9 $\;\;$ LAPLACE TRANSFORMS

PREVIEW

The Laplace transform is a linear integral transform used extensively to solve differential equations and characterize linear systems. For the initial value problem described by a linear ordinary differential equation with constant coefficients, the Laplace transform converts the equation into an algebraic equation in which any initial conditions are incorporated. The solution to the original differential equation is then found by inverting the transform. This chapter defines the Laplace transform and its properties and illustrates the application of the transform to analyze various physical systems.

$\qquad$ The chapter begins with the definition and properties of the Laplace transform. The discussion presents examples of the transform as well as the mathematical conditions for the existence of the transform. Next, techniques for the computation of the inverse transform are explored. Determining the inverse transform is a key step when differential equations are solved by the Laplace transform technique. Then, MATLAB commands are used to aid computation of Laplace Transforms and their inverses.

$\qquad$ An important section of the chapter demonstrates the Laplace transform method as applied to solution of ordinary differential equations. A related application is the characterization of linear systems. We show

that the transfer function and conditions of stability for such systems can be studied using Laplace transforms.

Table 9.1 summarizes example applications of Laplace transform techniques. This chapter emphasizes Laplace transform solution of differential equations but briefly describes applications to linear systems.

TABLE 9.1 *Applications of Laplace transforms*

Area	Application
Differential equations	Laplace transforms are used to solve linear, ordinary differential equations with constant coefficients and given initial conditions.
Linear systems	The Laplace transform can be used to characterize the response of a linear system
Stability	Stability of a linear system can be determined by analyzing the transfer function given by the Laplace transform.

For some purposes, the Laplace transform can be viewed as a generalization of the Fourier transform. A section of this chapter explores the relationship between the transforms. Another section presents a summary of basic Laplace transform properties.

9.1 DEFINITION AND PROPERTIES OF THE LAPLACE TRANSFORM

Let the function $f(t)$ be a function defined for $t \geq 0$. Such a function is called *causal* and generally such functions that begin at $t = 0$ are of interest in the analysis of physical systems. If $f(t)$ satisfies suitable conditions and is nonzero only for $0 \leq t < \infty$, the *unilateral or one-sided Laplace transform* is an operation, denoted by the symbol $\mathcal{L}$, which associates a function $F(s)$ with $f(t)$ according to the rule

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st} \, dt. \tag{9.1}$$

In this text, the function $F(s)$ will be called simply the *Laplace transform* of $f(t)$. Functions that are nonzero on the whole real axis $(-\infty, \infty)$ require

the two-sided (bilateral) Laplace transform which has $-\infty$ as the lower limit of integration. This form has limited use and is not considered here.

The variable of integration t in Equation 9.1 is a dummy variable and may be replaced by any other symbol. In order for the quantity st to be dimensionless in the factor e^{-st}, if t represents time, then s has dimensions of frequency . In general, s is complex and is written

$$s = \sigma + i\omega$$

in which σ and ω are real.

If $F(s)$ is the Laplace transform of the function $f(t)$, we say that $f(t)$ is the *inverse Laplace transform* when the inverse transform exists. In operator notation, the inverse transform will be denoted

$$f(t) = \mathcal{L}^{-1}[F(s)].$$

☐ **EXAMPLE 9.1**   *Laplace Transform Examples*

    a. Consider the piecewise continuous function $f(t)$ defined as

$$f(t) = \begin{cases} 0, & t < 0, \\ Ae^{-at}, & t \geq 0. \end{cases}$$

The Laplace transform is

$$
\begin{aligned}
F(s) = \mathcal{L}[f(t)] \quad &= \quad \int_0^\infty Ae^{-at}e^{-st}\, dt \\
&= \quad \int_0^\infty Ae^{-(s+a)t}\, dt \\
&= \quad A\frac{e^{-(s+a)t}}{-(s+a)}\Big|_0^\infty = \frac{A}{s+a} \quad\quad (9.2)
\end{aligned}
$$

provided that $s + a > 0$.[1] Thus, we make the transform correspondence

$$Ae^{-at} \ (t \geq 0) \Leftrightarrow \frac{A}{s+a}. \quad\quad (9.3)$$

    b. The Laplace transform of the unit step function,

$$U(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

is

$$F(s) = \mathcal{L}[U(t)] = \int_0^\infty 1e^{-st}\, dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \quad\quad (9.4)$$

for $s > 0$. Thus, we have the transform pair

$$U(t) \Leftrightarrow \frac{1}{s}. \qu\quad (9.5)$$

---

[1]Recall that this integral is an *improper integral* since the interval of integration is infinite. The integral is evaluated as $\int_0^\infty f(t)e^{-st}\, dt = \lim_{T \to \infty} \int_0^T f(t)e^{-st}\, dt$.

The unit step function is often called the *Heaviside unit step* in honor of the engineer Oliver Heaviside who applied Laplace transform techniques to engineering problems. The unit step function is often used as a test signal input to physical systems to determine the response as discussed in Chapter 6. Mathematically, a function that is zero for $t < 0$ can be written

$$f(t)U(t).$$

Thus, $Ae^{-at}U(t)$ represents the function with the Laplace transform given by Equation 9.2. The transform pair of Equation 9.3 can be written

$$Ae^{-at}U(t) \Leftrightarrow \frac{A}{s+a}.$$

□

W H A T I F ?   Consider the Fourier transforms of the functions in Example 9.1. Show that the Fourier transform of $f(t) = e^{-at}$ is given by $F(i\omega)$ where $F(s)$ is the Laplace transform of $f(t)$.

What happens if you apply the Fourier transform integral to the unit step function?

PROPERTIES OF THE LAPLACE TRANSFORM

Various important properties of the Laplace transform will be stated as theorems. These theorems can be employed to simplify the derivation of the Laplace transform for many functions.

■ **THEOREM 9.1**   ***Linearity***
    Assume that $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$. Then

$$\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s).$$

■ **THEOREM 9.2**   *s-**shifting***
    If $\mathcal{L}[f(t)] = F(s)$,
$$\mathcal{L}[e^{at}f(t)] = F(s-a). \tag{9.6}$$

This allows the computation of $e^{at}f(t)$ from the transform $F(s)$ of $f(t)$ by *shifting on the s-axis*. It follows that the inverse transform is

$$\mathcal{L}^{-1}[F(s-a)] = e^{at}f(t).$$

■ **THEOREM 9.3**   *Time Shifting*

Shifting $f(t)$ in time by t_0 simply multiplies the Laplace transform by e^{st_0}, so that

$$\mathcal{L}[f(t - t_0)] = e^{-st_0} F(s). \tag{9.7}$$

Each of these theorems is easily proven by substituting the definition of the Laplace transform in the theorem. Their use greatly simplifies the computation of many Laplace transform pairs.

