Fourier Series of a Pulse Train

Trig and Exponential Forms

FOURIER SERIES FORMULA

$\frac{a_0}{2}$

The Fourier series approximates a function $f(t)$ by using a *trigonometric polynomial* of degree N as follows:

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(nt) + b_n \sin(nt)] = s_N(t), \tag{8.1}$$

where $s_N(t)$ denotes the nth partial sum. Assuming that $f(t)$ is continuous on the interval $-\pi \leq t \leq \pi$, the coefficients a_n and b_n can be computed by the formulas

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt \tag{8.2}$$

for the constant term and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt)\, dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt)\, dt, \tag{8.3}$$

for $n = 1, 2, \ldots, N$. If $f(t)$ is continuous on the interval and the derivative of $f(t)$ exists, the series converges to $f(t)$ at the point t when $N \to \infty$.

Recall that in Chapter 2, the *inner product* of the trigonometric functions on the interval $[-\pi, \pi]$ was defined as

$$\langle f, g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt, \tag{8.9}$$

where the factor $1/\pi$ was introduced to normalize the inner product for the Fourier trigonometric functions.

Then, the trigonometric terms in the Fourier series consist of functions that form an orthonormal set, since for integers k and m

$$\langle \cos(kt), \cos(mt)\rangle = \begin{cases} 1, & k = m \neq 0, \\ 0, & k \neq m, \end{cases}$$

$$\langle \sin(kt), \sin(mt)\rangle = \begin{cases} 1, & k = m \neq 0, \\ 0, & k \neq m, \end{cases}$$

$$\langle \cos(kt), \sin(mt)\rangle = 0, \quad \text{for all } k, m. \tag{8.10}$$

Thus, the Fourier coefficients in the expansion of a function $f(t)$ from Equation 8.3 can be written as

$$\begin{aligned} a_k &= \langle f(t), \cos(kt) \rangle, \qquad k = 0, 1, \ldots, \\ b_k &= \langle f(t), \sin(kt) \rangle, \qquad k = 1, 2, \ldots. \end{aligned} \tag{8.11}$$

Notice that the constant term a_0 is computed as

$$a_0 = \langle f(t), 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, dt, \tag{8.12}$$

which is the inner product of $f(t)$ and the $\cos(kt)$ term in Equation 8.11 for $k = 0$.

How to Compute a FOURIER SERIES // Formulas & Full Example

27,635 views May 6, 2021 13:15

https://www.youtube.com/watch?v=ijQaTAT3kOg

How do you actually compute a Fourier Series? In this video I walk through all the big formulas needed to compute the coefficients in a Fourier Series. First we see three integrals that will really make everything easier, integrals of products of sin and cos terms of different frequencies. Then we will use these to compute out formulas for the coefficients of the in and cos terms.

Dr. Trefor Bazett

175K subscribers

[-T/2, T/2] INTERVAL

On the interval $[-T/2, T/2]$, the limits of integration for the Fourier series can be changed from $[-\pi, \pi]$ by assigning to the integration variable t the value $2\pi t/T$. The period of the function is thus T.

Assuming that $f(t)$ is continuous on the interval $-T/2 \le t \le T/2$, the coefficients a_n and b_n can be computed by the formulas

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt, \\ a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2n\pi t}{T}\right)\,dt, \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2n\pi t}{T}\right)\,dt, \end{aligned} \tag{8.20}$$

where $n = 1, 2, \ldots$ is any positive integer.

The Fourier series on the interval $[-T/2, T/2]$ is thus

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2n\pi t}{T}\right) + b_n\sin\left(\frac{2n\pi t}{T}\right)\right]. \tag{8.21}$$

Frequency Components Assuming the variable t represents time, the function $f(t)$ repeats every T seconds. The *frequency* associated with the fundamental sinusoid in the series of Equation 8.21 is $f_0 = 1/T$, measured in cycles per second, or hertz. The parameter

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

is the frequency in radians per second.

Since $2n\pi/T = 2n\pi f_0 = n\omega_0$, the series in Equation 8.21 can be written

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)], \end{aligned} \tag{8.22}$$

which emphasizes the components in terms of their frequencies.

☐ **EXAMPLE 8.7** ***Fourier spectrum***

Consider the even, periodic pulse train in Figure 8.6. This is an important test signal in electronics and the signal is also of interest because of the characteristics of its Fourier components. The period is T, the amplitude is A, and the pulse has duration τ in each period. The function is an even function with average value over a period of $A\tau/T$.

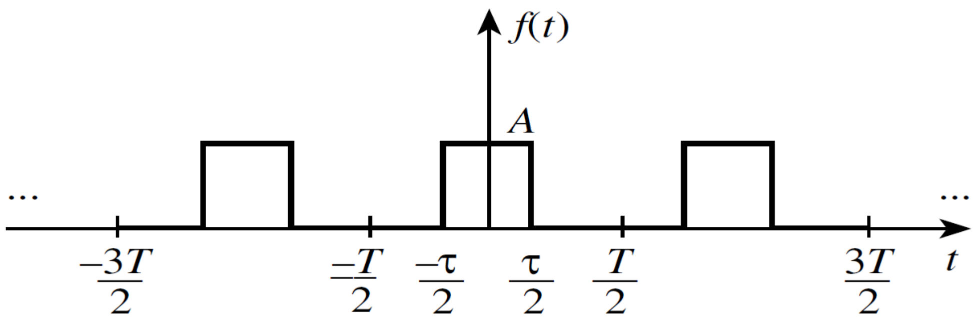


FIGURE 8.6 *Periodic train of rectangular pulses*

Letting $\omega_0 = 2\pi/T$, the coefficients of the complex series are

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} Ae^{-in\omega_0 t}\,dt = \frac{1}{T}\int_{-\tau/2}^{\tau/2} Ae^{-in\omega_0 t}\,dt.$$

