smoothing Filter, a = ', num2str(a)])
legend('Input x','Output y')
```

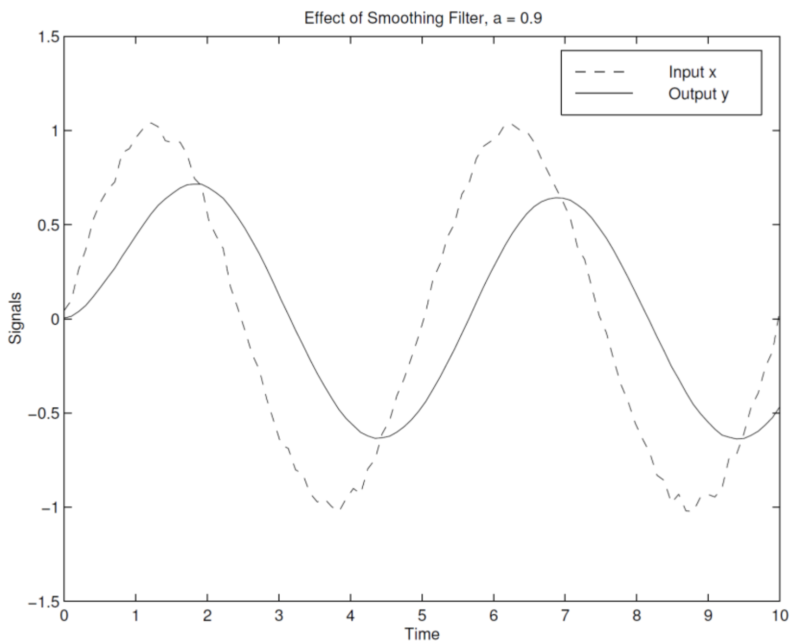

FIGURE 10.1 *Signal and Smoothed Version*

V3. Frequency Response of Digital Filters

As explained in Chapter 8, for continuous-time systems that are linear and time invariant, the frequency response is determined by assuming a sinusoidal signal as the input to the system. The steady state output is also a sinusoid of the same frequency, perhaps with an amplitude scaling and a phase shift compared to the input signal. In the differential equation model for the system with zero initial conditions, the ratio of the output sinusoidal signal to the input sinusoidal signal is called the *transfer function* of the system. The transfer function computed or plotted for a range of input frequencies represents the frequency response of the system.

In an analogous way, the frequency response of a linear, discrete-time system that is time-independent is determined by using a complex sampled sinusoid

$$x(n) = \hat{x}(nT_s) = Ae^{i\omega nT_s}$$

as the input. The value A is a real constant and T_s is the time between samples of the input signal. The ratio of the output $y(n)$ to $x(n)$ is the

transfer function for the discrete system. By varying the input frequency, the frequency response of the discrete system is determined.

Often the linear frequency variable (hertz) is normalized to yield the *digital frequency*

$$F = \frac{f}{f_s} = fT_s. \qquad \textbf{(10.30)}$$

Also, $\Omega = 2\pi F = \omega T_s$ is the digital radian frequency. The analog frequency $f = 1/T_s$ corresponds to the digital frequency $F = 1$ or $\Omega = 2\pi$.

Remember $\widehat{w} = \Omega = 2\,pi * \frac{f\_analog}{F\_sample}$; $F_{sample} = \frac{1}{Ts}$; f_max = $F_{sample}/2$

And I used in the this chapter fs= F_{sample} Hz or Samples/second

V4. Solutions for FIR and IIR filters

Difference Equations To determine the frequency response of a system described by the difference equation

$$y(n) + \sum_{m=1}^{N} a_m \, y(n-m) = \sum_{m=0}^{M} b_m \, x(n-m), \qquad \textbf{(10.31)}$$

we apply a unit sinusoid

$$x(n) = e^{i\Omega n}$$

as the discrete input and assume the solution to the difference equation has the form

$$y(n) = He^{i\Omega n}.$$

Thus, H is the transfer function for the system.