■ **THEOREM 9.4**   *Multiplication by t*

If $\mathcal{L}[f(t)] = F(s)$,

$$\mathcal{L}[tf(t)] = -\frac{dF(s)}{ds}. \tag{9.8}$$

Thus, multiplying $f(t)$ by t results in (negative) differentiation of the transform as shown by writing

$$t \exp(-st) = -\frac{d[\exp(-st)]}{ds}$$

and reversing the order of differentiation and integration in the transform so that

$$
\begin{aligned}
\mathcal{L}[tf(t)] &= \int_0^\infty f(t) \left[-\frac{d[\exp(-st)]}{ds} \right] dt \\
&= -\frac{d}{ds} \int_0^\infty f(t) e^{-st} \, dt = -\frac{dF(s)}{ds}.
\end{aligned} \tag{9.9}
$$

☐ **EXAMPLE 9.2**   *Examples Using Laplace Transform Properties*

a. Applying the linearity theorem, we find that

$$
\begin{aligned}
\mathcal{L}[\cos(\omega t)] &= \mathcal{L}\left[\frac{e^{i\omega t} + e^{-i\omega t}}{2} \right] \\
&= \frac{1}{2}\left[\frac{1}{s - i\omega} + \frac{1}{s + i\omega} \right] = \frac{s}{s^2 + \omega^2}
\end{aligned} \tag{9.10}
$$

using the result of Example 9.1 for $\mathcal{L}[e^{-at}]$ with $a = \pm i\omega$.

b. If the Laplace transform of $\cos(\omega t)$ is $s/(s^2 + \omega^2)$, then

$$\mathcal{L}[e^{-at}\cos(\omega t)] = \frac{s + a}{(s + a)^2 + \omega^2}$$

according to the s-shift theorem.

c. The rectangular pulse of height A and width τ defined as

$$P(t) = \begin{cases} A, & 0 \le t \le \tau, \\ 0, & |t| > \tau \end{cases}$$

can be written as $A[U(t) - U(t - \tau)]$ using the unit step function. The Laplace transform is

$$\begin{aligned} \mathcal{L}[P(t)] &= A\mathcal{L}[U(t) - U(t - \tau)] \\ &= A\left[\frac{1}{s} - \frac{e^{-s\tau}}{s}\right] \end{aligned}$$

using Equation 9.5 of Example 9.1 and the time-shift theorem.

d. Since the transform of the unit step function $U(t)$ is $1/s$, the transform of $tU(t)$ is

$$\mathcal{L}[tU(t)] = -\frac{d}{ds}\left[\frac{1}{s}\right] = \frac{1}{s^2}$$

$\square$

EXISTENCE OF THE LAPLACE TRANSFORM

In the definition of the Laplace transform,

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\, dt,$$

the integral exists if $f(t)e^{-st}$ is absolutely integrable. We expect that if $f(t)$ is a function that does not grow too rapidly with t, the integral will converge if the real part of s is sufficiently positive. Sufficient conditions for $f(t)$ to have a Laplace transform are as follows:

1. There are constants α, M and T such that

$$e^{-\alpha t}|f(t)| < M \qquad \text{for all } t > T.$$

2. $f(t)$ is *piecewise continuous* on $[0, \infty)$.

Functions that satisfy the first condition are said to be of *exponential order*. For example, even an increasing exponential such as $e^{2t}U(t)$ can satisfy the inequality in Part 1 and thus have a Laplace transform for any α greater than or equal to 2. Functions such as e^{t^2} do not satisfy the condition since

$$\lim_{t\to\infty} \frac{e^{t^2}}{e^{\alpha t}} = \lim_{t\to\infty} e^{t^2 - \alpha t} = \infty$$

no matter the value of α.

The second condition implies that even if $f(t)$ is piecewise continuous and thus has finite jump discontinuities, the function approaches finite limits as t approaches either endpoint from the interior of the interval where $f(t)$ is continuous.

The Laplace transform of a function is unique. For practical purposes, two functions that have the same transforms are identical.[2]

---

[2]The functions can only differ at isolated points. Two continuous functions with the same transform are identical. (This statement is known as Lerch's theorem).

TABLE LOOKUP   When computing a Laplace transform analytically, it is often useful to
                make use of a known transform and the properties of the transform such
                as in the transforms in Example 9.2. Problem 9.2 asks you to compute a
                number of Laplace transforms that are shown in Table 9.2. In the table,
                the functions $f(t)$ are causal because they are multiplied by the unit step
                function.

TABLE 9.2 *Example Laplace transforms*

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$U(t)$	$\dfrac{1}{s}$	$\cos(\omega t)U(t)$	$\dfrac{s}{s^2 + \omega^2}$
$tU(t)$	$\dfrac{1}{s^2}$	$\sin(\omega t)U(t)$	$\dfrac{\omega}{s^2 + \omega^2}$
$t^n U(t)$	$\dfrac{n!}{s^{n+1}}$	$\exp(-at)\cos(\omega t)U(t)$	$\dfrac{s + a}{(s + a)^2 + \omega^2}$
$\exp(-at)U(t)$	$\dfrac{1}{s + a}$	$\exp(-at)\sin(\omega t)U(t)$	$\dfrac{\omega}{(s + a)^2 + \omega^2}$

9.2 COMPUTATION OF INVERSE LAPLACE TRANSFORMS

The primary mathematical use of the Laplace transform is to solve differ-
ential equations. In particular, Laplace techniques solve the initial value
problem for ordinary differential equations with constant coefficients. The
basic idea is to convert the differential equation to an algebraic equation
which is solved in terms of the Laplace variable s. Then, the solution of the
original equation is found by determining the functions that correspond
to the Laplace solution. This involves inverting the Laplace solution.

The section concerning differential equations later in this chapter
elaborates on the technique of using Laplace transforms to find solutions.
First, it is necessary to understand how to determine the inverse function
corresponding to a Laplace transform. In other words, given a transform
$F(s)$ as in Equation 9.1, we wish to find the function $f(t)$ that is the
inverse Laplace transform.

When the Laplace transform $F(s)$ of a function is tabulated as for the
functions in Table 9.2, the inverse transform $f(t)$ is also given in the table.
In many cases, however, it is necessary to compute the inverse transform.
The general method is not presented in this chapter but we present the

method of inverting transforms that are the ratio of polynomials in the Laplace variable[3]

INVERSE TRANSFORMS FOR RATIONAL FUNCTIONS

Before considering various applications of Laplace transforms, we will explore methods to compute the inverse Laplace transform for a special class of transforms. These transforms arise from the study of linear time-invariant systems which are described by linear ordinary time-invariant differential equations as explained in Chapter 5.

