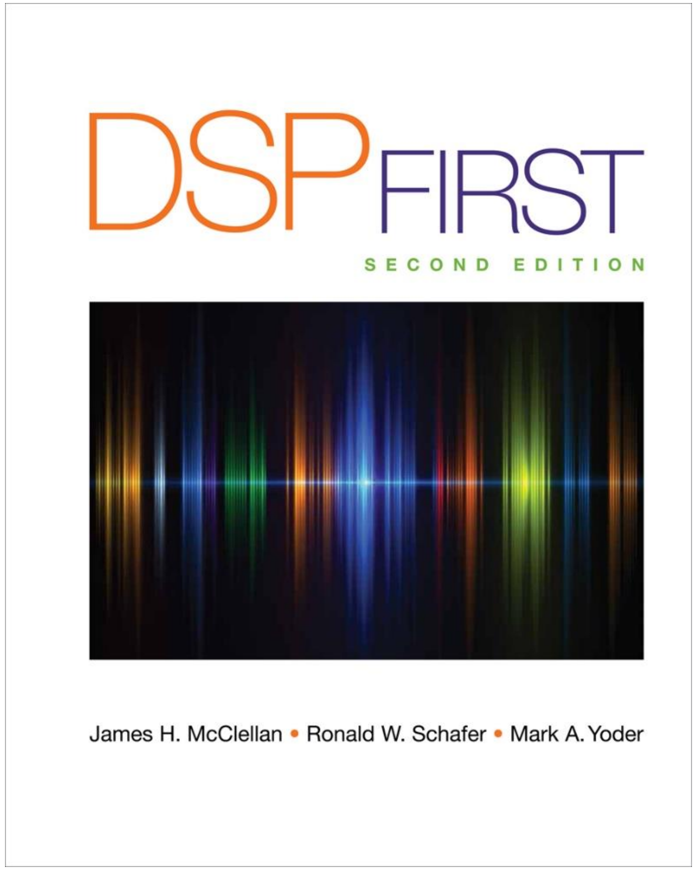
CENG 3315 REVIEW FOR QUIZ 1

February 21, 2022

Review of Jan 19 Lecture



ENG 3315 Spring 2022

Table 2-1: Basic Properties of the Sine and Cosine Functions

Property	Equation
Equivalence	$\sin\theta = \cos(\theta - \pi/2)$ or $\cos(\theta) = \sin(\theta + \pi/2)$
Periodicity	$\cos(\theta + 2\pi k) = \cos\theta$, when k is an integer
Evenness of cosine	$\cos(-\theta) = \cos\theta$
Oddness of sine	$\sin(-\theta) = -\sin\theta$
Zeros of sine	$\sin(\pi k) = 0$, when k is an integer
Ones of cosine	$\cos(2\pi k) = 1$, when k is an integer
Minus ones of cosine	$\cos[2\pi(k + \frac{1}{2})] = -1$, when k is an integer

Page 13

Table 2-2: Some Basic Trigonometric Identities

Page 14

Number	Equation
1	$\sin^2\theta + \cos^2\theta = 1$
2	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
3	$\sin 2\theta = 2\sin\theta\cos\theta$
4	$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$
5	$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

SINUSOIDAL SIGNAL

$$A\cos(\omega t + \varphi)$$

- **FREQUENCY** ω
  - Radians/sec
  - Hertz (cycles/sec)

$$\omega = (2\pi) f$$

- **PERIOD** (in sec)

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

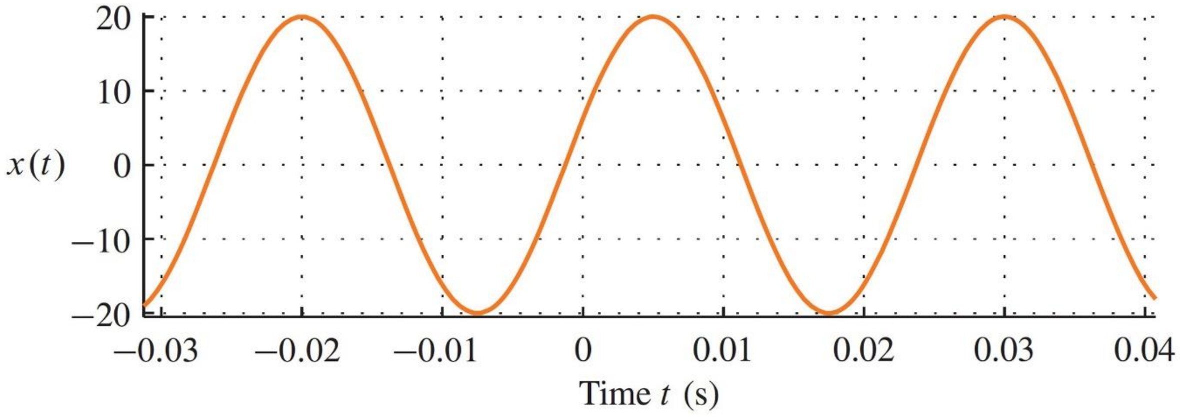
- **AMPLITUDE** A
  - Magnitude

- **PHASE** φ

Relation of Frequency to Period (1 of 2)

Time-Domain versus Frequency-Domain

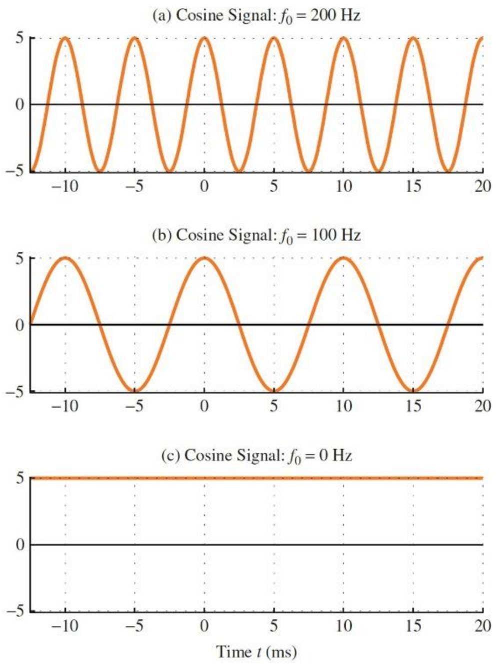
Figure 2-6: Sinusoidal signal with parameters *A* = 20, $\Omega_0 = 2\pi(40), F_0 = 40 \text{ Hz, and}$ $\phi = -0.4\pi rad$.



Relation of Frequency to Period (2 of 2)

Figure 2-7: Cosine Signals $X(t) = 5Cos(2\pi f_0 t)$ for Several Values of $F_0 : (A) F_0 = 200 \text{ Hz}; (B) F_0 = 100 \text{ Hz}; (C) F_0 = 0$

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$



$$\omega = 2\pi f$$

TIME-SHIFT

- In a mathematical formula we can replace t with $t-t_m$

$$x(t-t_m) = A\cos(\omega(t-t_m))$$

- Thus the $t=0$ point moves to $t=t_m$

- Peak value of $\cos(\omega(t-t_m))$ is now at $t=t_m$

PHASE ←→ TIME-SHIFT

- Equate the formulas:

$$A\cos(\omega(t - t_m)) = A\cos(\omega t + \varphi)$$

- and we obtain:

$$-\omega t_m = \varphi$$

- or,

$$t_m = -\frac{\varphi}{\omega}$$

PLOTTING COSINE SIGNAL from the FORMULA

$$5\cos(\underline{0.3}\pi t + 1.2\pi)$$

- Determine **period**:

$$T = 2\pi / \omega = 2\pi / 0.3\pi = 20/3$$

- Determine a **peak** location by solving

$$(\omega t + \varphi) = 0$$

$$0.3\pi t + 1.2\pi = 0$$

- **Peak at t=-4**

Time shift -4 sec

$$5\cos(0.3\pi t + 1.2\pi)$$

$$5\cos(0.3\,\pi\,[t+4])$$

MODIFIED TLH

DSP-First, 2/e

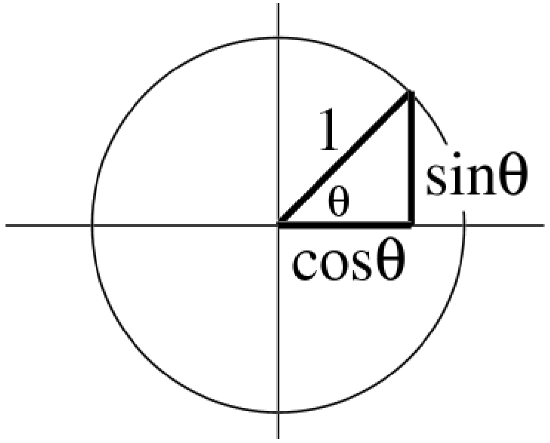
LECTURE 4 # Ch2

Phasor Addition Theorem

ADDING PHASORS WITH THE SAME FREQUENCY

Euler's FORMULA

- <u>**Complex Exponential**</u>
  - <u>**Real part is cosine**</u>
  - Imaginary part is sine
  - Magnitude is one



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

POP QUIZ: Complex Amp

- Find the COMPLEX AMPLITUDE for:

$$x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- Use EULER's FORMULA:

$$x(t) = \Re\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\}$$

$$= \Re\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\}$$

$$X = \sqrt{3}e^{j0.5\pi}$$

POP QUIZ-2: Complex Amp

- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is:

$$X = \sqrt{3} + j3$$

- Convert ***X*** to **POLAR**:

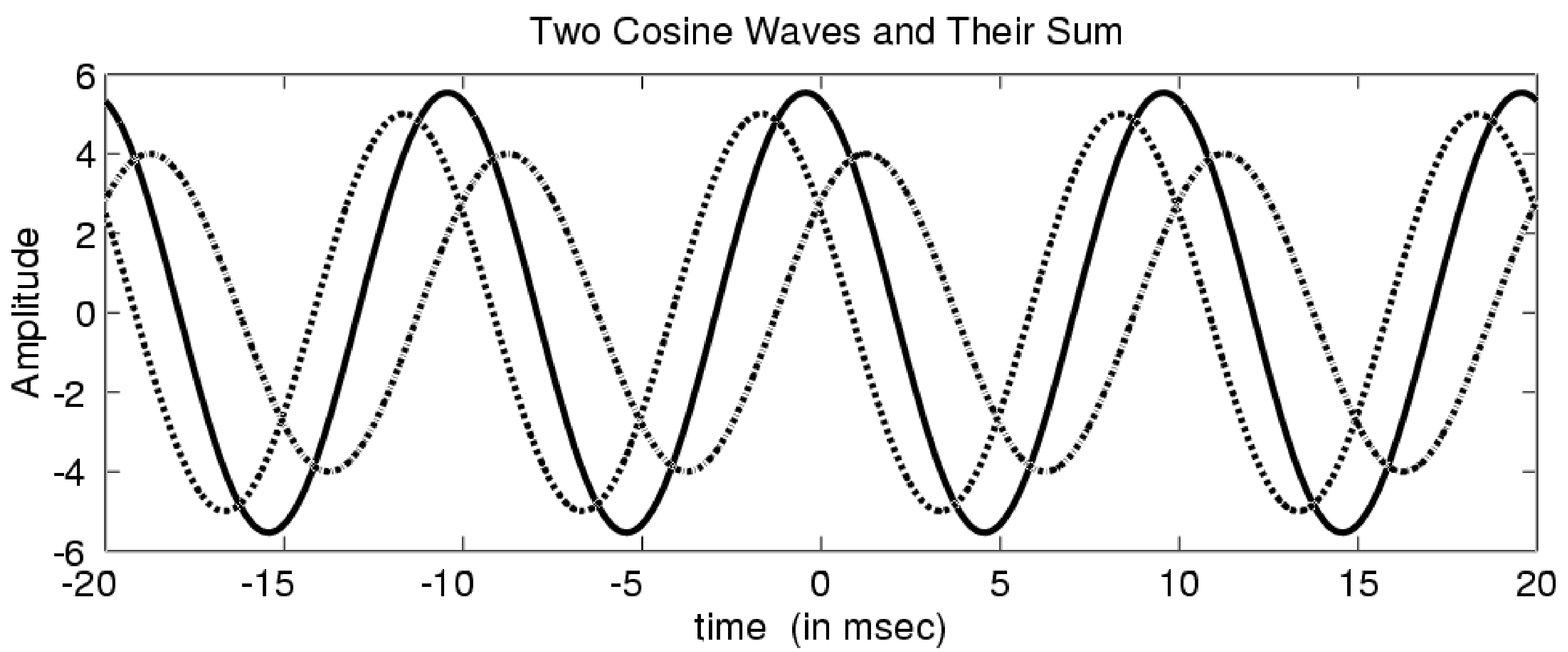
$$x(t) = \Re\{(\sqrt{3} + j3)e^{j(120\pi t)}\}$$

$$= \Re\{\sqrt{12}\, e^{j\pi/3} e^{j120\pi t}\}$$

$$\Rightarrow x(t) = \sqrt{12}\cos(120\pi t + \pi/3)$$

WANT to ADD SINUSOIDS

- **Main point to remember**: Adding sinusoids of common frequency results in sinusoid with **SAME** frequency