Integrating and substituting $-2\sin(n\omega_0\tau/2)$ for the resulting exponentials and then multiplying and dividing by the term $\omega_0\tau/2$ yields

$$\alpha_n = \frac{A\tau}{T}\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}.$$

The coefficient α_0 is determined as $A\tau/T$ by l'Hôpital's rule. Notice that the coefficient

$$\alpha_0 = \frac{A\tau}{T} = \frac{\text{area of pulse}}{\text{period}}.$$

Defining the function

$$\operatorname{sinc} x \equiv \frac{\sin x}{x} \tag{8.36}$$

with $x = n\omega_0\tau/2$ leads to the series

$$f(t) = \frac{A\tau}{T} + 2\frac{A\tau}{T}\sum_{n=1}^{\infty} \operatorname{sinc}(n\omega_0\tau/2)\cos(n\omega_0 t)$$

since the coefficients of the cosine series are twice the values of those in the complex series.

Comment: The sinc function is frequently defined as

$$\operatorname{sinc} t = \frac{\sin \pi t}{\pi t}.$$

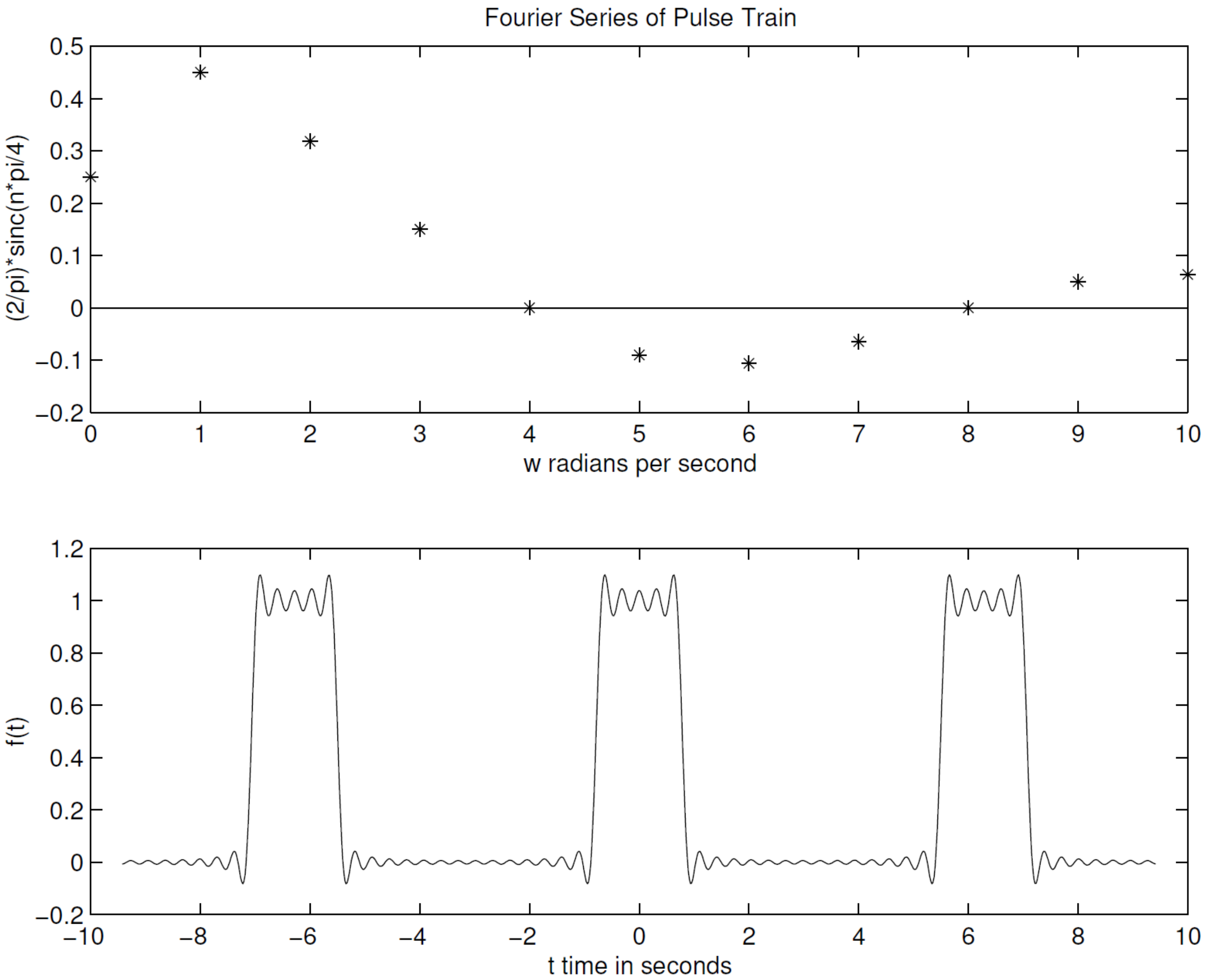


FIGURE 8.7 *Spectrum and approximation for periodic pulse train of Example 8.7*

FourierCh8_TLH   SEE PAGE 392 FOR MATLAB CODE

MATLAB_Fourier_EvenPulseTrain

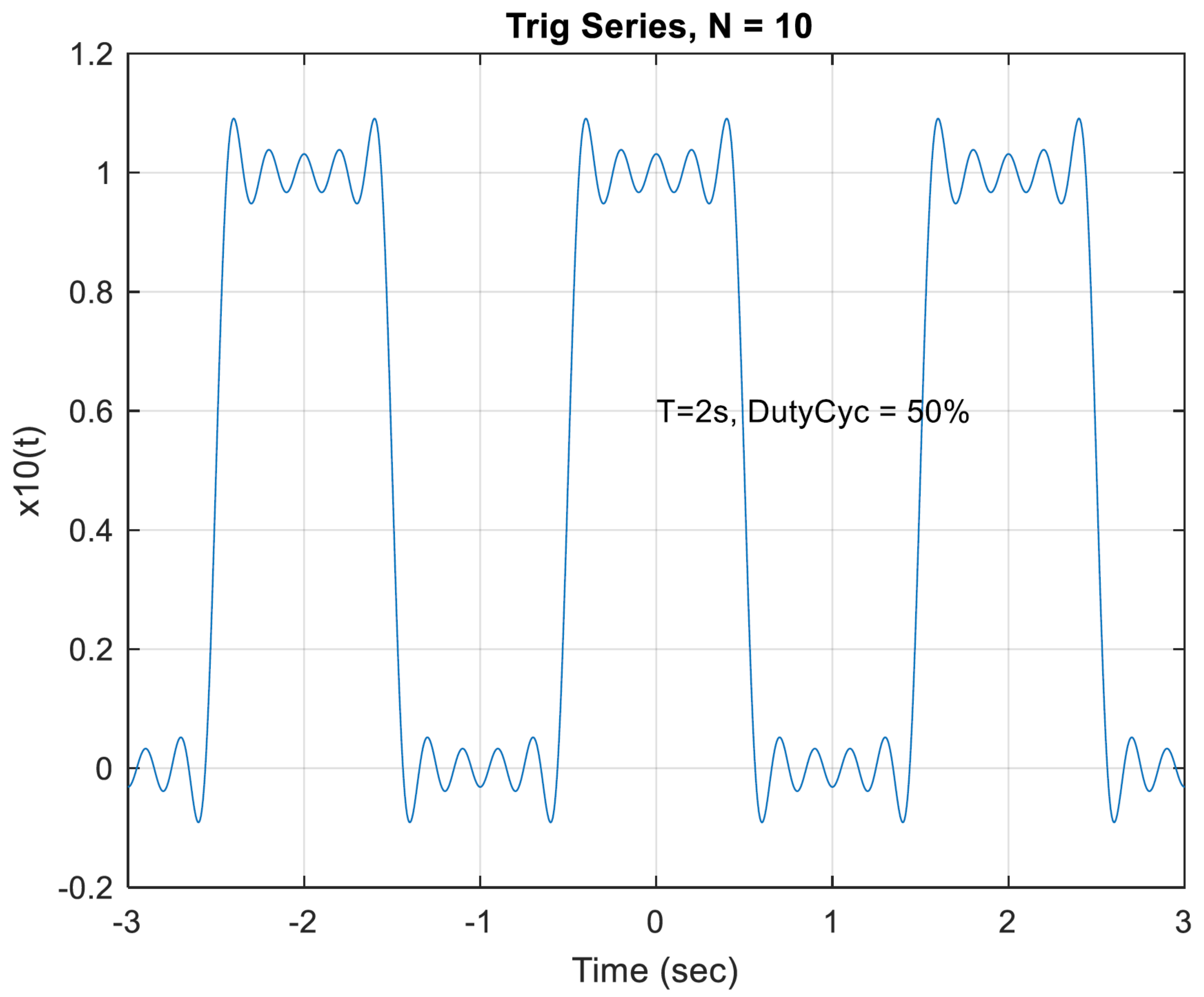
PulseTrain Computes trigonometric Fourier series for a Pulse Train with

Table of Contents

Period T=2 seconds so w = pi rad/sec for this even function.