In this section, the Laplace transforms we consider are a function of s that can be written as the ratio of polynomials in s, say

$$\frac{P(s)}{Q(s)} = \frac{b_0 + b_1 s + \ldots + b_p s^p}{a_0 + a_1 s + \cdots + a_r s^r} \tag{9.11}$$

for which

1. All coefficients are real;

2. $P(s)$ and $Q(s)$ have no common factors;

3. $Q(s)$ has higher degree than $P(s)$ $(r > p)$ so that $P(s)/Q(s)$ is said to be a *proper rational fraction*.

Elementary problems of interest will lead to a transform which is a proper rational fraction. Problem 9.8 presents a Laplace transform for which this is not the case.

It can be shown that any proper rational fraction in s can be written

$$F(s) = \frac{P(s)}{Q(s)} = F_1(s) + \cdots + F_n(s) \tag{9.12}$$

where each $F_i(s)$ is a rational function, called a *partial fraction*. The denominator $Q(s)$ can be factored into linear factors and quadratic factors having the form

$$(s - a)^m \qquad \text{or} \qquad (s^2 + \alpha s + \beta)^n \tag{9.13}$$

when a, α, and β are real.

When $m = 1$ in the first expression, the term is called a *simple factor*. Otherwise, where $m = 2, 3, \ldots$, the term is said to be a *repeated factor*. We will call the second form an *irreducible complex factor* if the denominator has no real roots.

The approach to expanding $F(s)$ in terms of partial fractions in s is an extension of the method of partial fraction expansion used in calculus to integrate functions of the form $P(x)/Q(x)$ where x is a real variable. The extension to functions of the complex variable s is straightforward

---

[3]The general method requires integration in the complex plane for which the techniques of complex variable theory are required.

and we will accept the result without proof. For the functions of interest, the expansion can be demonstrated by finding a common denominator in the expansion and showing the expansion equals the original function.

In this section, we write $P(s)/Q(s)$ as a sum of terms according to the form of the denominators in Equation 9.13 as follows:

1. To a simple factor $(s - a)$ assign the term

$$\frac{K}{(s - a)};$$

2. To factors $(s - a)^m$ with $m > 1$ assign the sum of terms

$$\frac{K_1}{s - a} + \frac{K_2}{(s - a)^2} + \cdots + \frac{K_m}{(s - a)^m};$$

3. To factors $(s^2 + \alpha s + \beta)^n$ with complex roots in the denominator assign the sum of terms

$$\frac{A_1 s + B_1}{(s^2 + \alpha s + \beta)} + \frac{A_2 s + B_2}{(s^2 + \alpha s + \beta)^2} + \cdots + \frac{A_n s + B_n}{(s^2 + \alpha s + \beta)^n}.$$

In the last case, we assume that the quadratic polynomial in the denominator has no real roots. Thus, the roots will occur in complex conjugates pairs since the coefficients are real. Once $P(s)/Q(s)$ is written in terms of its *partial fractions*, it is necessary to solve for the constants, $K, K_1, \ldots, K_m, A_1, \ldots, B_n$.

The coefficients for the factors just listed, such as $K, K_1, \ldots, K_m$, and the As and Bs, can be found by various techniques. The main idea in this section is to expand a proper rational fraction $P(s)/Q(s)$ in terms of the partial fractions by finding the coefficients and then use the results listed in Table 9.2 and the properties of the Laplace transform to find the inverse transform.[4]

Heaviside Expansion Technique   Several theorems will be presented that define a procedure to find the partial fraction expansion for a proper rational function with simple factors or repeated factors. The technique is called *Heaviside expansion*. Theorem 9.5 applies to the simple factors in the partial fraction expansion of a Laplace transform.

■ THEOREM 9.5   ***Inverse transforms for simple factors***
Suppose that $P(s)/Q(s)$ is a proper rational fraction. The coefficient of a simple factor in the partial fraction expansion such as

$$\frac{K}{s - a}$$

---

[4]The computation of the inverse Laplace transform for a general transform requires integration in the complex plane. Such integration techniques are not considered in this chapter.

is given by

$$K = \frac{P(a)}{Q'(a)} = \frac{P(a)}{q(a)}$$

where $Q'(a)$ is the derivative of $Q(s)$ evaluated at $s = a$ and $q(a)$ is the product of all the factors of $Q(s)$ except $s - a$, so that $q(s) = Q(s)/(s - a)$.

∎

Considering the theorem, it is not necessary to compute the derivative to find the coefficient of a simple factor when $Q(s)$ is a polynomial in s. In practice, the coefficient K is found analytically by multiplying the Laplace transform by the factor $s - a$ and evaluating the result at $s = a$ so that

$$K = \lim_{s \to a} \left[(s - a) \frac{P(s)}{Q(s)} \right] = \frac{P(a)}{q(a)}.$$

From the theorem and Table 9.2, the inverse transform corresponding to the simple factor would be

$$\mathcal{L}^{-1} \left[\frac{K}{s - a} \right] = \mathcal{L}^{-1} \left[\frac{P(a)}{q(a)(s - a)} \right] = \frac{P(a)}{q(a)} e^{at} U(t).$$

Thus, if the factors of the denominator in the partial fraction expansion are all simple, such as $(s - s_i)$, the transform is easily inverted. Suppose the Laplace transform can be written as

$$\begin{aligned} F(s) &= \frac{P(s)}{Q(s)} = \frac{P(s)}{(s - s_1)(s - s_2) \cdots (s - s_N)} \\ &= \frac{A_1}{(s - s_1)} + \frac{A_2}{(s - s_2)} + \cdots + \frac{A_N}{(s - s_N)}. \end{aligned} \qquad \textbf{(9.14)}$$

The ith coefficient can be found by multiplying both sides by $(s - s_i)$ and evaluating them at $s = s_i$ so that

$$A_i = \lim_{s \to s_i} \left[(s - s_i) F(s) \right] = (s - s_i) \left. F(s) \right|_{s = s_i}.$$

Each term in Equation 9.14 will yield an inverse transform of the form $A_i \exp(s_i t) U(t)$ so the inverse will be

$$f(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \cdots + A_N e^{s_N t} \qquad t \geq 0.$$

Table 9.3 summarizes the partial fraction expansion technique for transforms with simple factors. In this case, there are N simple roots of $Q(s)$ as in Equation 9.14.