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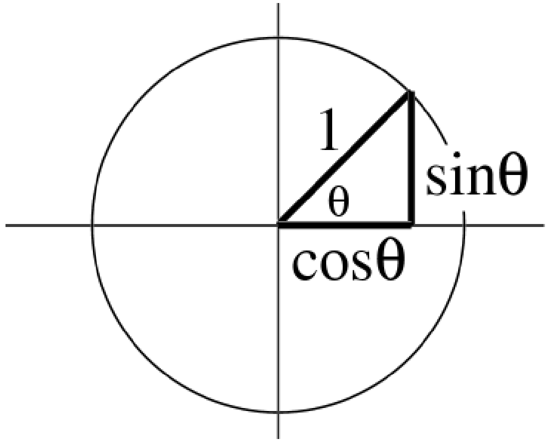
LECTURE # CH2-3
Complex Exponentials
& Complex Numbers

Euler's FORMULA

- **Complex Exponential**
  - **Real part is cosine**
  - Imaginary part is sine
  - Magnitude is one



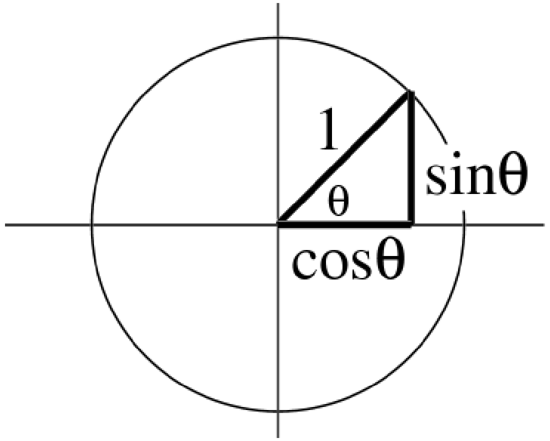
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

ROOTS OF UNITY

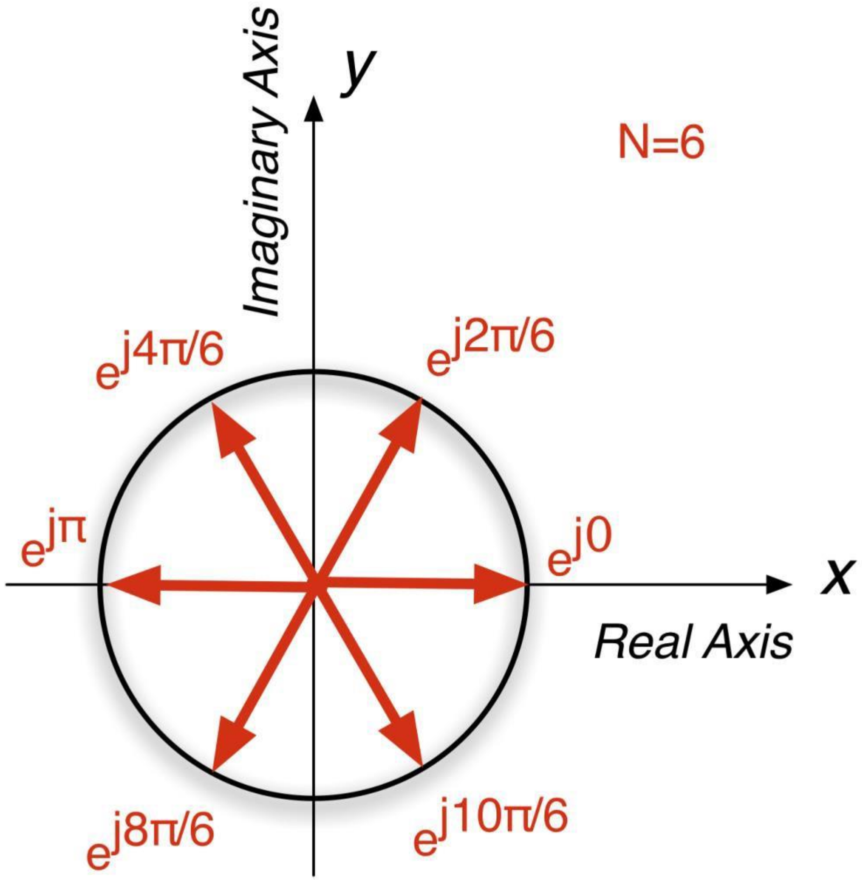
- We often have to solve $z^N=1$
- How many solutions?

$$z^N = r^N e^{jN\theta} = 1 = e^{j2\nu k}$$

$$\Rightarrow r = 1, \quad N\theta = 2\nu k \Rightarrow \theta = \frac{2\nu k}{N}$$

$$z = e^{j2\nu k/N}, \;\; k = 0, 1, 2, \ldots N-1$$

ROOTS OF UNITY for N=6



- Solutions to $z^N=1$ are N equally spaced vectors on the unit circle!
- What happens if we take the sum of all of them?

POP QUIZ-2: Complex Amp

- Determine the 60-Hz sinusoid whose COMPLEX AMPLITUDE is: $X = \sqrt{3} + j3$

- Convert ***X*** to **POLAR**:

$$x(t) = \Re\{(\sqrt{3} + j3)e^{j(120\nu t)}\}$$
$$= \Re\{\sqrt{12}\, e^{j\nu/3} e^{j120\nu t}\}$$

$$\Rightarrow x(t) = \sqrt{12}\cos(120\nu t + \nu/3)$$

Note atan $(3/\sqrt{3})$ –is atan$(\sqrt{3}) = \pi/3$

Remember 60 degree angle

Cos = 1/2 Sin = $\sqrt{3}/2$

PROBLEM SESSION 1

EXAMPLE 2-1 Plotting Sinusoids

0.2 × 2π

Figure 2-6 shows a plot of the signal

$$x(t) = 20\cos(2\pi(40)t - 0.4\pi) \tag{2.3}$$

In terms of our definitions, the signal parameters are $A = 20$, $\omega_0 = 2\pi(40)$ rad/s, $f_0 = 40$ Hz, and $\varphi = -0.4\pi$ rad. The signal size depends on the amplitude parameter A; its maximum and minimum values are $+20$ and -20, respectively. In Fig. 2-6 the maxima occur at

$$t = \ldots, -0.02, 0.005, 0.03, \ldots$$

and the minima at

$$\ldots, -0.0325, -0.0075, 0.0175, \ldots$$

The time interval between successive maxima in Fig. 2-6 is 0.025 s, which is equal to $1/f_0$. To understand why the signal has these properties, we will need to do more analysis.

PHASE ←→ TIME-SHIFT

- Equate the formulas:

$$A\cos(\omega(t-t_m)) = A\cos(\omega t+\varphi)$$

- and we obtain:

$$-\omega t_m = \varphi$$

- or,

$$t_m = -\frac{\varphi}{\omega}$$

2-3.1 Relation of Frequency to Period

The sinusoid plotted in Fig. 2-6 is a periodic signal. The *period* of the sinusoid, denoted by T_0, is the time duration of one cycle of the sinusoid. In general, the frequency of the sinusoid determines its period, and the relationship can be found by applying the definition of periodicity $x(t + T_0) = x(t)$ as follows:

$$A\cos(\omega_0(t + T_0) + \varphi) = A\cos(\omega_0 t + \varphi)$$

$$\cos(\omega_0 t + \underbrace{\omega_0 T_0}_{=2\pi?} + \varphi) = \cos(\omega_0 t + \varphi)$$

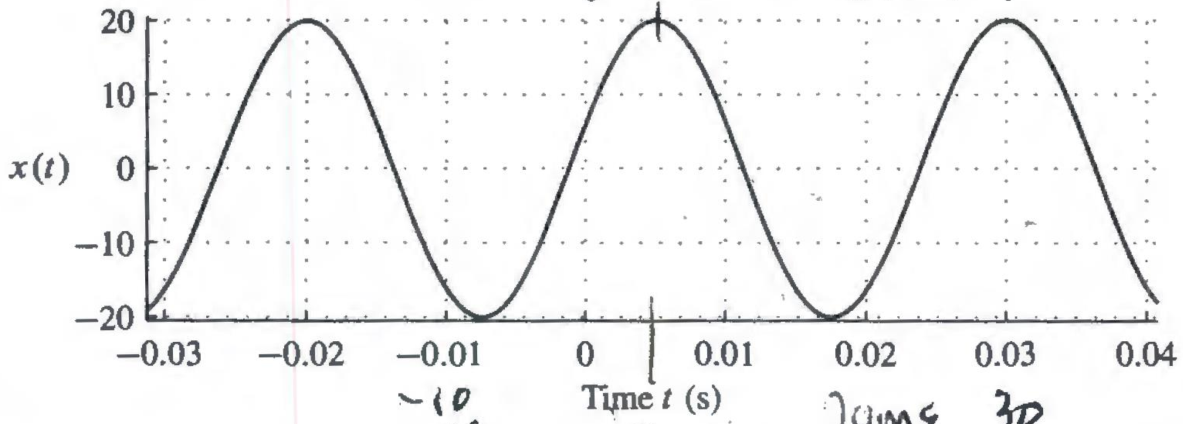
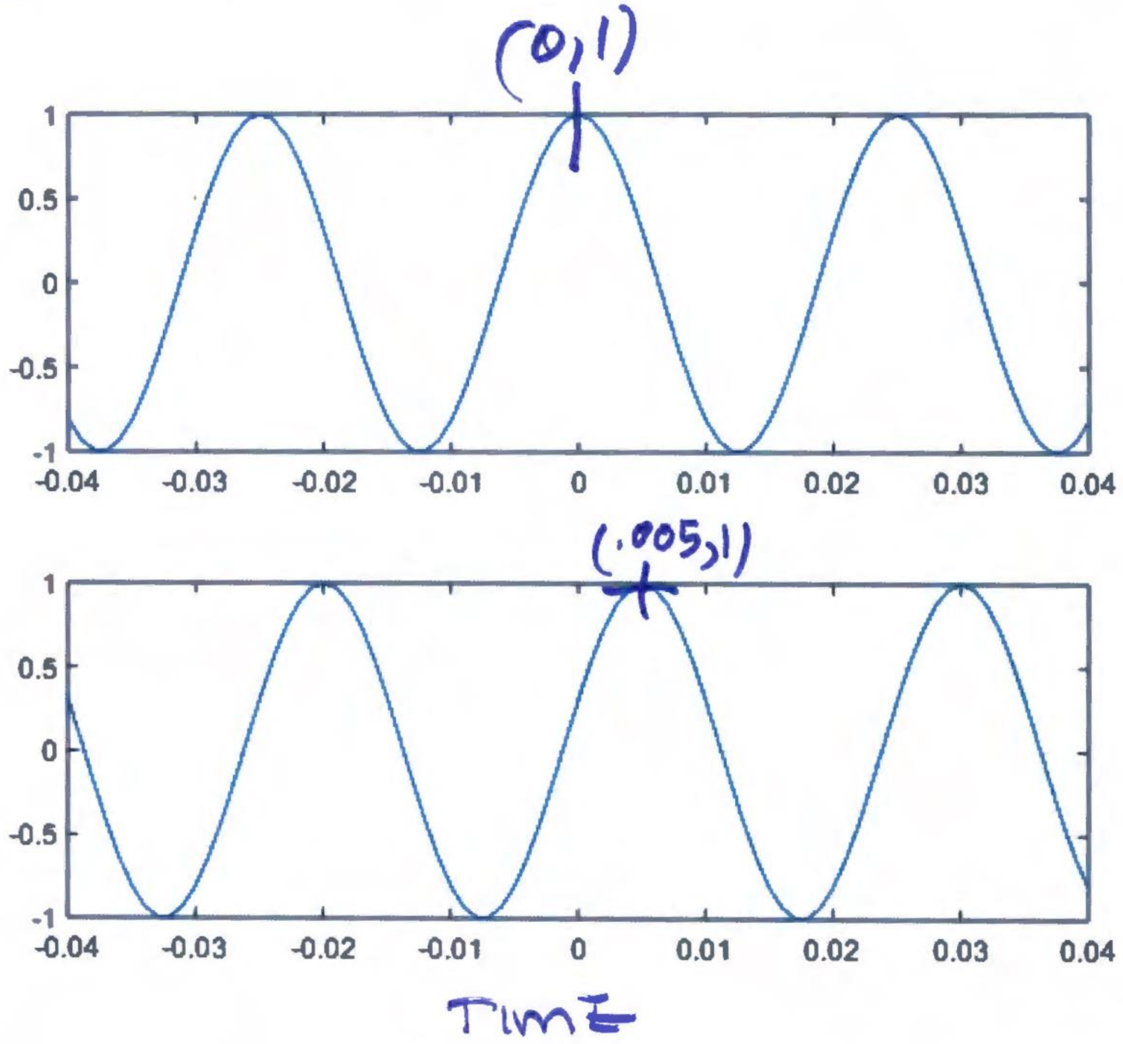