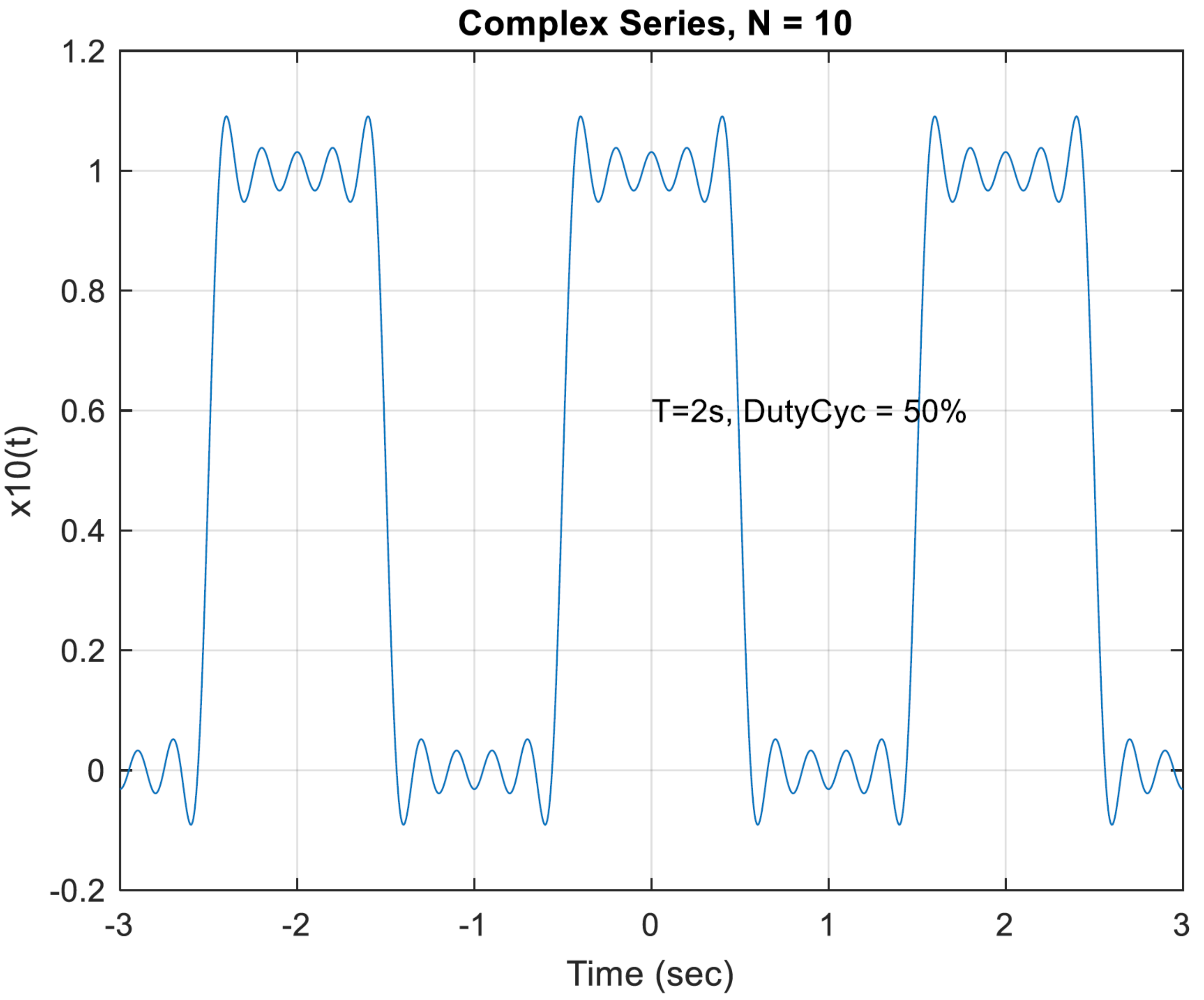
A= 1 and width tau = 1 so tau/T = 0.5; Duty cycle = 0.5

```
clear all,clf
A = 1.0; %Amplitude
t = -3:6/1000:3; % Plot 6 sec, 3 cycles
N = input('Number of harmonics ');
c0 = A* 0.5; % Duty cycle * Amplitude
w0 = pi;  % f = 1/2; T=2
xN = c0*ones(1,length(t));  % dc component added to each term
for k=1:2:N,                % Even harmonics are zero -ignore them.
   xN = xN + (2/k/pi)*sin(k*pi/2)*cos(k*w0*t); % Sum the components
end
```