TABLE 9.3 *Partial fraction expansions*

$F(s) = P(s)/Q(s)$	Expansion	Constants
$P(s) \displaystyle\prod_{i=1}^{N} \frac{1}{(s - s_i)}$	$\displaystyle\sum_{i=1}^{N} \frac{A_i}{(s - s_i)}$	$A_i = (s - s_i) \left. F(s) \right|_{s = s_i}$

For a transform with only simple factors as shown in the table, the inverse transform consists of a sum of exponential terms having the coefficients $A_1, A_2, \ldots, A_N$.

□ EXAMPLE 9.3   *Inverse Laplace Transform with Simple Roots*

As an example of a Laplace transform with simple factors, let

$$F(s) = \frac{s^2 + 4s + 2}{s(s+1)(s+2)}.$$ (9.15)

Equating this expression to terms with simple factors

$$\frac{s^2 + 4s + 2}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2},$$ (9.16)

the right-hand side could be summed and the three coefficients may be equated to those on the left. This would lead to three equations in three unknowns. In this case, $A = 1$, $B = 1$, and $C = -1$, which you should verify.

Another technique is to solve for a single unknown by multiplying both sides by the corresponding denominator and letting s be equal to the root of that denominator. For example, to find A, multiply both sides of Equation 9.16 by s and take the limit as $s \to 0$ so that

$$\left. \frac{s^2 + 4s + 2}{(s+1)(s+2)} \right|_{s \to 0} = \lim_{s \to 0} \left[A + \frac{sB}{s+1} + \frac{sC}{s+2} \right].$$

Thus, $A = 1$. Multiplying Equation 9.16 by $(s+1)$ and letting $s = -1$ yields $B = 1$. Finally, multiplying by $(s+2)$ and letting $s = -2$ yields $C = -1$.

The expansion of Equation 9.15 is thus

$$F(s) = \frac{1}{s} + \frac{1}{s+1} - \frac{1}{s+2}.$$

Using the linearity property of the Laplace transform and Table 9.2, the inverse transform is

$$f(t) = 1 + e^{-t} - e^{-2t} \qquad t \geq 0.$$

□

Repeated and Simple Factors   Consider a Laplace transform with simple factors and a repeated factor of multiplicity m and its partial fraction expansion

$$
\begin{aligned}
\frac{P(s)}{Q(s)} &= \frac{P(s)}{(s-a)^m (s-s_1) \cdots (s-s_N)} \\
&= \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \cdots + \frac{A_N}{s - s_N} \\
&\quad + \frac{K_1}{(s-a)} + \frac{K_2}{(s-a)^2} + \cdots + \frac{K_m}{(s-a)^m}.
\end{aligned}
$$ (9.17)

Table 9.4 describes a procedure for calculating the coefficients in Equation 9.17 that generalizes the procedure of Table 9.3. In the table,

the denominator $Q(s)$ contains N simple factors and a repeated factor of multiplicity m. Note that if we treat a derivative of order zero as the original function, all the terms in the expansion are of the same form.

TABLE 9.4 *Partial fraction expansions with repeated roots*

$F(s) = P(s)/Q(s)$	*Expansion*	*Constants*
$\dfrac{P(s)}{(s-a)^m} \displaystyle\prod_{i=1}^{N} \dfrac{1}{(s-s_i)}$	$\displaystyle\sum_{i=1}^{N} \dfrac{A_i}{(s-s_i)} + \sum_{j=1}^{m} \dfrac{K_j}{(s-a)^j}$	$K_j = \dfrac{1}{(m-j)!} \dfrac{d^{(m-j)}}{ds^{m-j}} [(s-a)^m F(s)]_{s=a}$

We list the steps to compute the coefficients of the repeated factors using the formal method shown in the table:

1. $K_m = \dfrac{1}{0!}(s-a)^m F(s)$ evaluated at $s = a$.

2. $K_{m-1} = \dfrac{1}{1!}\dfrac{d}{ds}[(s-a)^m F(s)]$ evaluated at $s = a$.

3. In general, $K_j, j < m$ is found by taking the $(m-j)$th derivative of the function $(s-a)^m F(s)$ evaluated at $s = a$, for $j = 1, 2, \ldots, m-1$ and dividing the result by $(m-j)!$.

☐ **EXAMPLE 9.4**   ***Inverse Laplace Transform with Repeated Roots***
Suppose a transform has repeated roots as in the example

$$Y(s) = \frac{s+3}{s(s+1)(s+2)^3}. \tag{9.18}$$

The expansion will be

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s+1} + \frac{K_3}{(s+2)^3} + \frac{K_2}{(s+2)^2} + \frac{K_1}{(s+2)} \tag{9.19}$$

The coefficients for the simple roots are $A_1 = 3/8$ and $A_2 = -2$. Using the prescription in Table 9.4, the coefficients for the repeated factors are

$$K_3 = \frac{1}{0!} (s+2)^3 Y(s)\Big|_{s=-2} = \frac{s+3}{s(s+1)}\Big|_{s=-2} = \frac{1}{2}$$

$$K_2 = \frac{1}{1!}\frac{d}{ds}\left[\frac{s+3}{s(s+1)}\right]_{s=-2} = \frac{(s^2+s)-(2s+1)(s+3)}{(s^2+s)^2}\Big|_{s=-2} = \frac{5}{4}$$

$$K_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{s+3}{s(s+1)}\right]_{s=-2}$$

$$= \frac{1}{2}\frac{-(s^2+s)(2s+6)+2(2s+1)(s^2+6s+3)}{(s^2+s)^3}\Big|_{s=-2} = \frac{13}{8}.$$

Thus, the Laplace transform of $Y(s)$ can be written

$$Y(s) = \frac{3/8}{s} + \frac{-2}{s+1} + \frac{1/2}{(s+2)^3} + \frac{5/4}{(s+2)^2} + \frac{13/8}{(s+2)} \qquad \textbf{(9.20)}$$

with the inverse transform

$$y(t) = \frac{3}{8} - 2e^{-t} + \frac{1}{4}t^2 e^{-2t} + \frac{5}{4}te^{-2t} + \frac{13}{8}e^{-2t} \quad t \geq 0. \qquad \textbf{(9.21)}$$

$\square$

Problem 9.4 presents another method to determine the partial fraction expansion when factors are repeated. Generally speaking, when $Q(s)$ is a polynomial, even with repeated factors, an algebraic method to find the coefficients in the expansion is less work than using derivatives.