Figure 2-6 Sinusoidal signal with parameters $A = 20$, $\omega_0 = 2\pi(40)$, $f_0 = 40$ Hz, and $\varphi = -0.4\pi$ rad.

```
% Plot of cosine and shifted cosine
% 20*cosine(2*pi*40 -0.4*pi) vs 20*cosine(2*pi*40)
% T0= 1/40 = 25ms. Consider a time axis from -.04 to +.04 seconds
taxis = -.04:.001: .04;
x1=cos(2*pi*40*taxis) ;
x2=cos(2*pi*40*taxis -0.4*pi);
subplot(2,1,1); plot(taxis,x1)
subplot(2,1,2); plot(taxis,x2)
grid on
```

FIG 2-6



DSP First, 2/e

Modified TLH

Lecture 5
Spectrum Representation

Chapter 3; 3-1

Example 3-1: To determine the spectrum of the following signal,

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

which is the sum of a constant and two sinusoids, we must convert from the general form in (3.2) to the two-sided form in (**3.4**). After we apply the inverse Euler formula, we get the following five terms:

$$\begin{aligned} x(t) =& 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ &+ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t} \end{aligned} \tag{3.1}$$

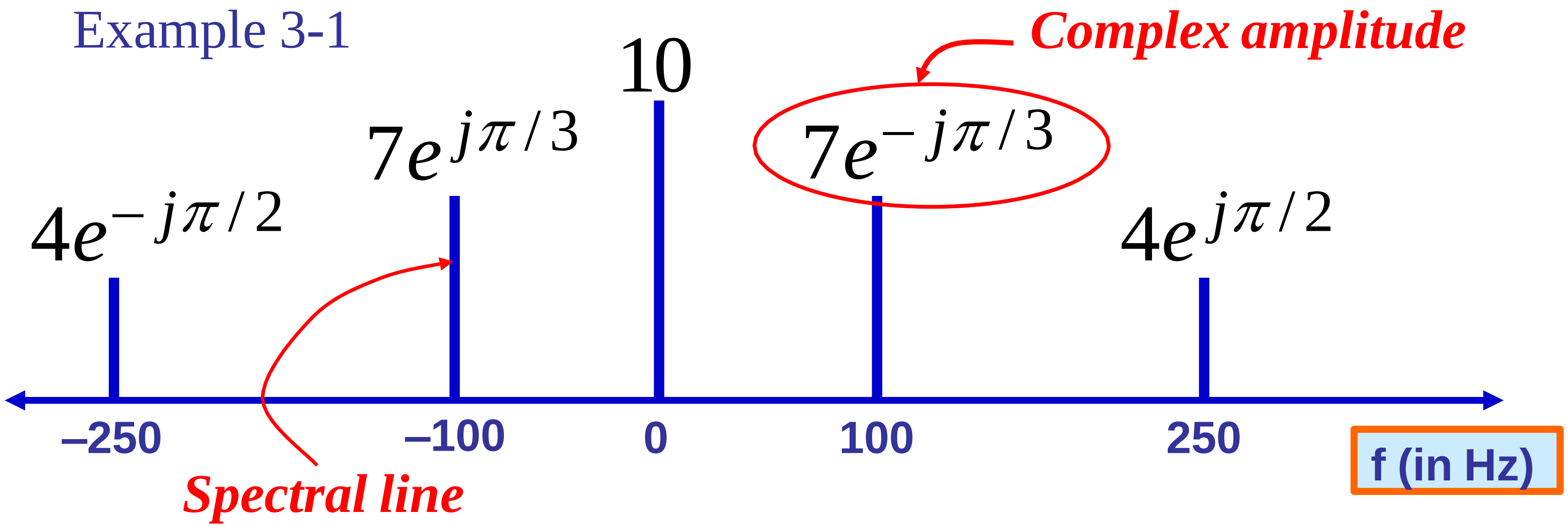
Note that the constant component of the signal, often called the ***DC component***, can be expressed as a complex exponential signal with zero frequency (i.e., $10e^{j0t} = 10$). Therefore, in the list form suggested in (3.5), the spectrum of this signal is the set of five rotating phasors represented by the frequency/complex amplitude pairs

$$\{(0,\ 10),\ (100,\ 7e^{-j\pi/3}),\ (-100,\ 7e^{j\pi/3}),\ (250,\ 4e^{j\pi/2}),\ (-250,\ 4e^{-j\pi/2})\}$$

Note: The terminology "DC" comes from electric circuits, where a constant value of current is called direct current, or DC. It is common to call $X_0 = A_0$ the DC component of the spectrum. Since the DC component is constant, its frequency is $f = 0$.

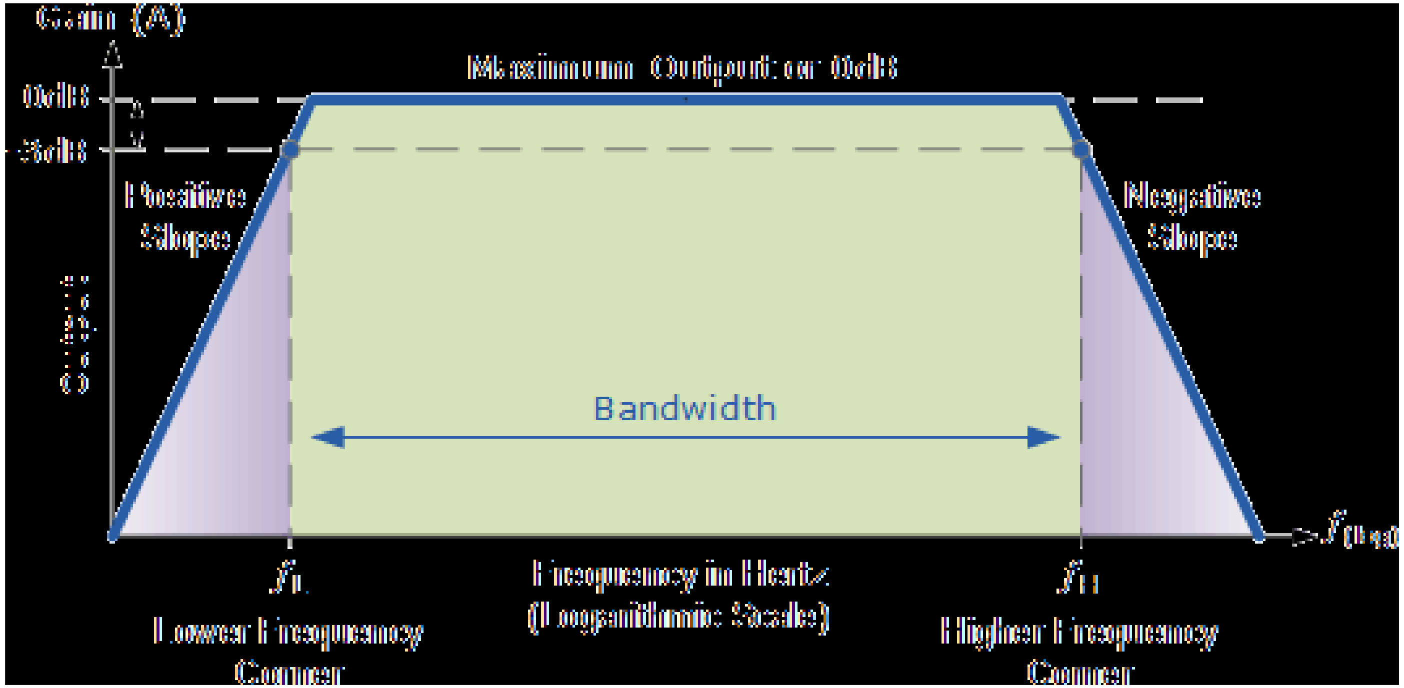
FREQUENCY DIAGRAM

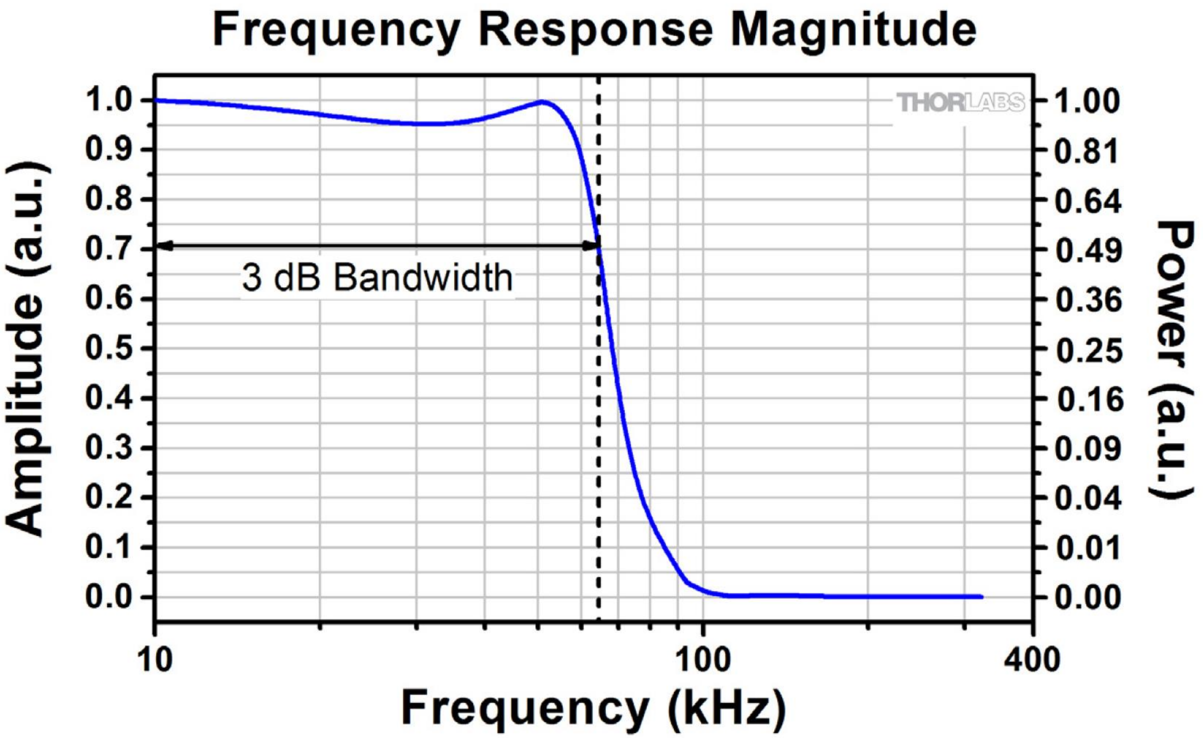
- Want to visualize relationship between frequencies, amplitudes and phases
- Plot Complex Amplitude vs. Frequency



WHAT IS BANDWIDTH?

https://www.youtube.com/watch?v=whUkZUORix0





Definition - For bits/second

The maximum amount of data transmitted over an internet connection in a given amount of time.

Bandwidth is often mistaken for internet speed when it's actually the volume of information that can be sent over a connection in a measured amount of time – calculated in megabits per second (Mbps).

Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)}$$

INVERSE Euler's Formula

- What is the "spectrum" representation for a single sinusoid?
- Solve Euler's formula for cosine (or sine)

$$\cos(\omega t) = \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right)$$

$$\sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

SPECTRUM Interpretation

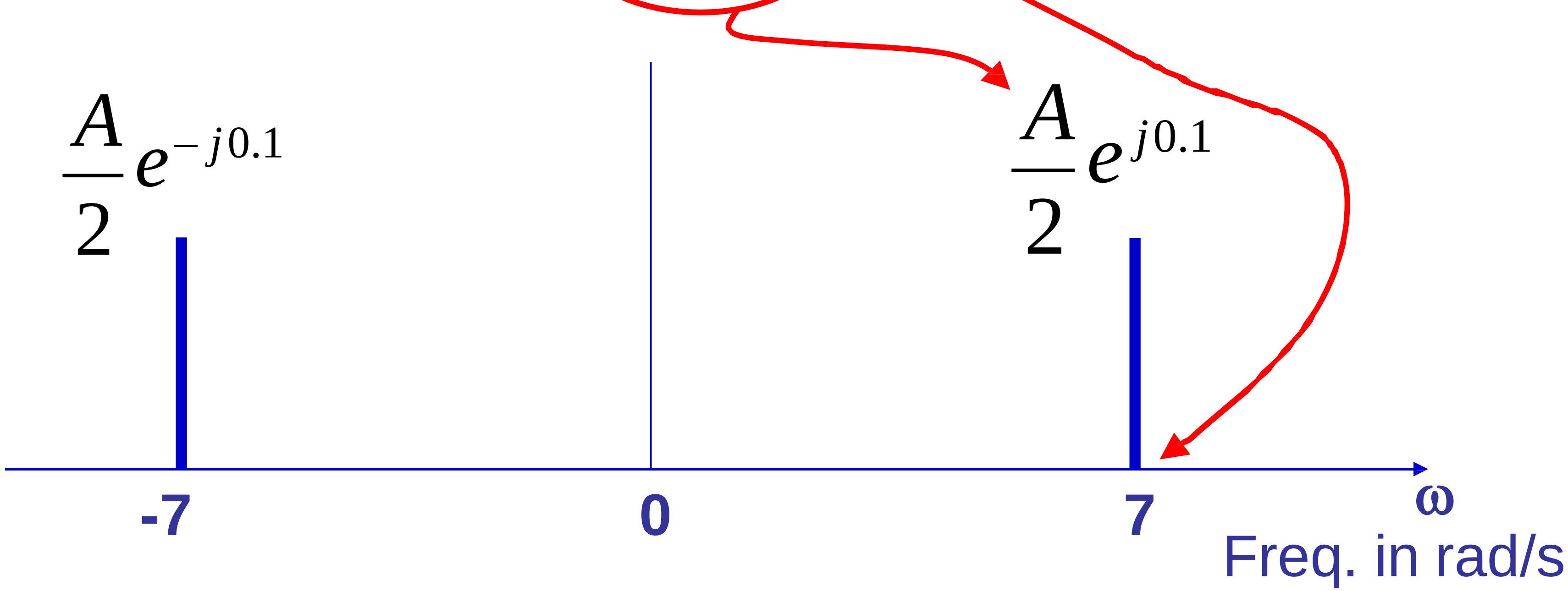
- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