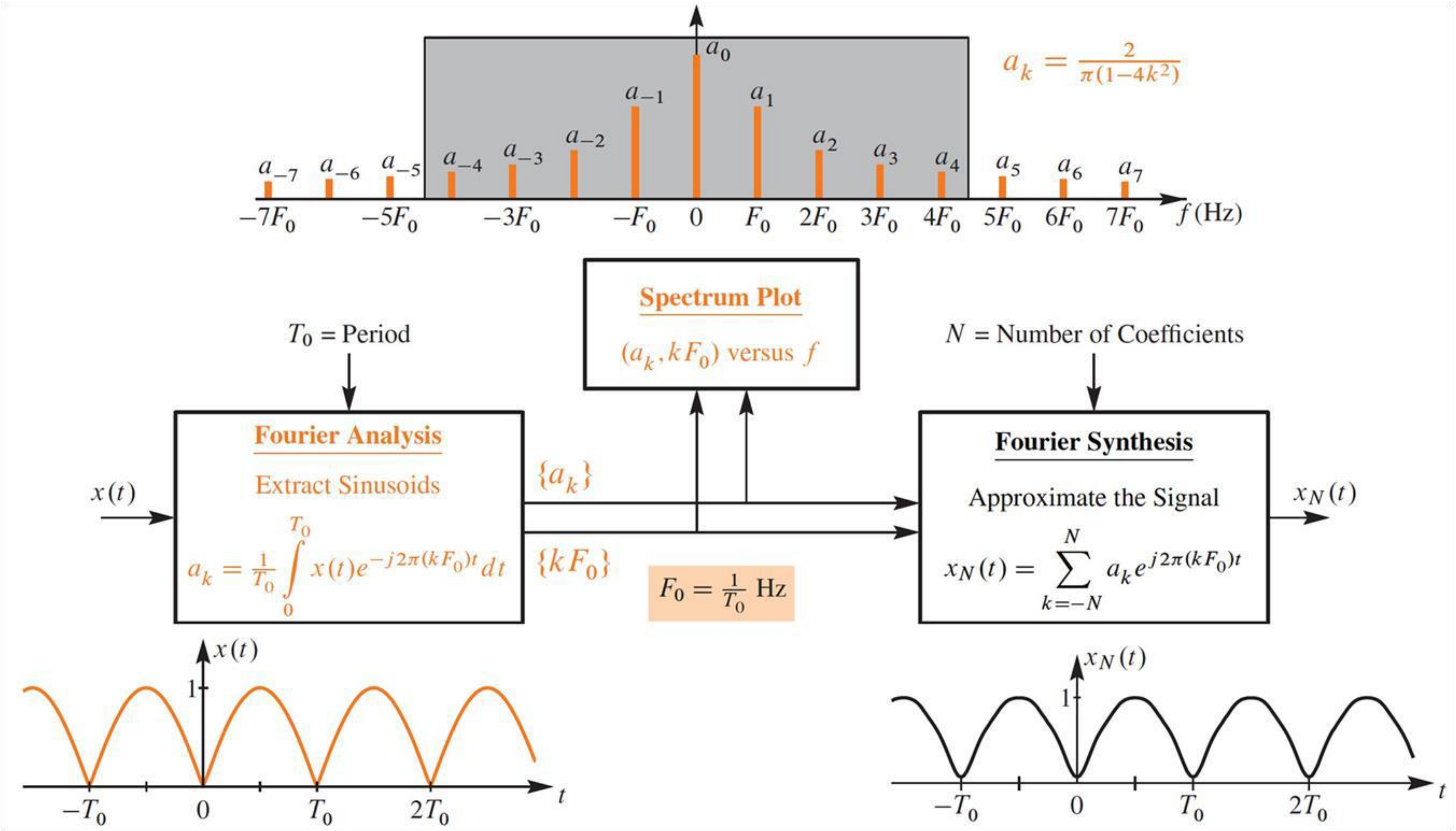

Trig Series, N = 10
T=2s, DutyCyc = 50%
x10(t)
Time (sec)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-3
-2
-1
0
1
2
3

Used to compare results from Trig series and Exponential

```
t = -3:6/1000:3;
% Use N harmonics input
c0 = 0.5;
w0 = pi;
xN = c0*ones(1,length(t));    % dc component
for k=1:N,
  ck = 1/k/pi*sin(k*pi/2);
  c_k = ck;
  xN = xN + ck*exp(j*k*w0*t) + c_k*exp(-j*k*w0*t);
end
```


Complex Series, N = 10
T=2s, DutyCyc = 50%
x10(t)
Time (sec)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-3
-2
-1
0
1
2
3

Fourier Series Synthesis

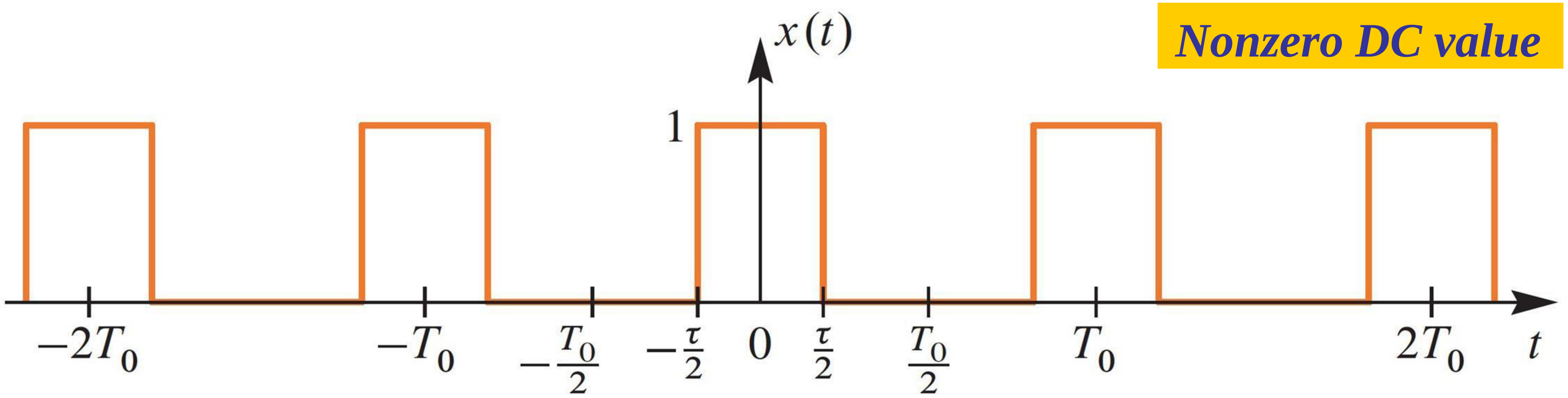


PULSE WAVE SIGNAL GENERAL FORM

Defined over one period

$$x(t)=\begin{cases}1 & 0\le |t| < \tau/2 \\ 0 & \tau/2 \le |t| \le T_0/2\end{cases}$$

Nonzero DC value



Pulse Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-j(2\pi/T_0)(k)t}\,dt$$