Substituting the assumed response into Equation 10.31 yields

$$He^{i\Omega n} + \sum_{m=1}^{N} a_m He^{i\Omega(n-m)} = \sum_{m=0}^{M} b_m e^{i\Omega(n-m)}.$$

Dividing out $e^{i\Omega n}$ and solving for H shows that the frequency response is

$$H = \frac{\sum_{m=0}^{M} b_m e^{-i\Omega m}}{1 + \sum_{m=1}^{N} a_m e^{-i\Omega m}}. \qquad \textbf{(10.32)}$$

Although H is a complex constant for each digital frequency, H will vary as the input frequency changes and it is customary to write the transfer function as $H(e^{i\Omega})$ to emphasize the dependence on $e^{i\Omega}$ in Equation 10.32. Notice that the discrete-time system frequency response is periodic with the sampling frequency

$$\omega_s = \frac{2\pi}{T_s}$$

since $H(e^{i(\omega + k\omega_s)T_s}) = H(e^{i\omega T_s})$ where k is an integer.

Note: For the FIR filters, the a coefficients would be zero.

Frequency Response of First-Order System

The frequency response of a first-order discrete system will be investigated. Consider the smoothing filter of Example 10.7

$$y(n) = a\, y(n-1) + (1-a)\, x(n)$$

where $0 < a < 1$. Using Equation 10.32 with $a_1 = -a$ and $b_0 = (1-a)$,

$$\begin{aligned} H &= \frac{1-a}{1-ae^{-i\Omega}} = \frac{1-a}{1-ae^{-i2\pi F}} \\ &= \frac{1-a}{1-a\cos 2\pi F + ia \sin 2\pi F} \end{aligned} \qquad \textbf{(10.33)}$$

since $\Omega = 2\pi F$. Writing H in polar form $H(e^{2\pi i F}) = A(F)e^{i\phi(F)}$ yields the amplitude and phase response of the system as

$$\begin{aligned} A(F) &= \frac{1-a}{\sqrt{1+a^2-2a\cos 2\pi F}} \\ \phi(F) &= -\tan^{-1} \frac{a \sin 2\pi F}{1-a\cos 2\pi F}. \end{aligned} \qquad \textbf{(10.34)}$$

Notice that the *dc gain* $H(0) = 1$ and that H is periodic. Figure 10.2 displays the amplitude and phase in degrees for three values of a.

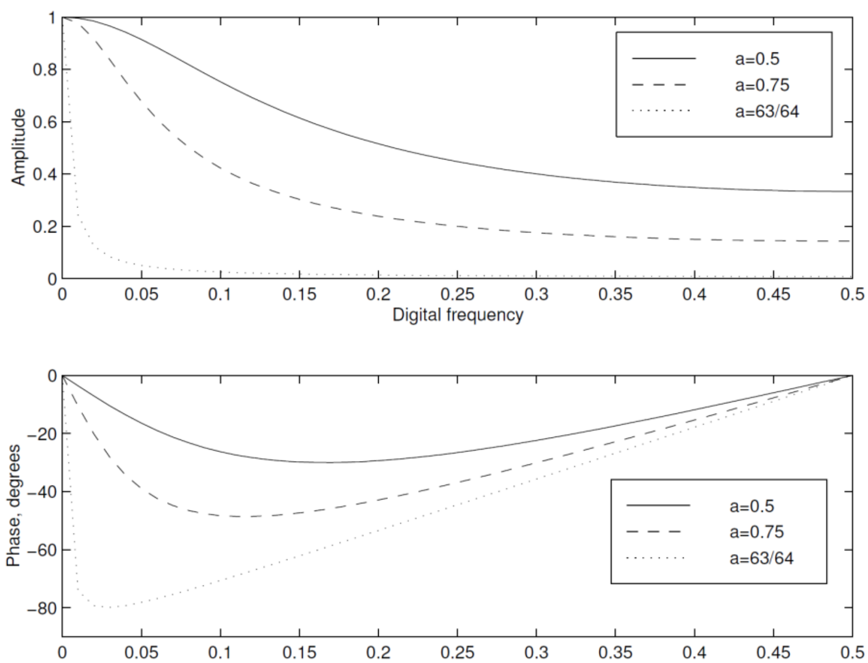

FIGURE 10.2 *Amplitude and phase plots*

From Figure 10.2, we see that this form of a digital filter acts as a low-pass filter since it attenuates the higher frequencies in an input signal relative to the lower frequencies. If a continuous input signal is sampled at time intervals T_s to form $x(n)$, Equation 10.30 shows that the sample interval T_s as well as the parameter a determine the frequency characteristic of the discrete system considered as a digital filter.