  - One has a positive frequency
  - The other has negative freq.
  - Amplitude of each is half as big

GRAPHICAL SPECTRUM

$$A\cos(7t+0.1) = \frac{A}{2}e^{j0.1}e^{j7t} + \frac{A}{2}e^{-j0.1}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are labels

DSP First, 2/e

Lecture 6

Periodic Signals, Harmonics & Time-Varying Sinusoids

Section 3-4

Harmonic Signal

Periodic signal : $x(t) = x(t+T)$

Can only have ***harmonic*** freqs : $f_k = k\, f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$x(t)$ is periodic if $f_0 T = 1$

$$\cos(2\pi k f_0 (t+T) + \varphi_k) = \cos(2\pi k f_0 t + 2\pi k f_0 T + \varphi_k)$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

Largest f_0 such that

$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

$$f_0 = \frac{1}{T_0}$$

$$f_0 = \text{fundamenta l Frequency}$$

$$f_k / f_0 = \text{integer, for all } k$$

$$T_0 = \text{fundamenta l Period}$$

<u>Main point</u>:
for periodic signals, all spectral lines have frequencies that are *<u>integer</u>* multiples of the fundamental frequency

Harmonic Signal Spectrum

$$\textbf{Harmonic}\ \text{freqs}: \ f_k = k f_0$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

Periodic Signal: Example

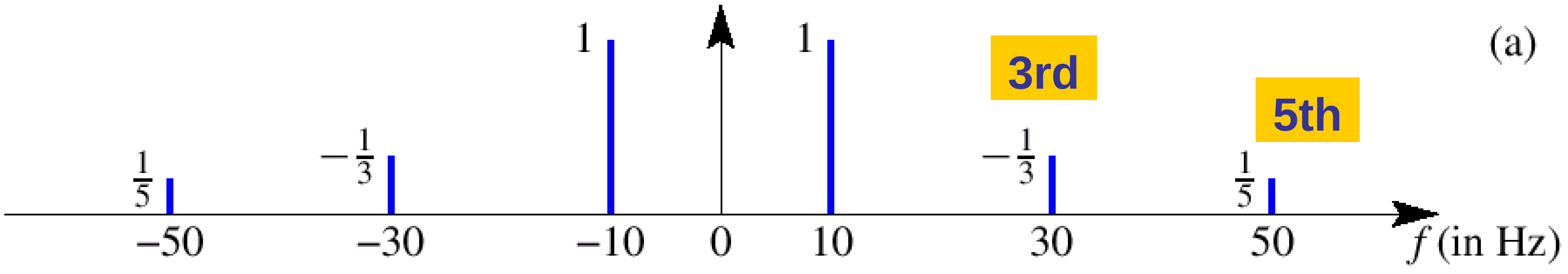
Fundamental frequency

- $e^{j\omega_0(t+T)} = e^{j\omega_0 t}e^{j\omega_0 T} = e^{j\omega_0 t}e^{j2\pi} = e^{j\omega_0 t}$
- $e^{j7\omega_0(t+T)} = e^{j7\omega_0 t}e^{j14\pi} = e^{j7\omega_0 t}$

$$\omega_0 = 2\pi / T$$
$$\Rightarrow \omega_0 T = 2\pi$$

$$x(t+T) = e^{j\omega_0(t+T)} + e^{j7\omega_0(t+T)} + e^{j10\omega_0(t+T)}$$
$$= e^{j\omega_0 t} + e^{j7\omega_0 t} + e^{j10\omega_0 t} = x(t)$$

Harmonic Spectrum (3 Freqs)

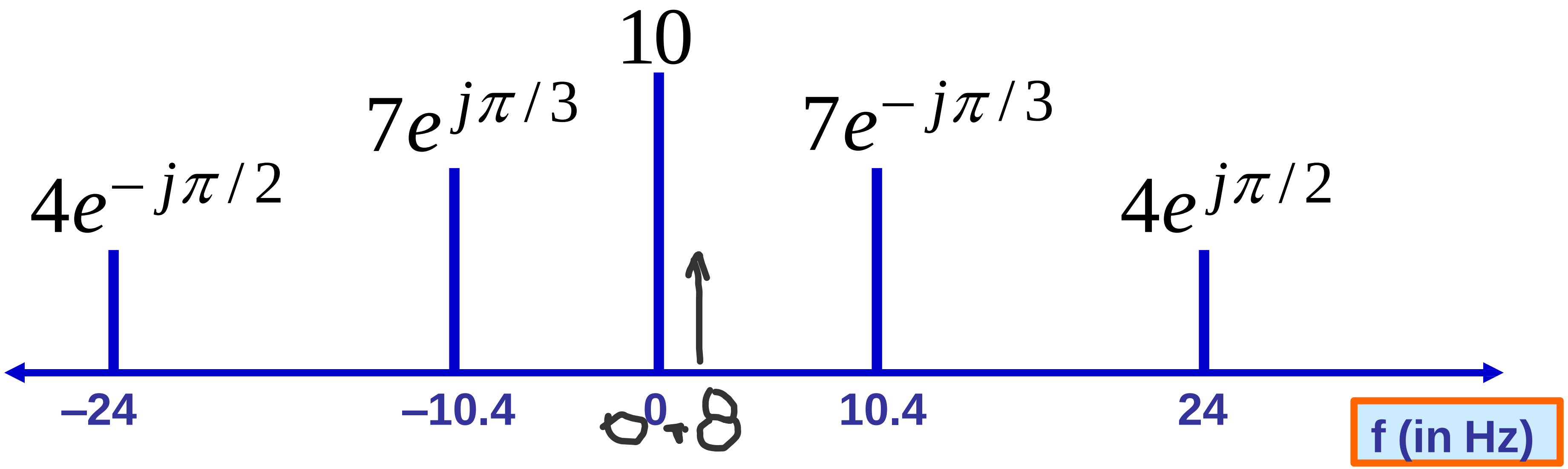


What is the fundamental frequency?

10 Hz

POP QUIZ: FUNDAMENTAL

- Here's another spectrum:



What is the fundamental frequency?

(0.1)GCD(104,240) = (0.1)(8)=0.8 Hz

Fundamental Frequency

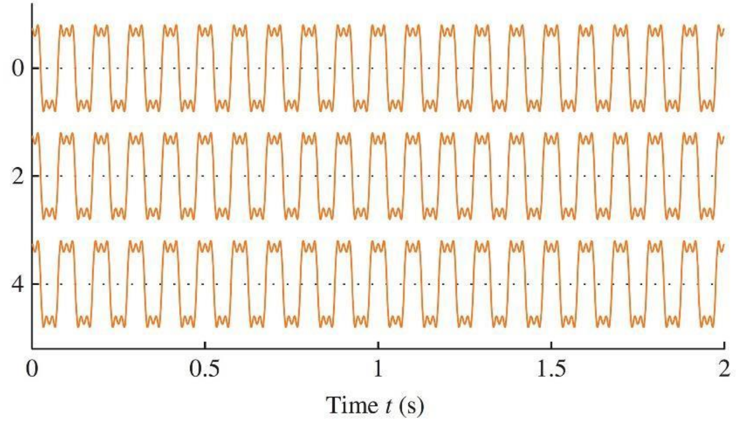
- Multiply and divide by 10
- 104, 240   8 divides 13, 30 –
- Now divide by 10 > 0.8

- 0.8, 1.6, 2.4,…8, 8.8, 9.6, **10.4**,… 16, …
- **24**,24.8, …

Example of a Periodic Signal (1 of 3)

Figure 3-16: Sum of three cosine waves with harmonic frequencies. The spectrum is shown in Figure 3-18(a), and the fundamental frequency of $x_h(t)$ is 10 Hz.

$$X(t) = 2\cos(20\pi t) - 2/3\cos(20\pi(3)t) + 2/5\cos(20\pi(5)t)$$



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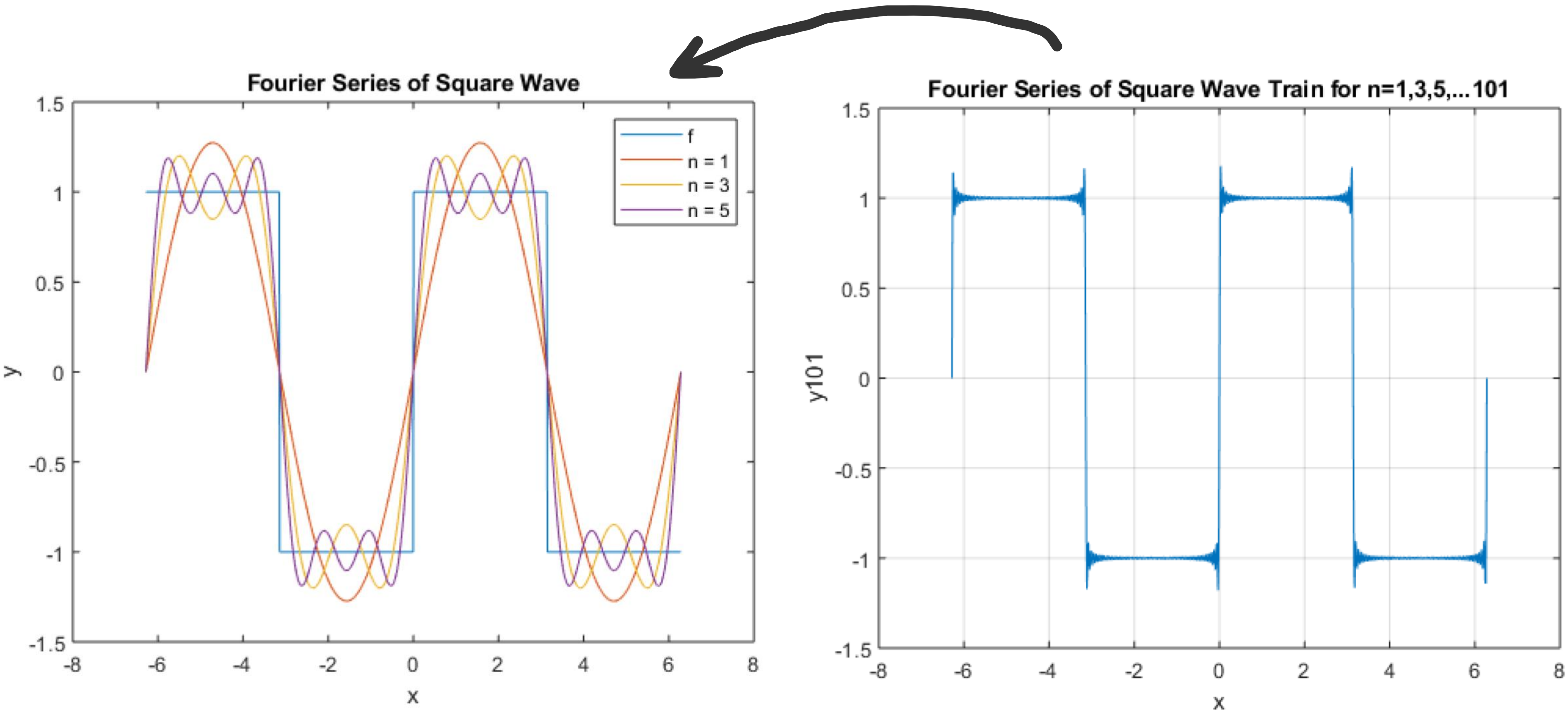
Lecture 7

Fourier Series Analysis

Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.