Factors with Complex Roots Consider the partial fraction term in the form

$$\frac{As + B}{(s^2 + bs + c)}$$

for which there are no real roots of the denominator polynomial. In the algebra of the real numbers, such a polynomial would be called *irreducible*. For the purposes of inverting a Laplace transform, the complex factors could be used. An alternative procedure is to recognize that the denominator can be written as

$$s^2 + bs + c = (s - s_1)(s - \bar{s}_1)$$

where $s_1 = \alpha + i\beta$ and $\bar{s}_1 = \alpha - i\beta$ is the complex conjugate, using the notation of Chapter 2. It then follows that

$$\begin{aligned} (s - s_1)(s - \bar{s}_1) &= s^2 - (s_1 + \bar{s}_1)s + s_1\bar{s}_1 \\ &= s^2 - 2\alpha s + \alpha^2 + \beta^2 \\ &= (s - \alpha)^2 + \beta^2. \end{aligned}$$

The Laplace transform can be rewritten as

$$F(s) = \frac{As + B}{(s - \alpha)^2 + \beta^2} = \frac{A(s - \alpha) + \alpha A + B}{(s - \alpha)^2 + \beta^2} \qquad \textbf{(9.22)}$$

and thus has the inverse transform

$$Ae^{\alpha t}\cos\beta t + e^{\alpha t}\left(\frac{\alpha A + B}{\beta}\right)\sin\beta t \qquad \textbf{(9.23)}$$

according to Table 9.2.

◻ **EXAMPLE 9.5**   *Inverse Laplace Transform with Complex Factors*

Consider the Laplace transform

$$Y(s) = \frac{s^2 + s + 1}{s(s^2 + 2s + 2)}.$$

The two methods described will be used to evaluate the inverse transform.

a. Since the quadratic in the denominator has the roots $-1 \pm i$,

$$Y(s) = \frac{K}{s} + \frac{B}{s + 1 + i} + \frac{\bar{B}}{s + 1 - i}.$$

Since the numerator and denominator of $Y(s)$ have real coefficients, and since the denominators of the second and third fractions are conjugates, the numerators B and $\bar{B}$ are conjugates, as you can easily verify. Evaluating the simple factors as in previous examples leads to the result

$$K = \frac{1}{2}, \quad B = \frac{i}{(-1 - i)(-2i)} = \frac{1}{4}\sqrt{2}e^{-i\pi/4}, \quad \bar{B} = \frac{1}{4}\sqrt{2}e^{i\pi/4}.$$

Taking the inverse transform and combining the complex exponential terms yields

$$y(t) = \frac{1}{2} + \frac{1}{2}\sqrt{2}\cos\left(t + \frac{\pi}{4}\right).$$

This can also be written as $y(t) = 0.5 + 0.5e^{-t}(\cos t - \sin t)$.

b. The other approach is to write $Y(s)$ as

$$Y(s) = \frac{K}{s} + \frac{As + B}{(s + 1)^2 + 1}$$

to allow evaluation of the inverse transform without using complex numbers. As before $K = 1/2$. To determine A and B, subtract the term in $1/s$ to yield

$$Y(s) - \frac{1/2}{s} = \frac{As + B}{(s + 1)^2 + 1} = \frac{s^2 + s + 1}{s(s^2 + 2s + 2)} - \frac{1/2}{s}$$

$$= \frac{1}{2}\frac{s^2}{s(s^2 + 2s + 2)}$$

Equating numerators to determine A and B yields

$$As + B = \frac{1}{2}s$$

so $A = 1/2$ and $B = 0$. Considering the form of Equation 9.22 with $\alpha = -1$ and $\beta = 1$, the transform becomes

$$Y(s) = \frac{1}{2s} + \frac{1}{2}\frac{(s + 1)}{(s + 1)^2 + 1} + \frac{(-1)(1/2)}{(s + 1)^2 + 1}$$

with the inverse $y(t) = 0.5 + 0.5e^{-t}(\cos t - \sin t)$, as before.

◻

9.3 MATLAB AND LAPLACE TRANSFORMS

MATLAB has commands that can be used to compute Laplace transforms and their inverse. The *Symbolic Math Toolbox* and the *Signal Processing Toolbox* have additional commands for this purpose.[5] Table 9.5 lists commands that are useful for Laplace analysis. The *Symbolic Math Toolbox* also contains commands useful for the manipulation of polynomials. Use the command **help symbolic** to see the list of commands.

TABLE 9.5 *MATLAB commands for Laplace analysis*

Command	Result
Numerical operations:	
conv	Convolution and polynomial multiplication
polyder	Derivative of a polynomial
roots	Roots of a polynomial
residue	Partial fraction expansion
Symbolic Math Toolbox operations:	
laplace	Laplace transform
ilaplace	Inverse Laplace transform
Special Symbolic functions:	
Dirac	Dirac delta function
Heaviside	Unit step function

The first example in this section will present the numerical commands that are useful to aid computation of inverse transforms. The **residue** command is used to compute the factors in a partial fraction expansion. It is so named because in complex variable theory the coefficients of the factors in the expansion are called *residues*. Then, we will use the symbolic commands **laplace** and **ilaplace** to compute Laplace transforms and inverse transforms. The special symbolic functions Dirac and Heaviside will be described later in the chapter.

MATLAB AND
INVERSE
TRANSFORMS

If a Laplace transform is a ratio of polynomials in s, MATLAB commands can aid computation of the inverse. The numerical commands in Table 9.5 perform much of the algebra involved in inverting a rational transform.

---

[5]These toolboxes are included in the student version of MATLAB. They must be purchased separately from the professional version of MATLAB.

□ **EXAMPLE 9.6** **_MATLAB Inverse Laplace Transform Examples_**

This example shows how to use MATLAB commands to aid in determining the partial fraction expansion for the Laplace transform

$$F(s) = \frac{s^2 + 4s + 2}{s(s+1)(s+2)}$$

of Example 9.3. The factors of the polynomial in the denominator are multiplied by the **conv** (convolution) command. It is used twice to find the coefficients of the third-degree polynomial in the denominator.

The M-file shown then uses the **residue** command in the form

```
[K,S,p]=residue(P,Q)
```

to determine the roots of the denominator and numerator. The **residue** command determines the coefficients of the partial fraction expansion and the roots of simple or repeated factors. The column vector K contains the coefficients and S is the vector of corresponding roots. If the Laplace transform is a proper rational fraction, the column vector p will be empty.