General Pulse Wave

$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau/2 \\ 0 & \tau/2 \le |t| \le T_0/2 \end{cases}$$

$$a_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1 e^{-j(2\pi/T_0)kt}\,dt$$

$$= \left(\tfrac{1}{T_0}\right) \frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k}\Bigg|_{-\tau/2}^{\tau/2} = \frac{e^{-j(2\pi/T_0)k(\tau/2)} - e^{-j(2\pi/T_0)k(-\tau/2)}}{-j(2\pi)k}$$

$$= \frac{e^{j(\pi/T_0)k(\tau)} - e^{-j(\pi/T_0)k(\tau)}}{(j2)\pi k} = \frac{\sin(\pi k\tau/T_0)}{\pi k}$$

Pulse Wave

$\{a_k\}$ = sinc

Pulse Wave

$$a_k = \frac{\sin(\pi k \tau / T_0)}{\pi k} \qquad k = 0, \pm 1, \pm 2, \ldots$$

Double check the DC coefficient:

$$a_0 = \tfrac{1}{T_0} \int_{-\tau/2}^{\tau/2} 1 e^{-j(2\pi/T_0)(0)t} dt$$

$$= \tfrac{1}{T_0} \int_{-\tau/2}^{\tau/2} 1 dt = \tfrac{1}{T_0} \left[\tfrac{\tau}{2} - \tfrac{-\tau}{2} \right] = \tfrac{\tau}{T_0}$$

$$\text{Note, } \lim_{k \to 0} \frac{\sin(\pi k \tau / T_0)}{\pi k} \to \frac{\tau}{T_0}$$

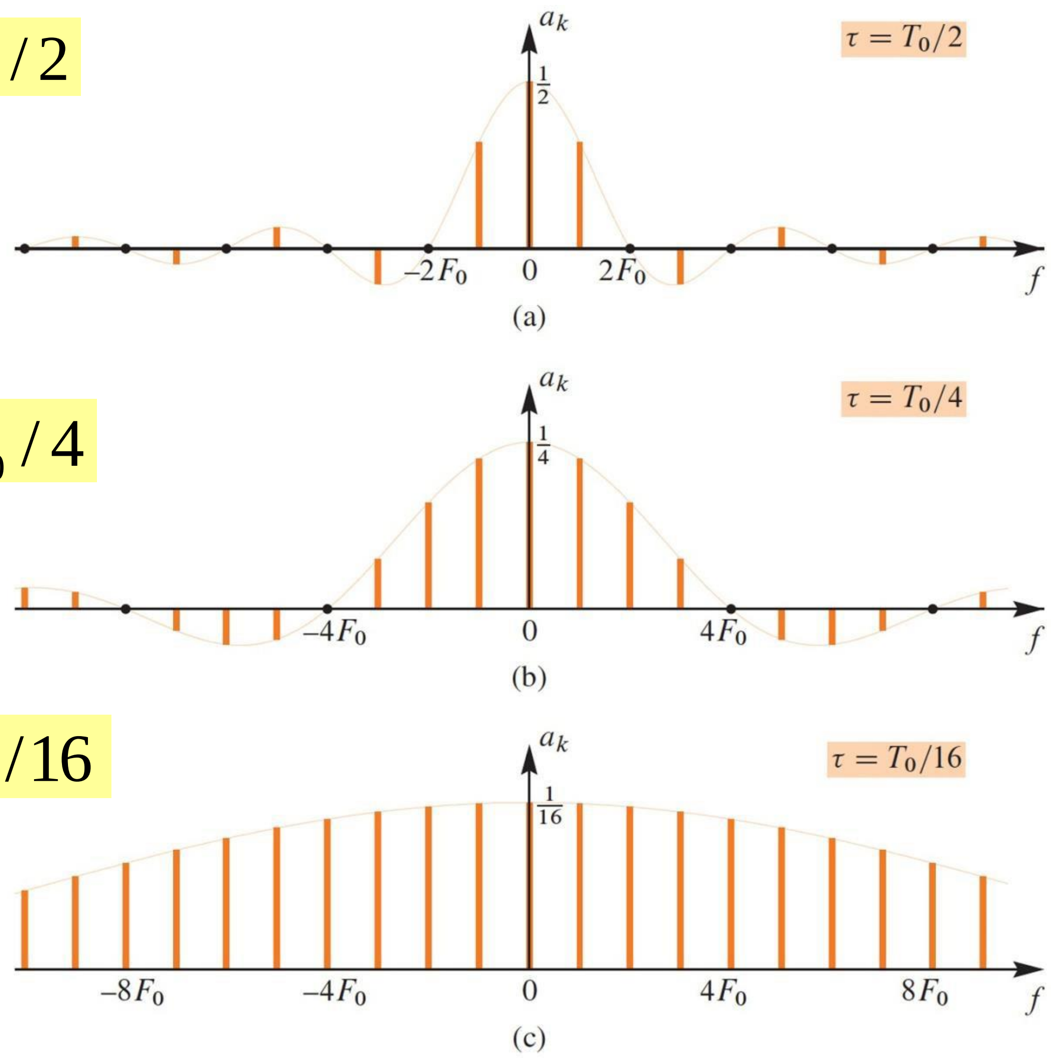
Where do you go if sick?