Fourier Series of Square Wave
f
n = 1
n = 3
n = 5
y
x
1.5
1
0.5
0
-0.5
-1
-1.5
-8
-6
-4
-2
0
2
4
6
8
Fourier Series of Square Wave Train for n=1,3,5,...101
y101
x

Harmonic Signal->Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

*Sums of **Harmonic** complex exponentials are **Periodic** signals*

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

STRATEGY: x(t) → a_k

- **<u>ANALYSIS</u>**
  - Get representation from the signal
  - Works for **<u>PERIODIC</u>** Signals
  - Measure similarity between signal & harmonic
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}\,dt$$

Fourier Series: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Given a **PERIODIC Signal**
  - Fourier Series coefficients are obtained via an **INTEGRAL over one period**

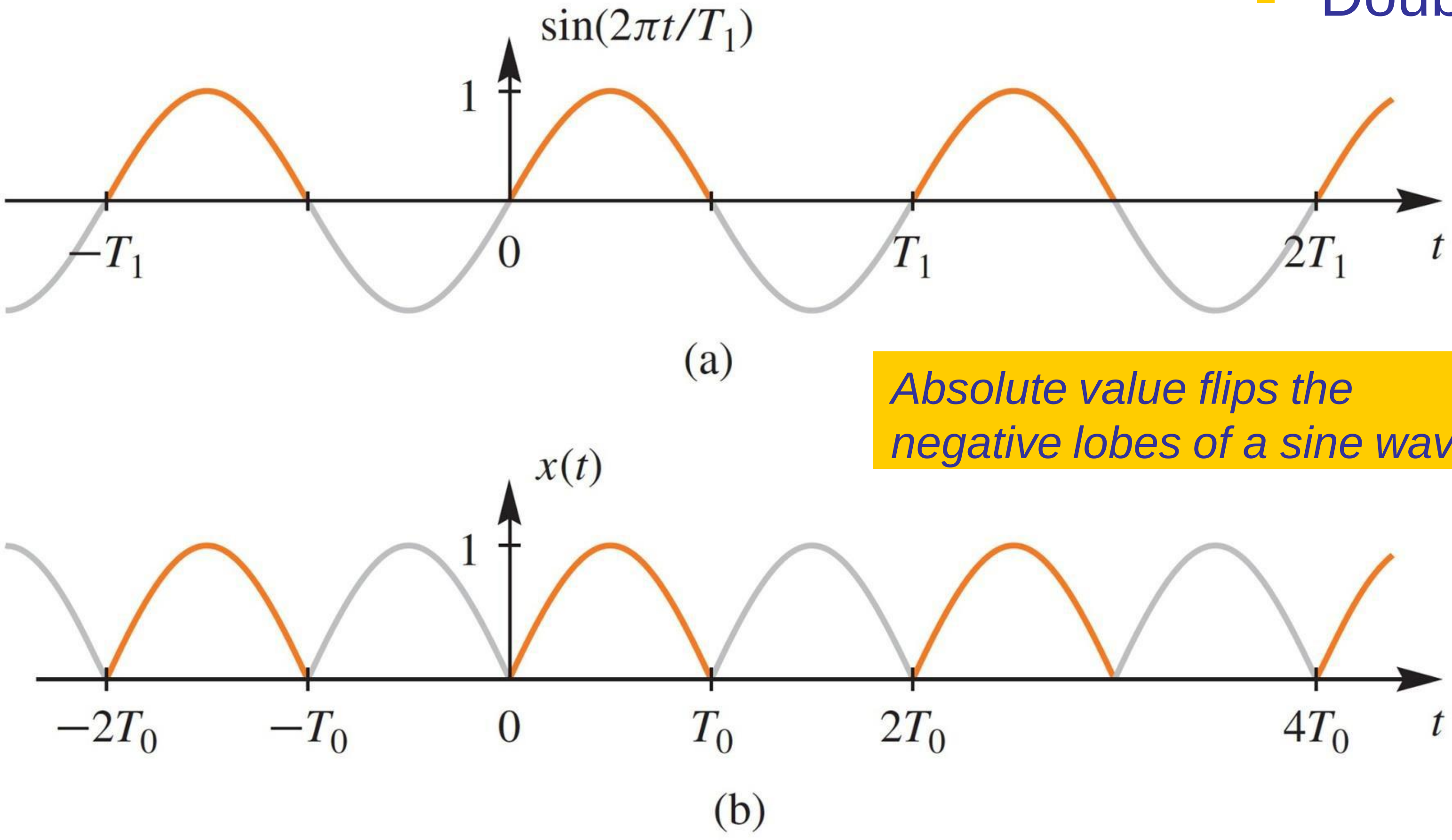
INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

Full-Wave Rectified Sine

$$x(t) = \left|\sin(2\pi t / T_1)\right| \qquad \text{Period is } T_0 = \tfrac{1}{2} T_1$$

- Frequency
- Doubles



Absolute value flips the negative lobes of a sine wave

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Full-Wave Rectified Sine

$$x(t) = \left|\sin(2\pi t/T_1)\right|$$

$$Period: T_0 = \tfrac{1}{2}T_1$$

$$\Rightarrow x(t) = \left|\sin(\pi t/T_0)\right|$$

$$a_k = \frac{1}{T_0}\int_0^{T_0} \sin(\tfrac{\pi}{T_0}t)e^{-j(2\pi/T_0)kt}dt$$

$$= \frac{1}{T_0}\int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}dt$$

$$= \frac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k-1)t}dt - \frac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k+1)t}dt$$

$$= \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

$$= \frac{1}{\frac{2\pi}{(2k-1)}}\left(e^{-j(\pi/T_0)(2k-1)T_0} - 1\right) - \frac{1}{\frac{2\pi}{(2k+1)}}\left(e^{-j(\pi/T_0)(2k+1)T_0} - 1\right)$$

$$= \frac{1}{\pi(2k-1)}\left(e^{-j\pi(2k-1)} - 1\right) - \frac{1}{\pi(2k+1)}\left(e^{-j\pi(2k+1)} - 1\right)$$

$$= \left(\frac{2k+1-(2k-1)}{\pi(4k^2-1)}\right)\left(-(-1)^{2k} - 1\right) = \frac{-2}{\pi(4k^2-1)}$$

Fourier Coefficients: a_k

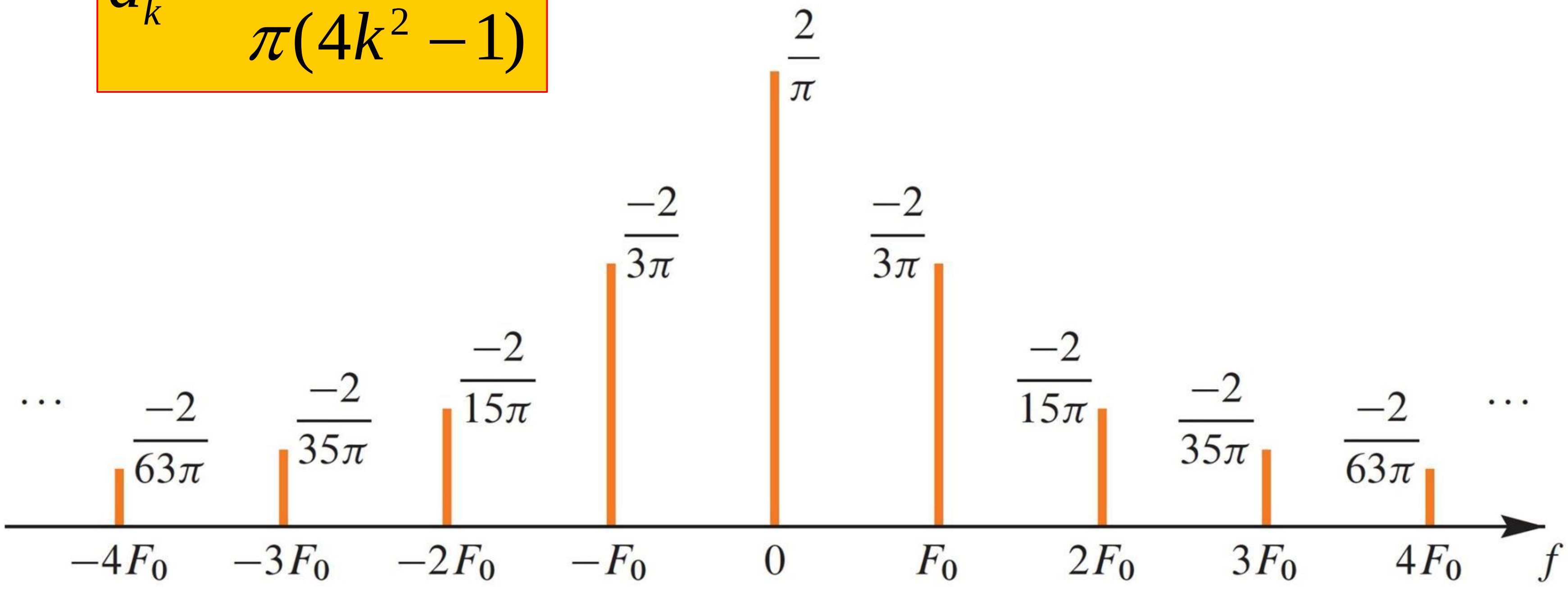
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - $$a_k = \frac{-2}{\pi(4k^2 - 1)}$$ NOTE: $\frac{1}{k^2}$ *for large k*
  - Does not depend on the period, T_0
  - DC value is $a_0 = 2/\pi = 0.6336$

Spectrum from Fourier Series

Plot a_k for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$$F_0 = 1/T_0 \quad \text{and} \quad \omega_0 = 2\pi F_0$$



Reconstruct From Finite Number of Harmonic Components

Full-Wave Rectified Sinusoid $x(t) = \left|\sin(\pi t / T_0)\right|$

$$T_0 = 10 \text{ ms}$$
$$\Rightarrow F_0 = 100 \text{ Hz}$$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$$a_0 = 2/\pi = 0.6336$$

$$x_N(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k \, e^{j2\pi k F_0 t} + a_k^* \, e^{-j2\pi k F_0 t} \right\}$$

How close is $x_N(t)$ to $x(t) = \left|\sin(\pi t / T_0)\right|$?

Reconstruct From Finite Number of Spectrum Components

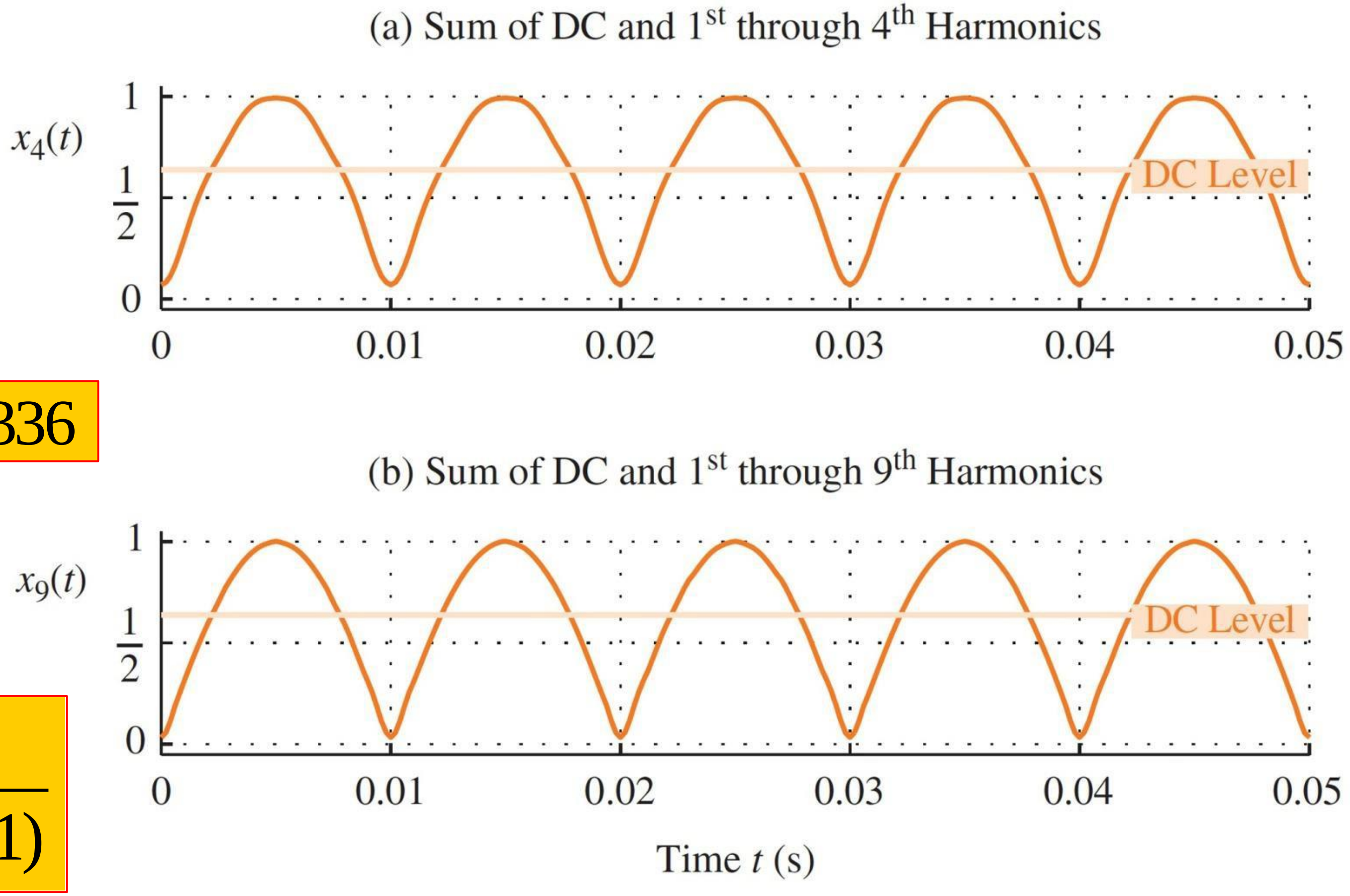
Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