MATLAB Script

```
Example 9.6
% INVLAP.M Factors of Inverse Laplace transform of
%  (s^2+4s+2)/s(s+1)(s+2) using conv and residue
%
% Compute the coefficients of denominator polynomial
Q1=[1 0];              % s
Q2=[1 1];              % s+1
QD1=conv(Q1,Q2);
Q3=[1 2];              % s+2
Q=conv(QD1,Q3)         % Coefficients of denominator
% Compute partial fraction coefficients
P=[1 4 2]              % Coefficients of numerator
[K,S,p]=residue(P,Q) % K coefficients, S roots of Q
%
% The edited results are as follows
Q =     1    3    2    0 % Coefficients of denominator
P =     1    4    2        % Coefficients of numerator
K = -1   % Coefficients of expansion
     1
     1
S = -2   % Roots
    -1
     0
p =[]
```

The results are edited from the diary file. From the results of **residue**, we can write

$$\frac{s^2 + 4s + 2}{s(s+1)(s+2)} = \frac{-1}{s+2} + \frac{1}{s+1} + \frac{1}{s}.$$

□

Other examples using MATLAB to find inverse transforms are given in Problem 9.15. Also, MATLAB plotting commands are useful to represent a Laplace transform graphically. This is helpful in analyzing systems described by the transform as indicated in the section of this chapter concerning the application of Laplace transforms to linear systems.

SYMBOLIC
LAPLACE
TRANSFORMS

If you have the *Symbolic Math Toolbox* there are MATLAB symbolic commands to compute Laplace transforms and their inverses for functions defined symbolically. The command **laplace** computes the transform and **ilaplace** the inverse as shown in Example 9.7.

☐ **EXAMPLE 9.7**   *MATLAB Symbolic Laplace Transform Examples*

This example presents the MATLAB solution of several previous examples. The edited results taken from the diary file are shown.

MATLAB Script _____

```
Example 9.7
% LAPEX.M Test of MATLAB Laplace function
%    NOTE: This M-file requires the Symbolic Math Toolbox
%
syms A a s t w
% (a) Exponential
f1=sym('A*exp(-a*t)'), F1=laplace(f1,t,s)
% (b) Cosine
f2=sym('cos(w*t)'), F2=laplace(f2,t,s)
pretty(F2) % Redisplay as P(s)/Q(s)
% (c) Inverse transform
F4=sym('(s^2+4*s+2)/(s^3+3*s^2+2*s)'), f4=ilaplace(F4,s,t)
%
F5=sym('(s+3)/(s*(s+1)*(s+2)^3)'), f5=ilaplace(F5,s,t)
%
% The results (edited) are as follows:
% (a)
f1 =A*exp(-a*t)
F1 =A/(s+a)
% (b)
f2 =cos(w*t)
F2 =s/(s^2+w^2)
% Pretty                                       s
                                            -------
                                             2    2
                                            s  + w
% (c) Inverse
F4 =(s^2+4*s+2)/(s^3+3*s^2+2*s), f4 = 1-exp(-2*t)+exp(-t)
%
F5 =(s+3)/(s*(s+1)*(s+2)^3)
f5 =3/8-2*exp(-t)+1/4*t^2*exp(-2*t)+5/4*t*exp(-2*t)+13/8*exp(-2*t)
```

☐

9.4 APPLICATIONS TO DIFFERENTIAL EQUATIONS

The Laplace transform is often used to solve differential equations and also to analyze linear time-invariant systems. In the cases we consider in this section, use of the Laplace transform converts a differential equation to an algebraic equation in the complex variable s. Inverting the transform leads to the solution of the differential equation in terms of the original variables. A diagram of the procedure is given in Figure 9.1. This technique is particularly useful when solving initial value problems.

FIGURE 9.1 *Procedure for solving linear differential equations*

LAPLACE
TRANSFORMS
OF DERIVATIVES
AND INTEGRALS

Before applying Laplace transforms to differential equations, we will compute the transform for derivatives and integrals of functions. The transform of the first and higher derivatives of a function will be derived. The notation for the derivative of $f(t)$ with respect to t will be $\dot{f}$ in this section.

■ **THEOREM 9.6**   ***Laplace transform of the Derivative***

Suppose that $f(t)$ is continuous for all $t \geq 0$ and that $f(t)$ is of exponential order, and its derivative $\dot{f}(t)$ is piecewise continuous on every interval. Then, for a function $f(t)$ with Laplace transform $F(s)$

$$\mathcal{L}\left[\frac{df}{dt}\right] = \mathcal{L}\left[\dot{f}(t)\right] = sF(s) - f(0). \qquad \textbf{(9.24)}$$

To show this result for $\dot{f}(t)$ continuous, apply the definition of the Laplace transform and integrate by parts so that

$$\mathcal{L}\left[\dot{f}(t)\right] = \int_0^\infty e^{-st}\dot{f}(t)\,dt = e^{-st}f(t)\big|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt.$$

The first term on the right reduces to $-f(0)$ since it is zero at the upper limit and the second term is the Laplace transform multiplied by s. Thus, the theorem holds for a function with a continuous derivative. If

the derivative is piecewise continuous, the range of integration must be separated into intervals so that $\dot{f}(t)$ is continuous in each interval.

Applying Theorem 9.6 to the second derivative $d^2 f/dt^2 \equiv \ddot{f}(t)$ yields

$$
\begin{aligned}
\mathcal{L}\left[\ddot{f}(t)\right] &= s\mathcal{L}[\dot{f}(t)] - \dot{f}(0) \\
&= s\{s\mathcal{L}[f(t)] - f(0)\} - \dot{f}(0) \\
&= s^2\mathcal{L}[f(t)] - sf(0) - \dot{f}(0).
\end{aligned} \tag{9.25}
$$

In general, the nth derivative $f^{(n)}(t)$ has the Laplace transform

$$
\mathcal{L}[f^{(n)}(t)] = s^n \mathcal{L}[f(t)] - s^{n-1} f(0) - s^{n-2}\dot{f}(0) - \cdots - f^{(n-1)}(0) \tag{9.26}
$$

if $f(t)$ and the derivatives of $f(t)$ through order $n-1$ are continuous and $f^{(n)}(t)$ is piecewise continuous.

■ THEOREM 9.7   **Laplace transform of the Integral**

If $f(t)$ is piecewise continuous and is of exponential order with Laplace transform $F(s)$,

$$
\mathcal{L}\left\{ \int_0^t f(\tau)\, d\tau \right\} = \frac{1}{s}\mathcal{L}[f(t)] = \frac{F(s)}{s}. \tag{9.27}
$$

These theorems are useful in the solution of equations that contain derivatives or integrals as well as to aid in inverting transforms containing the factors s or $1/s$.

We summarize the results as follows: for certain functions, differentiation of functions corresponds to the multiplication of the transforms by s, and integration of functions corresponds to the division of the transforms by s.