PULSE WAVE SPECTRA

$\tau = T_0 / 2$

$\tau = T_0 / 4$

$\tau = T_0 / 16$



50% duty-cycle (Square) Wave

$$\tau = T_0/2 \;\Rightarrow a_k = \frac{\sin(\pi k(T_0/2)/T_0)}{\pi k} = \frac{\sin(\pi k/2)}{\pi k} \qquad k = 0, \pm 1, \pm 2, \ldots$$

- Thus, a_k=0 when k is odd
  - Phase is zero because x(t) is centered at t=0
    - different from a previous case

PulseWave starting at t=0

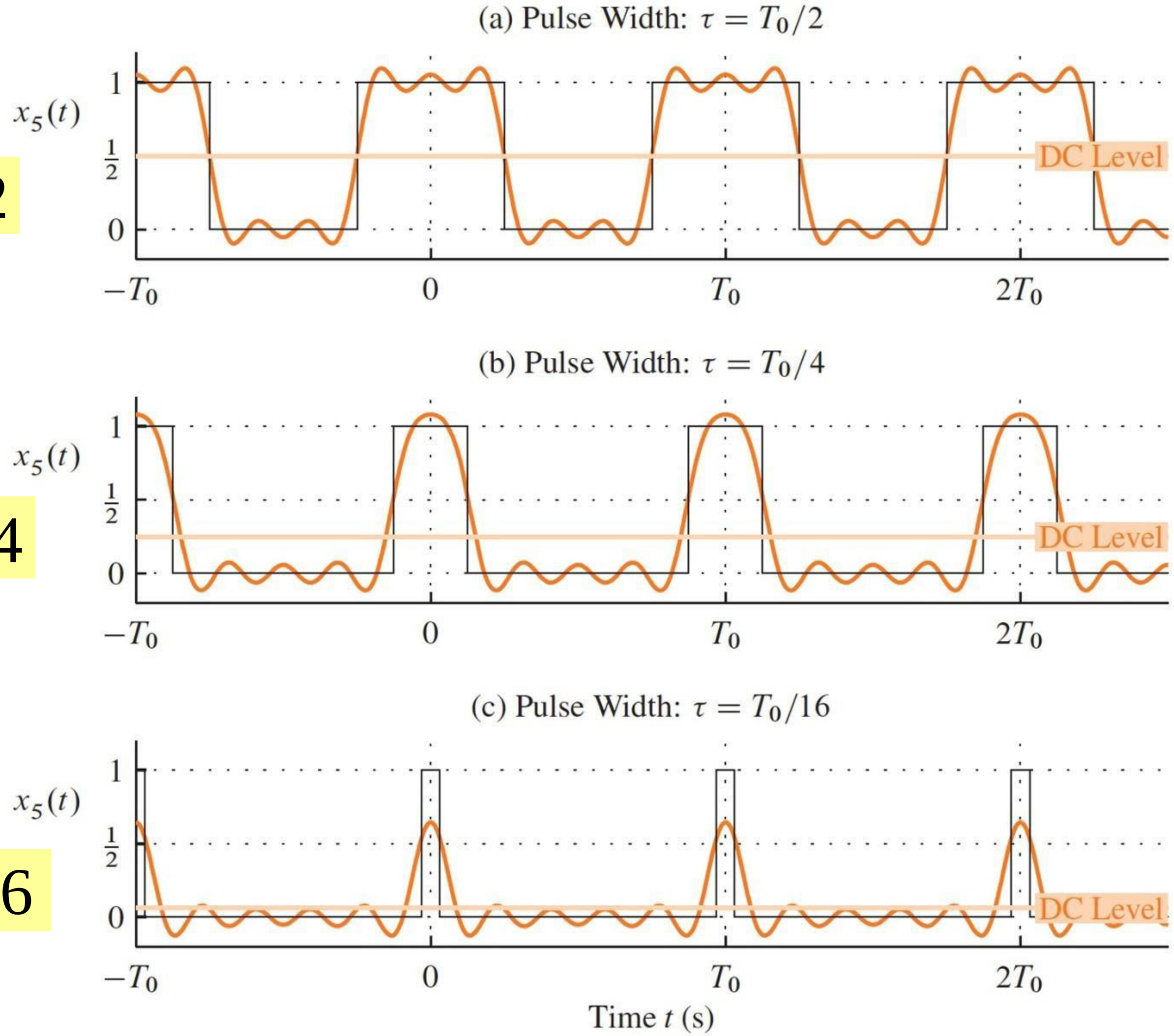
$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau \\ 0 & \tau \le |t| \le T_0 \end{cases} \;\leftrightarrow\; a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