$T_0 = 10 \text{ ms}$

$\Rightarrow F_0 = 100 \text{ Hz}$

$a_0 = 2/\pi = 0.6336$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$



- MODIFIED BY TLH

DSP First, 2/e

Lecture Ch3

Fourier Series Analysis

SEE COURSE WEBSITE FourierCh8_TLH

LECTURE OBJECTIVES

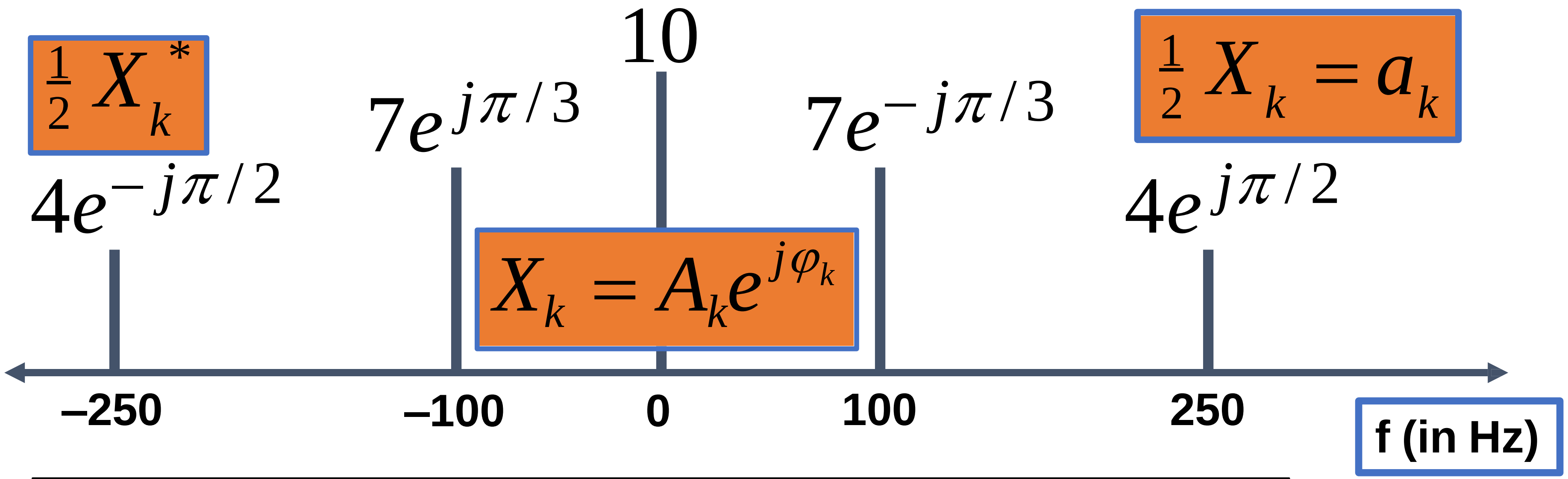
- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k/T_0)t} dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\mathbf{x(t+T_0) = x(t)}$
  - Draw spectrum from the Fourier Series coefficients

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal->Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k F_0 t}$$

*Sums of **Harmonic** complex exponentials are **Periodic** signals*

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

Fourier Trig Series

☐ EXAMPLE 8.4 ***Fourier series square wave example***

A square wave of amplitude A and period T shown in Figure 8.4 can be defined as

$$f(t) = \begin{cases} A, & 0 < t < \dfrac{T}{2}, \\ -A, & -\dfrac{T}{2} < t < 0, \end{cases}$$

with $f(t) = f(t + T)$, since the function is periodic.

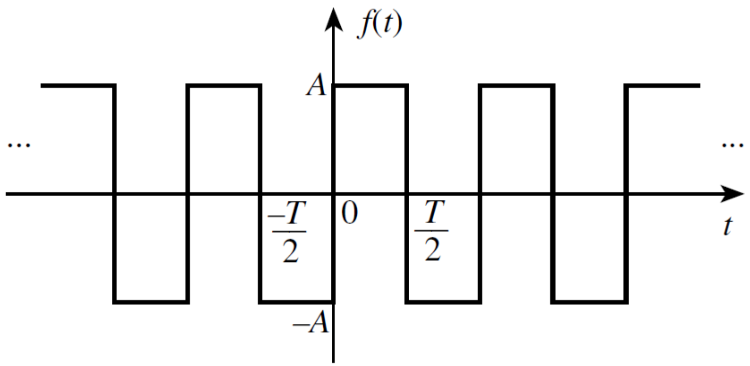


FIGURE 8.4 *Square wave of Example 8.4*

The first observation is that $f(t)$ is odd, which yields the result that $a_0 = 0$ and $a_i = 0$ for every coefficient of the cosine terms. Letting $\omega_0 = 2\pi/T$, the coefficients b_n are

$$b_n = 2\left(\frac{2}{T}\right)\int_0^{T/2} A\sin(n\omega_0 t)\,dt.$$

The result is

$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

where $(2n-1)$ is introduced to assure that only odd terms are included in the summation. The sine waves that make up the Fourier series for the odd square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right],$$

so the series consists not only of sine terms, as expected, but also odd harmonics appear. This is due to the rotational symmetry of the function since the wave shapes on alternate half-cycles are identical in shape but reversed in sign. Such waveforms are produced in certain types of rotating electrical machinery.

□

☐ EXAMPLE 8.5 ***Complex Series Square Wave Example***

Consider the odd square wave of Example 8.4 and the complex Fourier coefficients

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{0} (-A)e^{-in\omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2} (A)e^{-in\omega_0 t}\,dt, \qquad (8.29)$$

which leads to the series

$$f(t) = \frac{2A}{i\pi}\sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}, \qquad (8.30)$$

as defined in Equation 8.23.

This form contains complex coefficients, but the series can be written in terms of sine waves by combining the corresponding terms for positive and negative arguments. To determine the coefficients, the amount of difficulty is about the same for the trigonometric series and the complex series. However, the complex series perhaps has an advantage when the magnitude of the coefficients are of interest.

Each coefficient has the form

$$\alpha_n = \frac{2A}{in\pi} = \frac{2A}{n\pi}e^{-i\pi/2}, \qquad n = \pm 1, \pm 3, \ldots,$$

and the coefficients for even values, $n = 0, \pm 2, \ldots$, are zero. Notice that the coefficients decrease as the index n increases. The use of these coefficients to compute the *frequency spectrum* of $f(t)$ is considered later.

The trigonometric series is derived from the complex series by expanding the complex series of Equation 8.30 as

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t} \\ &= \cdots - \frac{2A}{3\pi i}e^{-i3\omega_0 t} - \frac{2A}{\pi i}e^{-i\omega_0 t} + \frac{2A}{\pi i}e^{i\omega_0 t} + \frac{2A}{3\pi i}e^{i3\omega_0 t} + \cdots \end{aligned}$$

and recognizing the sum of negative and positive terms for each n as $2\sin(n\omega_0 t)$. The trigonometric series becomes

$$f(t) = \frac{4A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

which is the result of Example 8.4.

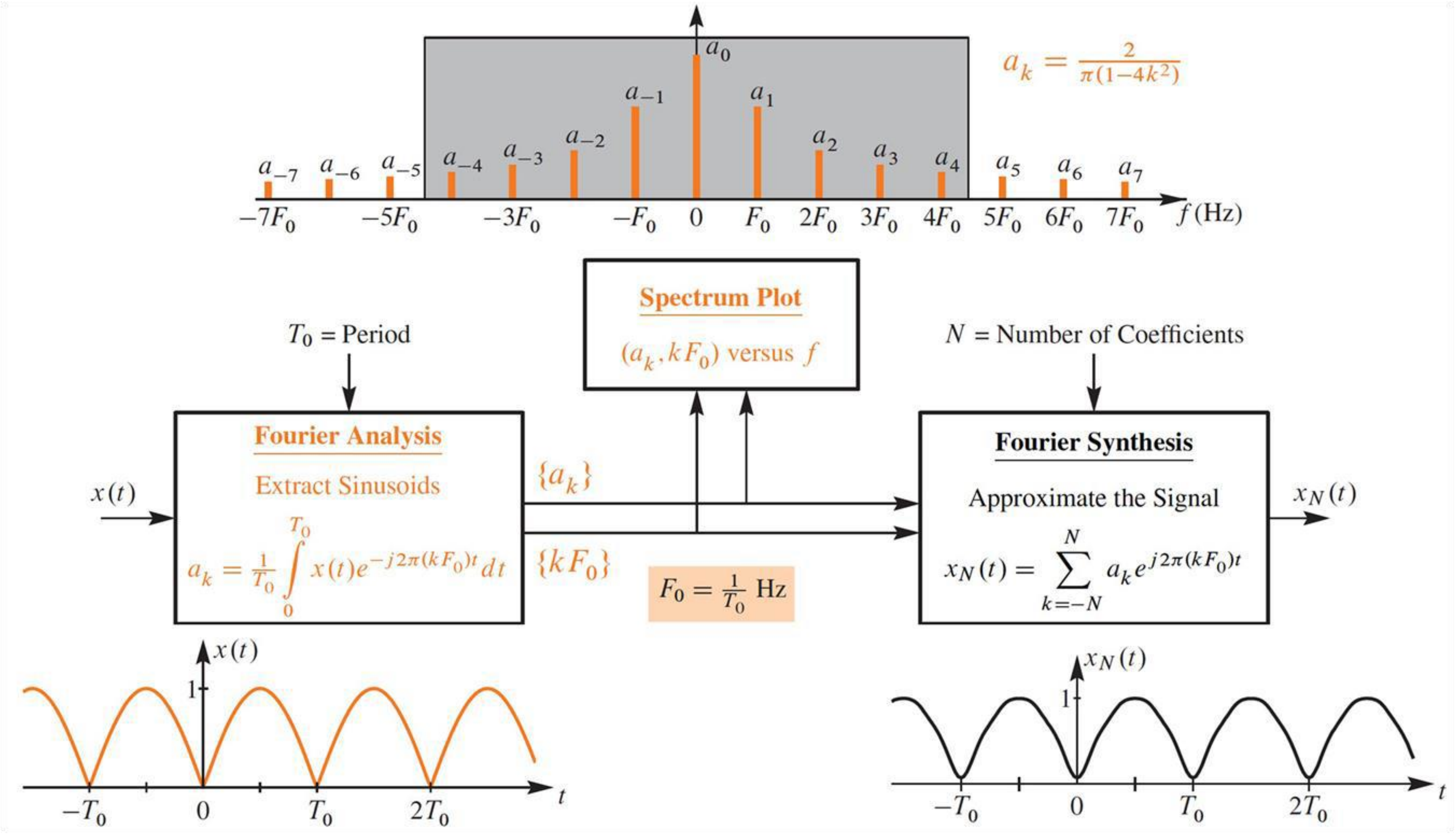
□

Fourier Series of a Pulse Train

Trig and Exponential Forms

Fourier Pulse Train Lecture on Course Website

Fourier Series Synthesis

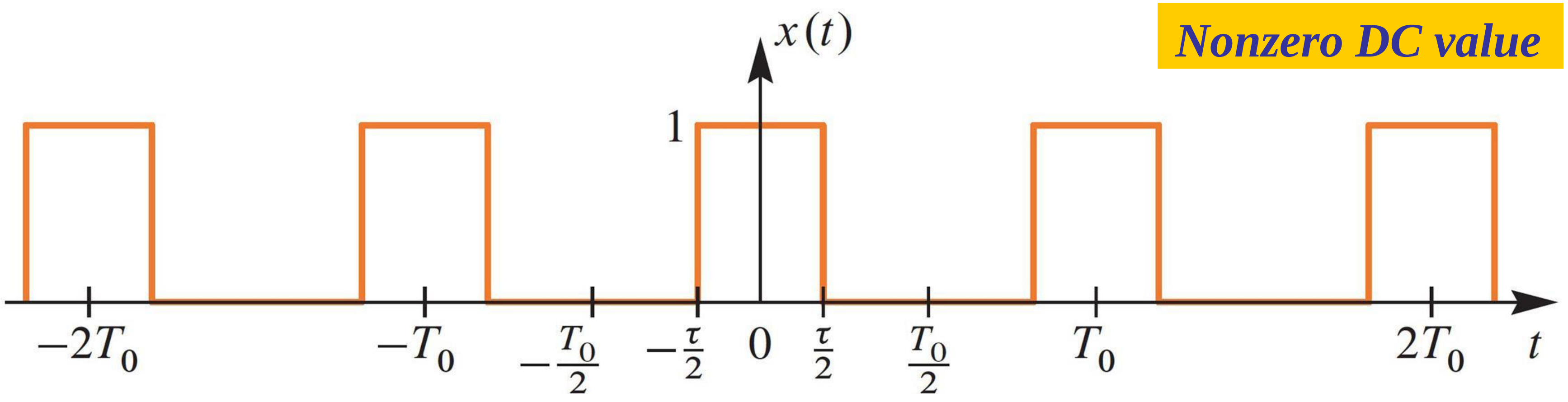


PULSE WAVE SIGNAL
GENERAL FORM

Defined over one period

$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau/2 \\ 0 & \tau/2 \le |t| \le T_0/2 \end{cases}$$