☐ EXAMPLE 9.8   **Inverse Laplace Transforms by Differentiation or Integration**

a. Let $\mathcal{L}[f(t)] = F(s)$. If $f(0) = 0$, then

$$
\mathcal{L}^{-1}\{sF(s)\} = \dot{f}(t)
$$

by Theorem 9.6. Since

$$
\mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 1} \right\} = \sin t \quad \text{and} \quad \sin 0 = 0
$$

then

$$
\mathcal{L}^{-1}\left\{ \frac{s}{s^2 + 1} \right\} = \frac{d}{dt}(\sin t) = \cos t.
$$

b. Using Theorem 9.7, and given that

$$
\mathcal{L}^{-1}\left\{ \frac{2}{s^2 + 4} \right\} = \sin 2t
$$

then

$$
\mathcal{L}^{-1}\left\{ \frac{2}{s(s^2 + 4)} \right\} = \int_0^t \sin 2\tau\, d\tau = \frac{1}{2}(1 - \cos 2t).
$$

☐

Consider the n-th order linear differential equation with constant coefficients

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t). \qquad \textbf{(9.28)}$$

The Laplace transform approach to the solution is to first transform the differential equation and the forcing function $f(t)$. The transformed equation is then solved for $Y(s)$ and the inverse transform is taken to yield $y(t)$. Example 9.9 shows the steps to solve a second-order differential equation.

☐ **EXAMPLE 9.9**　　*Laplace Solution of Differential Equation*

Consider the second-order equation

$$m\ddot{y}(t) + \beta\dot{y}(t) + ky(t) = f(t), \qquad \textbf{(9.29)}$$

for which it is desired to compute the Laplace transform solution. The equation can be rewritten as

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n{}^2 y(t) = \frac{f(t)}{m} \qquad \textbf{(9.30)}$$

using the substitutions $\omega_n{}^2 = k/m$ and $2\zeta\omega_n = \beta/m$ as was done in Chapter 5. This form of the equation is written in terms of the *natural frequency* of oscillation ω_n that would be obtained if the damping factor β were zero. The factor ζ is a measure of the damping and thus can range from zero in an ideal undamped system to any positive value in a physical system.

Taking the Laplace transform of Equation 9.30 yields the *subsidiary equation* in terms of the variable s as

$$s^2 Y(s) - sy(0) - \dot{y}(0) + 2\zeta\omega_n[sY(s) - y(0)] + \omega_n{}^2 Y(s) = \frac{F(s)}{m}.$$

Solving for $Y(s)$ gives the Laplace transform of $y(t)$ as

$$Y(s) = \frac{F(s)/m}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{(s + 2\zeta\omega_n)y(0) + \dot{y}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \qquad \textbf{(9.31)}$$

As discussed in detail in Chapter 5, the solution to the differential equation consists of the sum of the particular and complementary solution. The first term in Equation 9.31 is the Laplace transform of the solution to the nonhomogeneous differential equation and the second term is the transform of the complementary solution.

As a specific example, let

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 2, \qquad y(0) = \dot{y}(0) = 1.$$

The Laplace solution of the equation is

$$Y(s) = \frac{2/s + (s+3)(1) + 1}{s^2 + 3s + 2} = \frac{s^2 + 4s + 2}{s(s+1)(s+2)}.$$

Taking the inverse transform $y(t)$ as shown in Example 9.3 yields the solution to the differential equation

$$y(t) = 1 + e^{-t} - e^{-2t}.$$

$\square$

Discussion of Laplace Solution   The Laplace transform solution of a linear ordinary differential equation with constant coefficients has the following properties:

1. Solves initial value problems uniquely.

2. Determines both the complementary and particular solution.

WHAT IF?   Considering the method of undetermined coefficients to find particular solutions to differential equations as described in Chapter 5, it might appear that there is no advantage to using the Laplace transform method to determine these solutions. Do you agree? Which method is more general?

Laplace Transfer Functions   Consider the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t). \qquad \textbf{(9.32)}$$

but with zero initial conditions. In that case, taking the Laplace transform of both sides yields an equation of the form

$$(s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0)Y(s) = F(s). \qquad \textbf{(9.33)}$$

If $f(t)$ is considered the *input* function and $y(t)$ the *output* or response, then we define the rational fraction

$$H(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

from Equation 9.33 as the *transfer function* for the system[6]. It is the ratio of the transform of the output function and the transform of the input function, so that

$$H(s) = \frac{Y(s)}{F(s)} = \frac{\mathcal{L}(\text{Output})}{\mathcal{L}(\text{Input})}. \qquad \textbf{(9.34)}$$

Alternatively, the transform of the output can be written as

$$Y(s) = F(s)H(s). \qquad \textbf{(9.35)}$$

---

[6]Chapter 8 presented the equivalent transfer function using Fourier transform techniques

The transform of a linear differential equation with constant coefficients can be written in terms of the transfer function as $F(s)H(s)$ with $F(s)$ being the transform of the input function, plus a term due to the initial conditions. Assuming the initial conditions are zero, the transfer function gives the Laplace model of the system described by the differential equation. From Equation 9.31 in Example 9.9, the transfer function for a second-order system is

$$H(s) = \frac{1/m}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

The transfer function has the properties:

1. The transfer function is a mathematical model of the differential equation describing a system.

2. The transfer function depends only on the system; it is independent of the input function or any initial conditions.

3. It is possible to determine the transfer function of a system experimentally from the input and output functions.

The transfer function will be considered again when we treat applications to linear systems in this chapter.

SPECIAL FUNCTIONS

This section presents the Laplace transform of step, impulse, and periodic functions. Two important functions used to test or analyze linear systems are the *step function* and the *impulse function*. Applying a step function to a system might consist of applying a constant voltage to a circuit or a constant force to a mechanical system. An *impulsive* voltage or force is a relatively large excitation which acts over a short interval of time. A system given a hammer blow experiences an impulsive force. The *impulse* of force is defined in physics as the integral of the force over its duration. This section presents the transforms of these special functions but applications will be presented later.

It is useful to take the Laplace transform of a periodic function since this is a common input function to systems. Chapter 8 introduced the Fourier series representation of periodic signals. Many times the Laplace transform approach to analyze a system is simpler.

Unit step The unit step function or Heaviside function was introduced in Chapter 6 as an example of a forcing function for differential equations. The Laplace transform of the shifted unit step function,

$$U(t - t_0) = \begin{cases} 0, & t < t_0, \\ 1, & t \geq t_0. \end{cases} \tag{9.36}$$

has the form

$$\mathcal{L}[U(t - t_0)] = \frac{e^{-t_0 s}}{s} \tag{9.37}$$

based on Equation 9.5 and the time-shifting Theorem 9.3.

From this result, the pulse of height A from $t = 0$ to $t = t_0$ can be written as

$$P(t) = A(U(t) - U(t - t_0))$$ (9.38)

with the Laplace transform

$$\mathcal{L}[P(t)] = P(s) = A\frac{1 - e^{-t_0 s}}{s}.$$ (9.39)

This function is useful as an input function to a linear system. The pulse with suitable characteristics can also be used to introduce the impulse function.