VI. Z-transforms

10.5 INTRODUCTION TO Z TRANSFORMS

In preceding chapters, we have studied differential equations by direct solution as well as by using Fourier and Laplace transform techniques. We now turn to a transform that is called the *Z-transform* due to the use of the complex variable $z = x + iy = \rho e^{i\Omega}$ in the transform. The Z-transform plays the same role for discrete systems as the Laplace transform does for continuous systems. Table 10.3 summarizes some of the applications of the Z-transform.

TABLE 10.3 *Applications of Z-transform*

Area	*Application*
Definition	The Z-transform is defined as a sum that transforms discrete signals to the complex frequency (Z) domain.
System analysis	The Z-transform converts convolutions to a product and difference equations to algebraic equations.
Stability	Stability of a discrete linear system can be determined by analyzing the transfer function $H(z)$ given by the Z-transform.
Frequency response	The transfer function $H(z)$ can be evaluated to determine the frequency response of a discrete system.
Digital filters	Digital filters can be analyzed and designed using the Z-transform.
Control	Digital control systems can be analyzed and designed using Z-transforms.

TABLE 10.4 *Example Z Transforms*

$f(n)$	$F(z)$	$f(n)$	$F(z)$
$u(n)$	$\frac{z}{z-1}, \lvert z\rvert > 1$	$\cos(n\omega)u(n)$	$\frac{z^2 - z\cos\omega}{z^2 - 2z\cos\omega + 1}, \lvert z\rvert > 1$
$a^n u(n)$	$\frac{z}{z-a}, \lvert z\rvert > a$	$\sin(n\omega)u(n)$	$\frac{z\sin\omega}{z^2 - 2z\cos\omega + 1}, \lvert z\rvert > 1$
$nu(n)$	$\frac{z}{(z-1)^2}, \lvert z\rvert > 1$	$na^n u(n)$	$\frac{za}{(z-a)^2}, \lvert z\rvert > a$

TABLE 10.5 *MATLAB commands for Discrete Analysis*

Command	*Result*
Numerical operations:	
conv	Convolution and polynomial multiplication.
roots	Roots of a polynomial.
residue	Partial fraction expansion.
Symbolic Math Toolbox operations:	
ztrans	Z transform.
iztrans	Inverse Z transform.
Signal Processing Toolbox operations:	
freqz	Frequency response of discrete system.
residuez	Partial-fraction expansion.

VI.1 Z-transforms and Frequency Response

Relate Z and F.

If the transfer function $H(z)$ is evaluated for values of

$$z = \exp(i2\pi F) = \exp(i\Omega)$$

we obtain the *frequency response*, $H(i2\pi F)$, of the system. This is equivalent to evaluating $H(z)$ on the unit circle in the z-plane. Note that the function $H(i2\pi F)$ is periodic with period 1 since $\exp(i2\pi F)$ is periodic with period 1. According to Equation 10.30 the *digital frequency*

$$F = \frac{f}{f_s} = fT_s,$$

where T_s is the sampling time or time between samples. The analog frequency $f = 1/T_s$ corresponds to the digital frequency $F = 1$ or $\Omega = 2\pi$.

z-plane and Frequency Response

The accompanying MATLAB script and figures show the pole-zero plot and frequency response for a system with transfer function

$$H(z) = \frac{z}{z - 0.9}.$$

After defining the numerator and denominator of $H(z)$, the **subplot** command in the script reduces the size of the pole-zero plot but leaves the labels full size. The command **zplane** produces the plot of Figure 10.3. Then, the command **freqz** plots the magnitude in dB and the phase of the frequency response.