Nonzero DC value



Pulse Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j(2\pi/T_0)(k)t}\,dt$$

General Pulse Wave

$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau/2 \\ 0 & \tau/2 \le |t| \le T_0/2 \end{cases}$$

$$a_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} 1e^{-j(2\pi/T_0)kt}\,dt$$

$$= \left(\tfrac{1}{T_0}\right)\frac{e^{-j(2\pi/T_0)kt}}{-j(2\pi/T_0)k}\Bigg|_{-\tau/2}^{\tau/2} = \frac{e^{-j(2\pi/T_0)k(\tau/2)} - e^{-j(2\pi/T_0)k(-\tau/2)}}{-j(2\pi)k}$$

$$= \frac{e^{j(\pi/T_0)k(\tau)} - e^{-j(\pi/T_0)k(\tau)}}{(j2)\pi k} = \frac{\sin(\pi k\tau/T_0)}{\pi k}$$

Pulse Wave $\{a_k\}$ = sinc

PulseWave

$$a_k = \frac{\sin(\pi k \tau / T_0)}{\pi k} \qquad k = 0, \pm 1, \pm 2, \ldots$$

Double check the DC coefficient:

$$a_0 = \tfrac{1}{T_0} \int_{-\tau/2}^{\tau/2} 1 e^{-j(2\pi/T_0)(0)t} dt$$

$$= \tfrac{1}{T_0} \int_{-\tau/2}^{\tau/2} 1 dt = \tfrac{1}{T_0} \left[\tfrac{\tau}{2} - \tfrac{-\tau}{2} \right] = \tfrac{\tau}{T_0}$$

Note, $\lim_{k \to 0} \frac{\sin(\pi k \tau / T_0)}{\pi k} \rightarrow \frac{\tau}{T_0}$

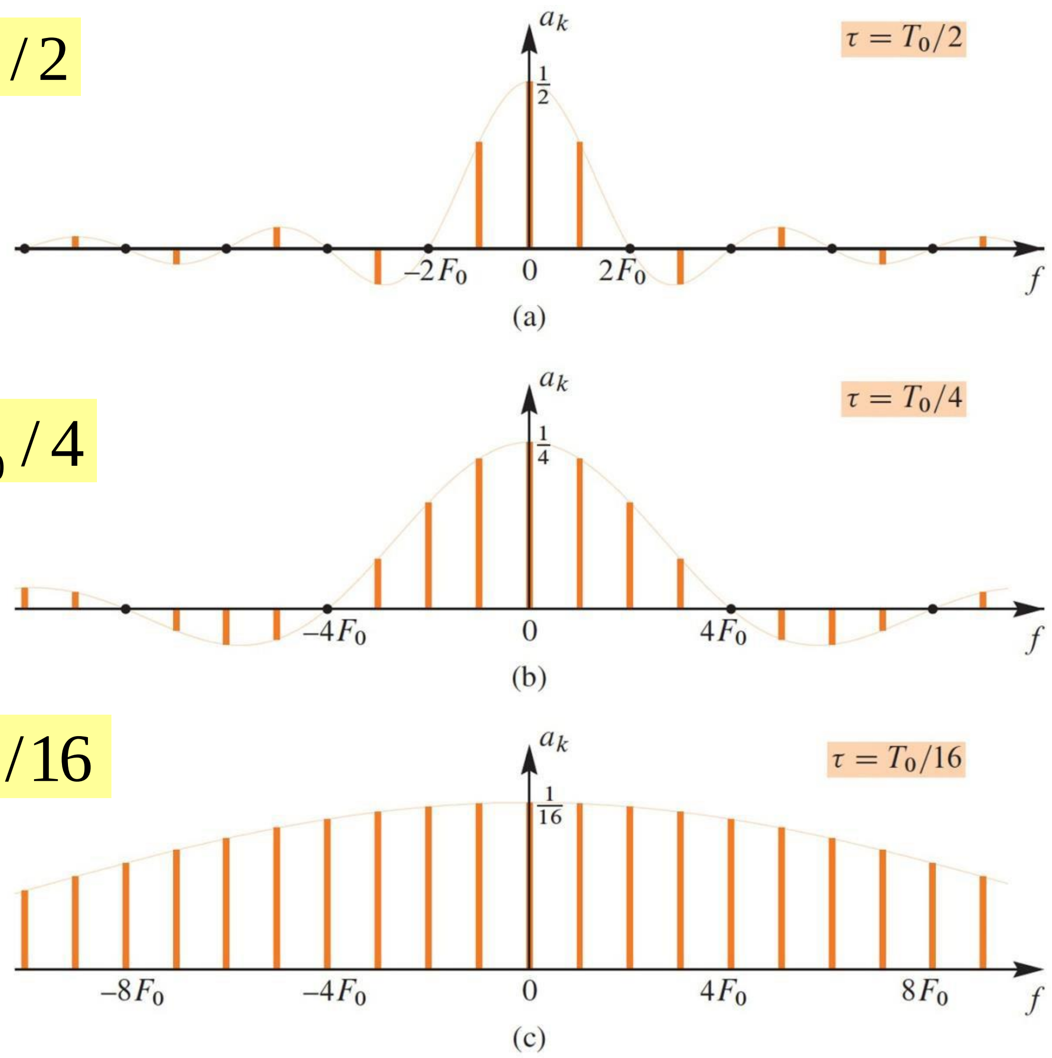
Where do you go if sick?

PULSE WAVE SPECTRA

$\tau = T_0 / 2$

$\tau = T_0 / 4$

$\tau = T_0 / 16$



50% duty-cycle (Square) Wave

$$\tau = T_0/2 \;\Rightarrow a_k = \frac{\sin(\pi k (T_0/2)/T_0)}{\pi k} = \frac{\sin(\pi k/2)}{\pi k} \qquad k = 0, \pm 1, \pm 2, \ldots$$

- Thus, $a_k=0$ when k is odd
  - Phase is zero because x(t) is centered at t=0
    - different from a previous case

PulseWave starting at t=0

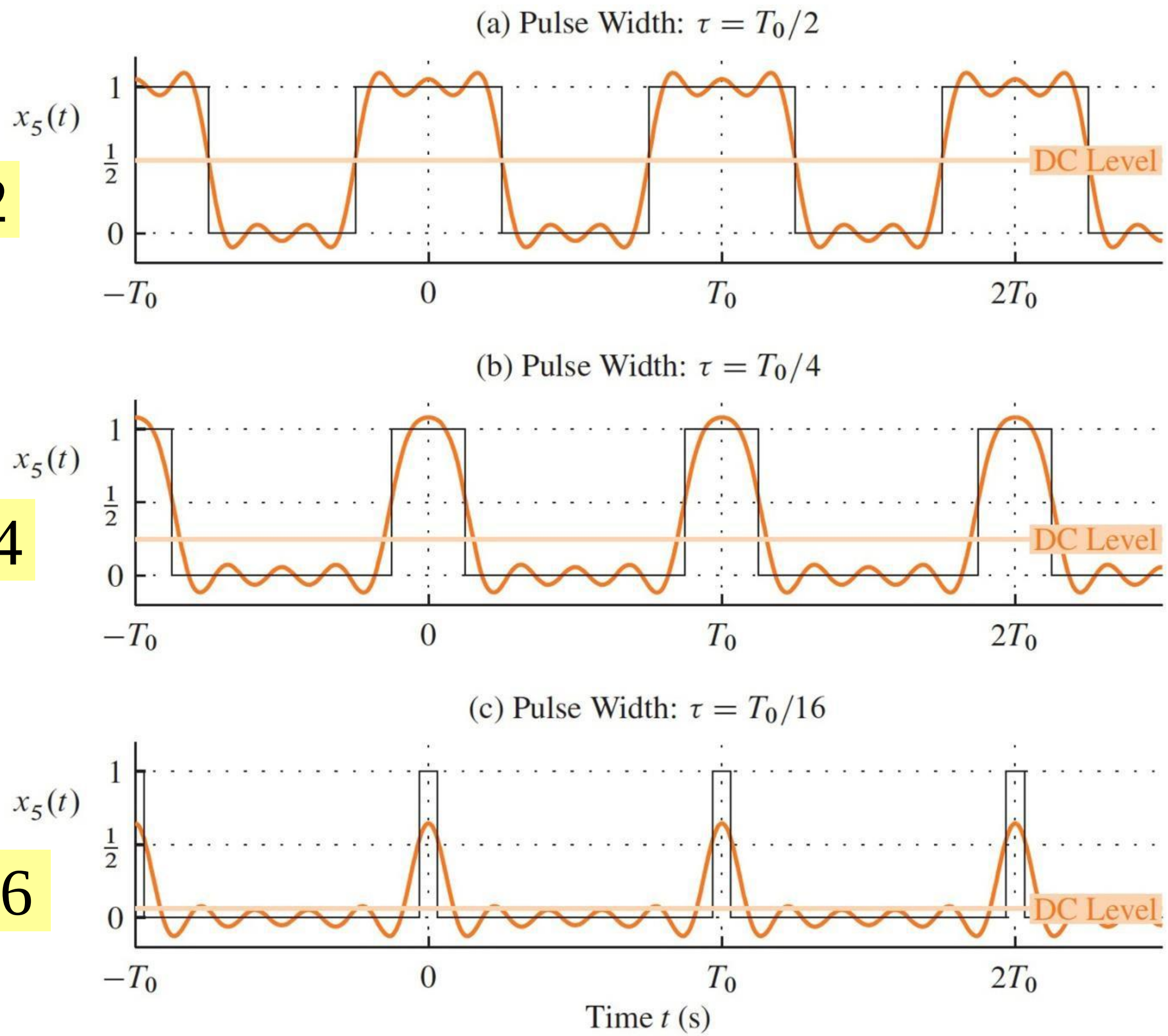
$$x(t) = \begin{cases} 1 & 0 \le |t| < \tau \\ 0 & \tau \le |t| \le T_0 \end{cases} \;\leftrightarrow\; a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