PULSE WAVE SYNTHESIS
with first 5 Harmonics

$\tau = T_0 / 2$

$\tau = T_0 / 4$

$\tau = T_0 / 16$

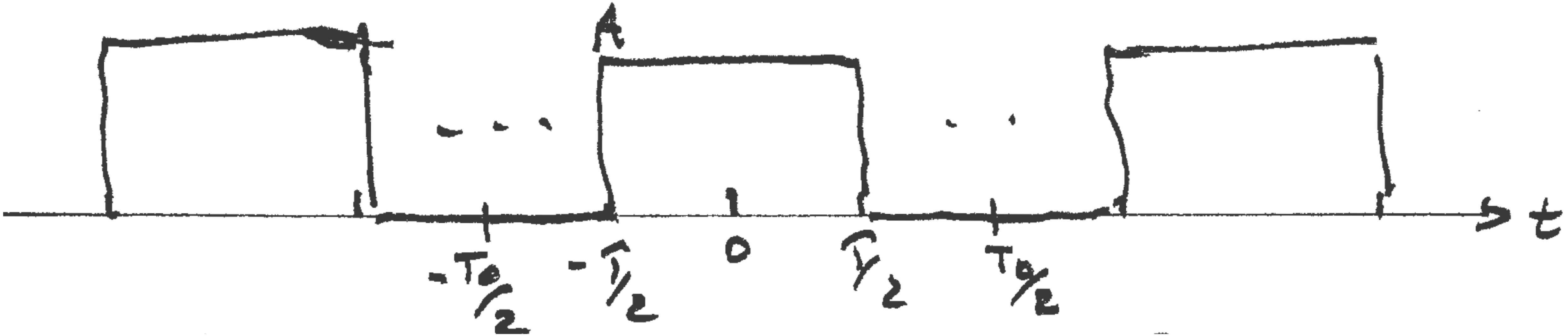


FourierProblemSession_EvenPulseTrain

PULSE WAVE

Lecture 3 ~~5~~ 2 slide

$x(t) = x(t+T)$



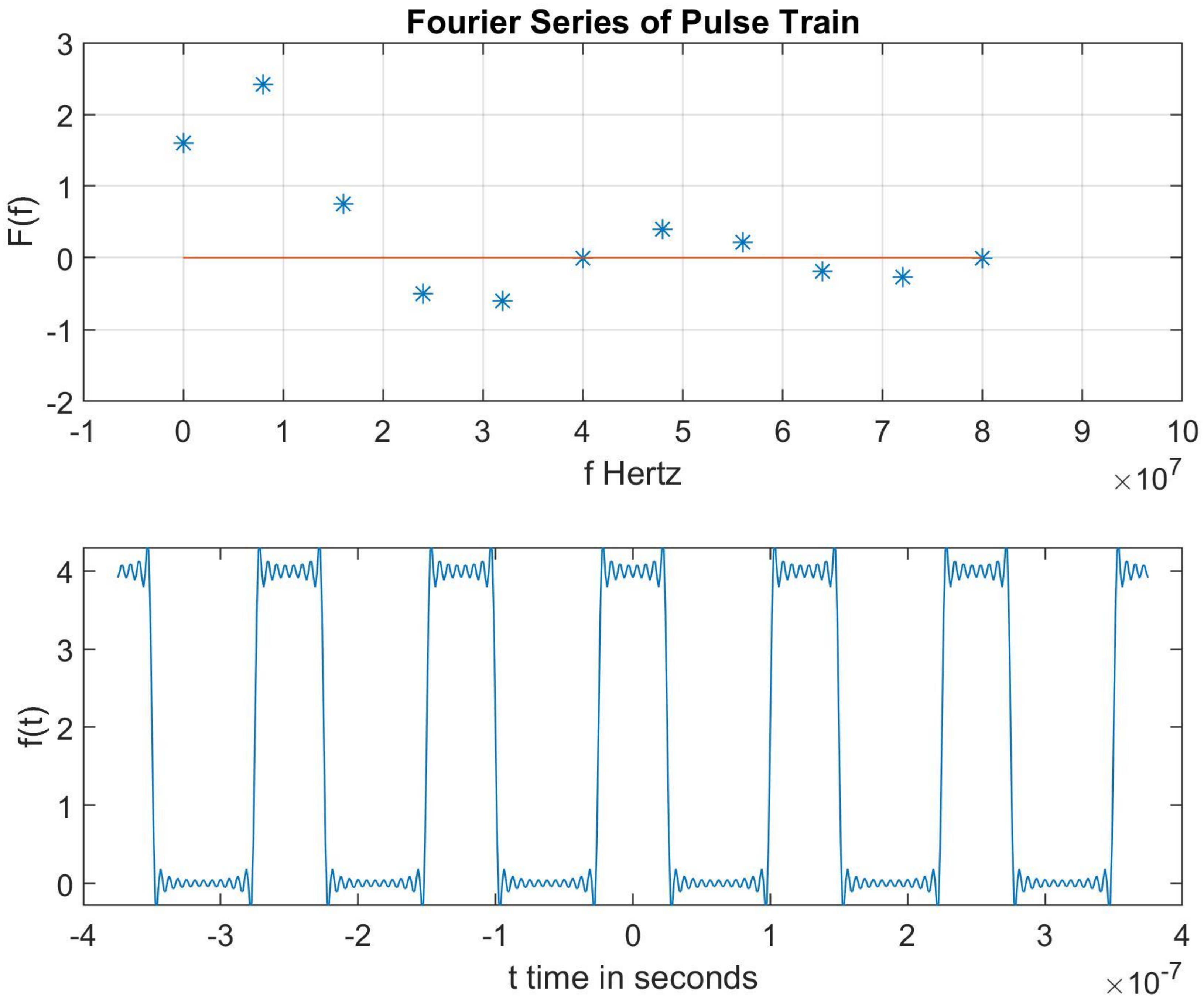
HOMEWORK HELP

Problem 4 30

Fourier series of clock signal Consider the computer clock signal shown in the Figure, with a pulse rate of 8 million pulses per second ($f_c = 8$ Megahertz) and amplitude of 4 volts and a pulse width of 0.05 microseconds. NOTE: The figure does not show the signal to scale.

1. Find the Fourier series by hand calculation using the basic definitions of the coefficients.

BONUS POINTS 20 See MATLAB_Fourier_Even_PulseTrain on our website for help.


Fourier Series of Pulse Train
F(f)
f Hertz
×10^7
f(t)
t time in seconds
×10^-7