Impulse function   It is often convenient to define a narrow pulse input to a system. One approach is to consider the pulse of Equation 9.38 defined with an area of unity as the pulse width is decreased to zero. Thus, assume the pulse is defined as

$$P_\epsilon(t) = \frac{1}{\epsilon}(U(t) - U(t - \epsilon))$$ (9.40)

with Laplace transform

$$\mathcal{L}[P_\epsilon(t)] = \frac{1 - e^{-\epsilon s}}{\epsilon s}.$$ (9.41)

Such a pulse is shown in Figure 9.2.

FIGURE 9.2 *(a) Narrow pulse of unit area (b) impulse function*

In the limit as $\epsilon \to 0$, the Laplace transform of Equation 9.38 becomes

$$\lim_{\epsilon \to 0} \frac{1 - e^{-\epsilon s}}{\epsilon s} = \lim_{\epsilon \to 0}\left[1 - \frac{\epsilon s}{2!} + \frac{\epsilon^2 s^2}{3!} - \cdots\right] = 1.$$

In the limit, the pulse $P_\epsilon(t)$ is denoted $\delta(t)$ and called the *Impulse function* or the *Dirac delta function*.[7] Considering the relationship between $\delta(t)$ and the Laplace transform, we define the transform pair

$$\delta(t) \Longleftrightarrow 1.$$ (9.42)

---

[7]More formally, the impulse function is described as the *Dirac delta function*, after the famous physicist Paul Dirac who used the function in advanced physics problems.

Generalized Functions. Strictly speaking, $\delta(t)$ is not a function in the ordinary sense. It is in a class of functions called *generalized functions* or *distributions*. A definition is

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0, \qquad \int_{-\infty}^{\infty} \delta(t) \, dt = 1 \qquad \textbf{(9.43)}$$

with the *sifting* property

$$\int_{-\infty}^{\infty} \delta(t) f(t) \, dt = f(0). \qquad \textbf{(9.44)}$$

An equivalence useful in the solution of differential equations is that

$$f(t)\delta(t) = f(0)\delta(t) \qquad \textbf{(9.45)}$$

provided that $f(t)$ is continuous at $t = 0$.

The textbooks by Kaplan and Lighthill listed in the Annotated Bibliography for this chapter treat generalized functions in some detail. We will accept the properties of the delta function and its usefulness in Laplace analysis.

Even though the function $\delta(t)$ has the strange property that it is zero everywhere except at the point zero, it is useful in studying the properties of linear systems. Also, since the unit step transform pair is

$$U(t) \Longleftrightarrow \frac{1}{s},$$

it is reasonable to associate the integral of the impulse function with the unit step function.

Periodic Signal   Consider a periodic function as shown in Figure 9.3.

FIGURE 9.3 *Periodic function*

The equation for the function can be written

$$f(t) = f_T(t)U(t), \quad f(t + T) = f(t), \qquad \textbf{(9.46)}$$

where the period is a constant $T > 0$. This periodic function is defined for all positive t and is zero for $t < 0$. Let $f(t)$ be piecewise continuous

so the Laplace transform of the function exists and we can compute the transform as the sum of transforms over successive periods as follows:

$$
\begin{aligned}
\mathcal{L}[f(t)] &= \int_0^\infty e^{-st} f(t)\, dt \\
&= \int_0^T e^{-st} f(t)\, dt + \int_T^{2T} e^{-st} f(t)\, dt + \int_{2T}^{3T} e^{-st} f(t)\, dt + \cdots.
\end{aligned}
$$

Setting $t = \tau + (n-1)T$ in the nth integral in the sum and recognizing that

$$
\begin{aligned}
f(\tau) &= f(\tau + T) = f(\tau + 2T) \\
&= \cdots = f(\tau + nT) = \cdots
\end{aligned}
$$

leads to the result

$$
\begin{aligned}
\mathcal{L}[f_T(t)U(t)] &= \int_0^T e^{-s\tau} f(\tau)\, d\tau + \int_0^T e^{-s(\tau+T)} f(\tau)\, d\tau \\
&\quad + \int_0^T e^{-s(\tau+2T)} f(\tau)\, d\tau + \cdots.
\end{aligned}
$$

Taking out the factors that do not depend on τ from under the integral sign and setting the dummy variable of integration τ to t yields

$$
\mathcal{L}[f_T(t)U(t)] = \left[1 + e^{-sT} + e^{-2sT} + \cdots \right] \int_0^T e^{-st} f(t)\, dt. \qquad \textbf{(9.47)}
$$

The series in brackets is a geometric series as described in Chapter 6 with the sum $1/(1 - e^{-sT})$. Thus, the Laplace transform of a piecewise continuous periodic function with period T is

$$
\frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} f(t)\, dt \qquad \textbf{(9.48)}
$$

Laplace Transforms of Special Functions   Table 9.6 summarizes the transforms of the special functions treated in this section. These functions are used in examples in following sections of this chapter and in the problems.

TABLE 9.6 *Laplace Transforms of Special Functions*

Function	$f(t)$	$F(s)$
Unit Step	$U(t)$	$\dfrac{1}{s}$
Impulse	$\delta(t)$	1
Periodic	$f(t) = f(t+T)$	$\dfrac{1}{1 - e^{-Ts}} \displaystyle\int_0^T e^{-st} f(t)\, dt$

Differential Equations with Special Inputs

Consider the differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = f(t), \qquad y(0) = \dot{y}(0) = 0. \qquad \textbf{(9.49)}$$

We wish to find the Laplace transform method solution of the equation for the inputs:

   a. $f(t) = \delta(t)$

   b. $f(t) = U(t)$

The Laplace transform of the differential equation with zero initial condition is

$$Y(s) = \frac{F(s)}{s^2 + 3s + 2}. \qquad \textbf{(9.50)}$$

   a. The response to the impulse input is determined by setting $F(s) = 1$ in Equation 9.50 and inverting $Y(s) = 1/(s^2 + 3s + 2)$ to find

$$y(t) = e^{-t} - e^{-2t} \qquad t \geq 0.$$

   b. The response to a unit step is found by setting $F(s) = 1/s$ in Equation 9.50 and inverting the transform so that

$$y(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \qquad t \geq 0.$$

Figure 9.4 shows the results for comparison.

FIGURE 9.4 *Impulse and step response of the second-order differential equation of Example 9.10*

□

W H A T I F ?　Suppose you wish to determine the impulse response from the step function response for the differential equation in Example 9.10. What is the relationship between the two solutions?