PULSE WAVE SYNTHESIS
with first 5 Harmonics

$\tau = T_0 / 2$

$\tau = T_0 / 4$

$\tau = T_0 / 16$

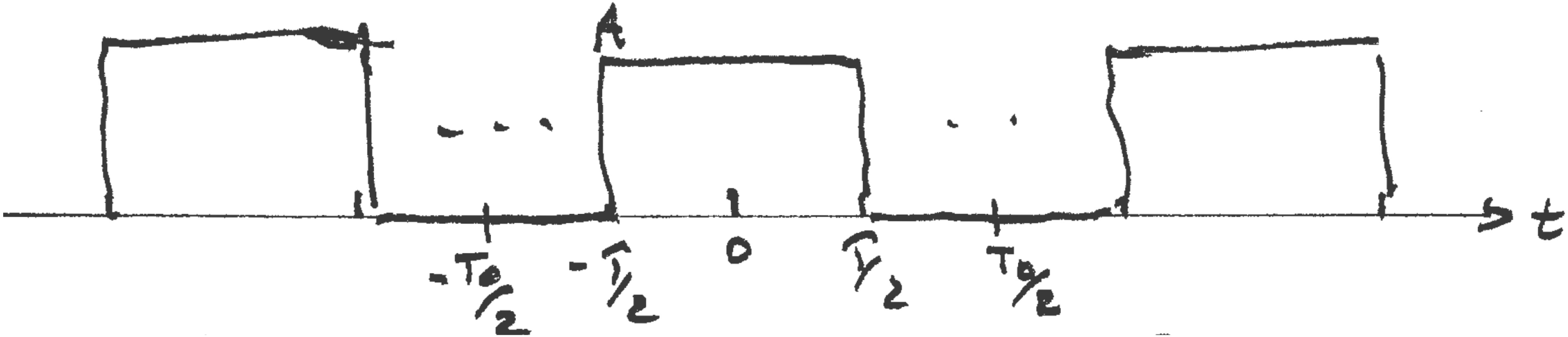


FourierProblemSession_EvenPulseTrain

PULSE WAVE

Lecture 352 slide

$x(t) = x(t+T)$



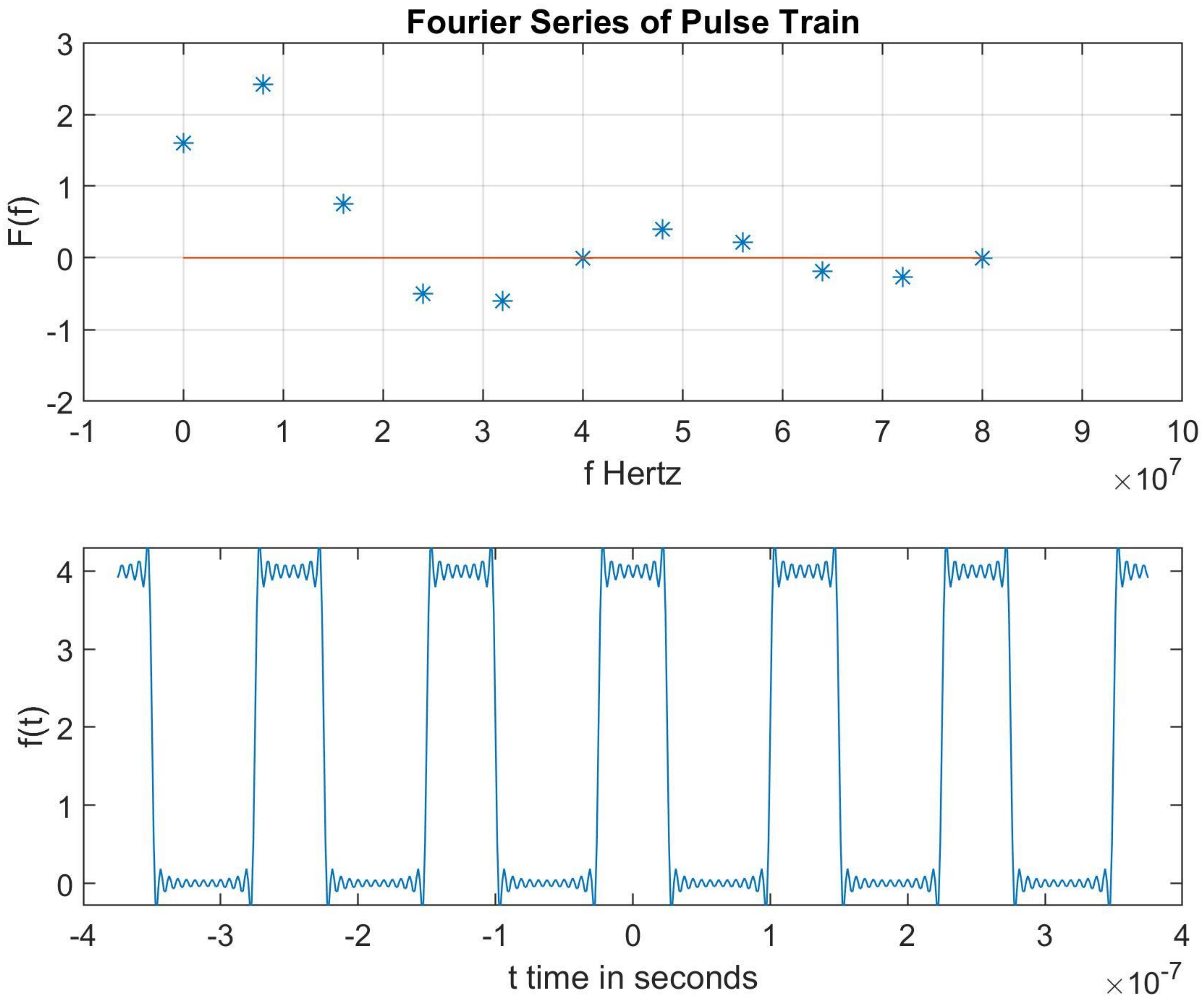
HOMEWORK HELP

Problem 4 30

Fourier series of clock signal Consider the computer clock signal shown in the Figure, with a pulse rate of 8 million pulses per second ($f_c = 8$ Megahertz) and amplitude of 4 volts and a pulse width of 0.05 microseconds. NOTE: The figure does not show the signal to scale.

1. Find the Fourier series by hand calculation using the basic definitions of the coefficients.

BONUS POINTS 20 See MATLAB_Fourier_Even_PulseTrain on our website for help.

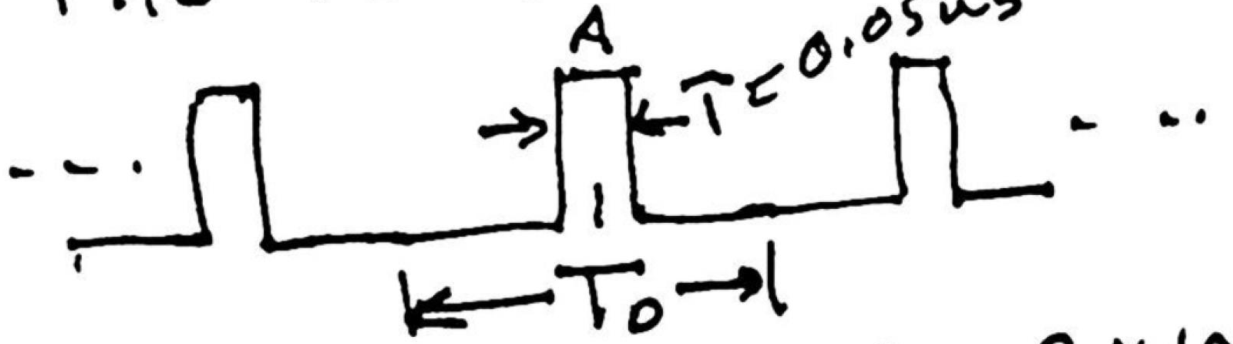


Fourier Series of Pulse Train
F(f)
f Hertz
×10^7
f(t)
t time in seconds
×10^-7

FourierProblemSessionEvenPulseTrain.pdf

https://sceweb.sce.uhcl.edu/harman/CENG3315_Sp2019/0_ProblemSessions/FourierProblemSession.pdf

FOR THE PULSE TRAIN



① PERIOD T_0 SINCE $f_0 = 8 \times 10^6$ Hz , $\omega_0 = 2\pi \times 8 \times 10^6$ rad/sec

$$T_0 = \frac{1}{f_0} = 0.125\mu s \; ; \; \tau = 0.05\mu s \; ; \; \frac{\tau}{T_0} = 0.4$$

② APPLY PAGE 380 Fourier Ch8_TLH

$$a_0 = \frac{2}{T_0}\int_0^{T_0} x(t)\,dt = \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} A\,dt = \frac{A}{T_0}\,t\Big]_{-\tau/2}^{\tau/2}$$

$$= 2\frac{A\tau}{T_0} = 2\cdot\frac{4v\,(0.5us)}{0.125us} = 1.6v \times 2$$

Even Pulse cosine only + dc (2)

$\frac{f}{0}$ $dc = A\frac{\tau}{T} = 0.4 \times 4 = 1.6V$ AVERAGE

8 mHz $a_k = \frac{2A}{k\pi}\sin\left(k\pi \frac{\tau}{T_0}\right) = \frac{8}{k\pi}\sin(0.4k\pi)$

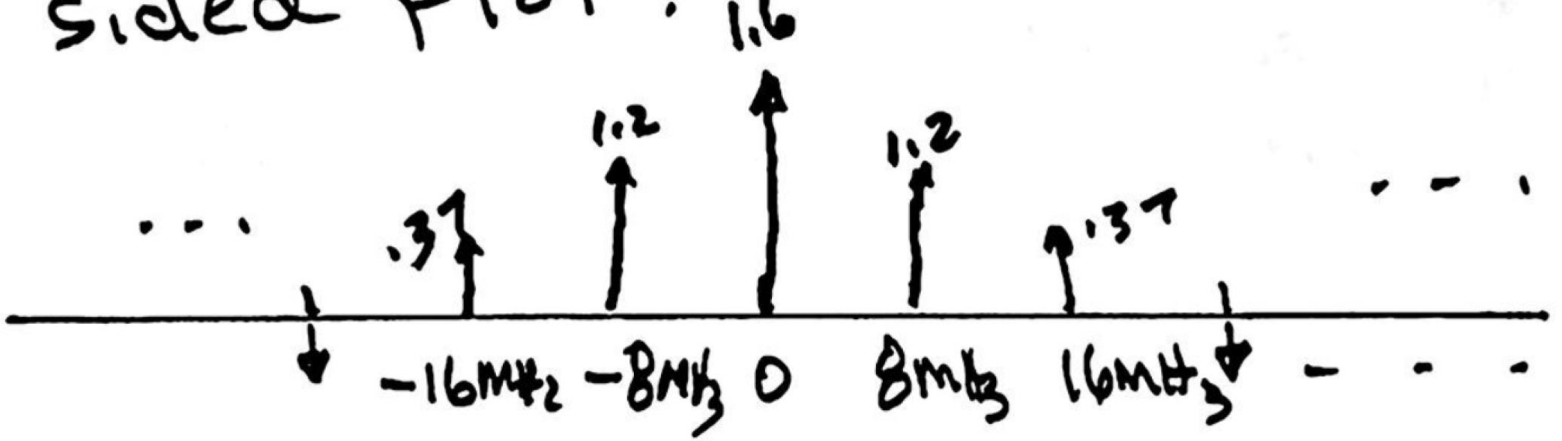
8 mHz $a_1 = \frac{8}{\pi}\sin(0.4k\pi) = 2.4218$

16 MHz $a_2 = \frac{8}{\pi}\cdot\frac{1}{2}\sin(0.8\pi) = 0.7484$

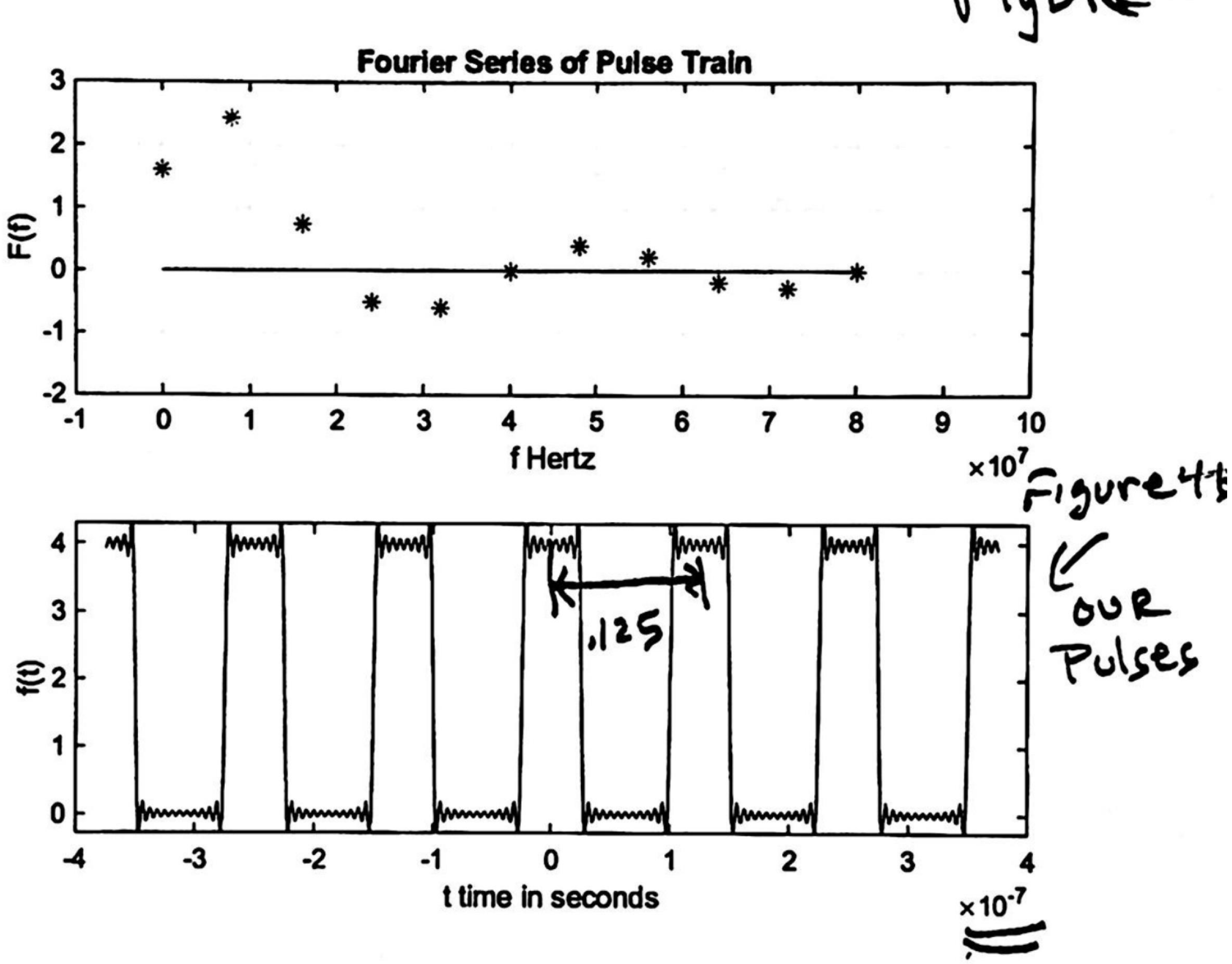
⋮

80 MHz $a_{10} = \frac{8}{\pi}\cdot\frac{1}{10}\sin(4\overset{10}{k}\pi) \equiv 0$

Two sided plot:



For one sided plot - Double values for spectrum except dc value



Fourier Series of Pulse Train
F(f)
f Hertz
×10^7
Figure 4-5
our Pulses
f(t)
.125
t time in seconds
×10^-7