MATLAB Script

```
Example 10.17
% ZEX.M Show the Z-plane and frequency response of the function
%   H(z)=z/(z-.9)
clear,clf
num=[1 0]; den=[1 -0.9]; % Define numerator and denominator of H(z)
subplot(2,2,1)            % Keep plot small
figure(1)
zplane(num,den)  % Draw the z-plane
figure(2)
freqz(num,den)            % Plot frequency response
```

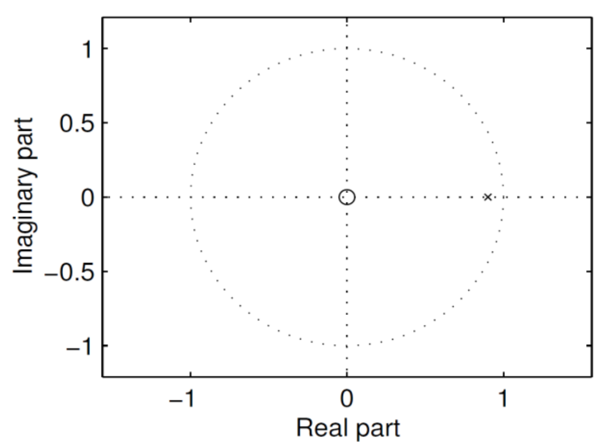

FIGURE 10.3 *z-plane plot of* $H(z) = z/(z - 0.9)$

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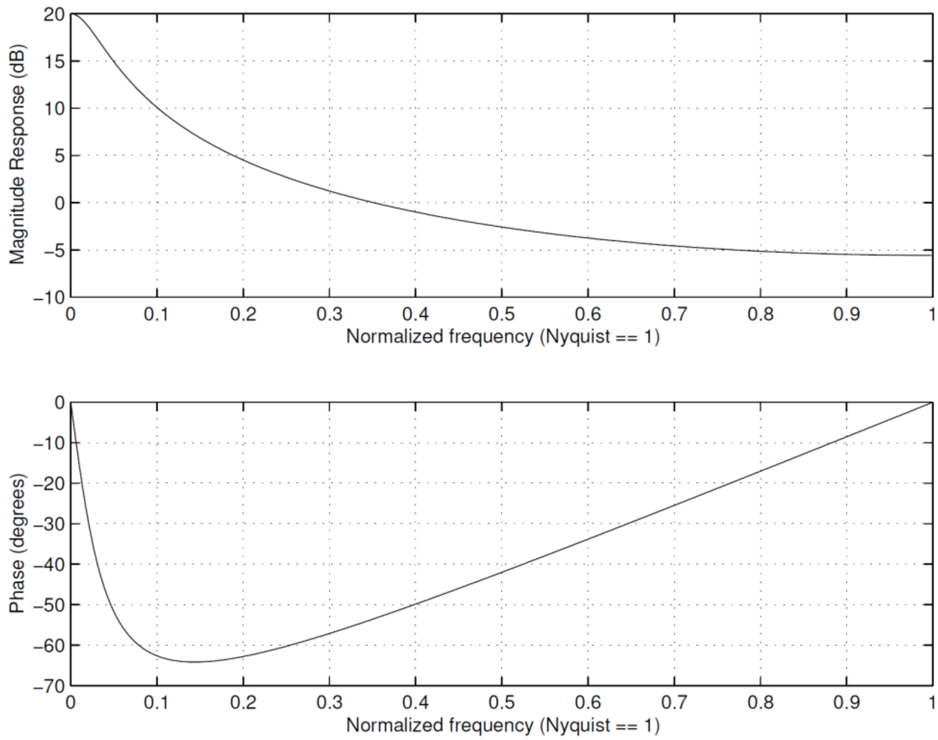

FIGURE 10.4 *Amplitude and phase plots of* $H(i2\pi F)$