DSP First, 2/e

Modified TLH

Lecture 5
Spectrum Representation

Chapter 3; 3-1

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, Section 3-1

- Other Reading:
  - Appendix A: Complex Numbers

LECTURE OBJECTIVES

- Sinusoids with DIFFERENT Frequencies
  - SYNTHESIZE by Adding Sinusoids

$$x(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

- SPECTRUM Representation
  - Graphical Form shows **DIFFERENT** Freqs

Example 3-1: To determine the spectrum of the following signal,

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

which is the sum of a constant and two sinusoids, we must convert from the general form in (3.2) to the two-sided form in (**3.4**). After we apply the inverse Euler formula, we get the following five terms:

$$\begin{aligned} x(t) = 10 &+ 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} \\ &+ 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t} \end{aligned} \quad (3.1)$$

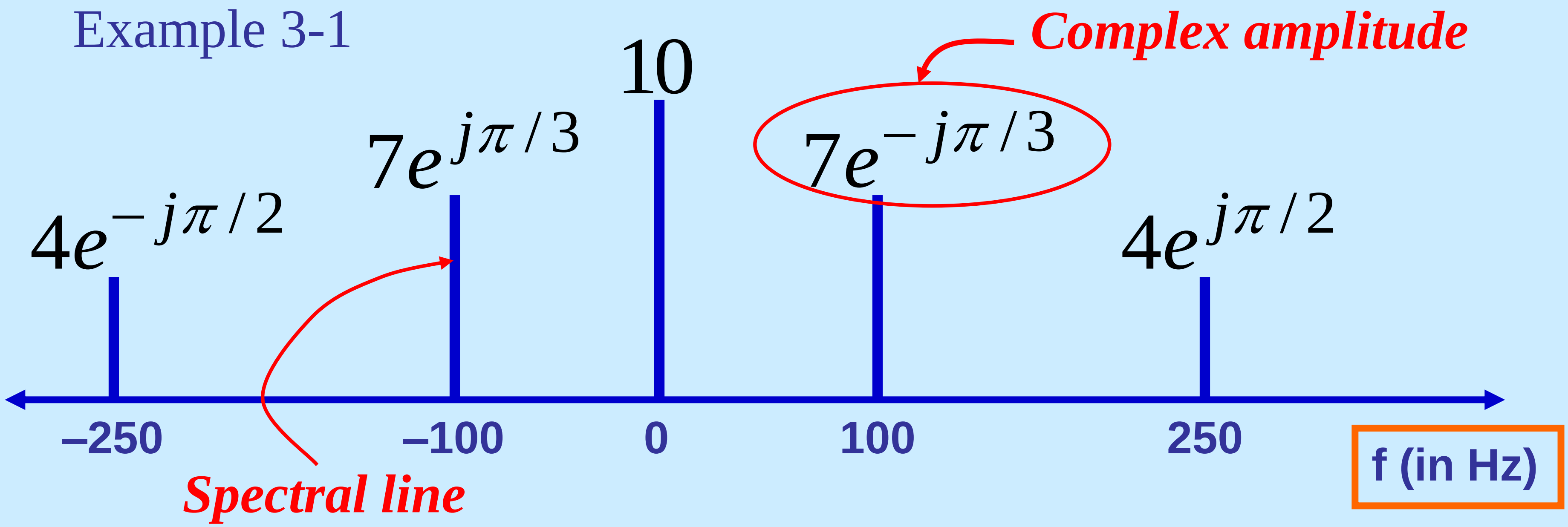
Note that the constant component of the signal, often called the ***DC component***, can be expressed as a complex exponential signal with zero frequency (i.e., $10e^{j0t} = 10$). Therefore, in the list form suggested in (3.5), the spectrum of this signal is the set of five rotating phasors represented by the frequency/complex amplitude pairs

$$\{(0,\ 10),\ (100,\ 7e^{-j\pi/3}),\ (-100,\ 7e^{j\pi/3}),\ (250,\ 4e^{j\pi/2}),\ (-250,\ 4e^{-j\pi/2})\}$$

Note: The terminology "DC" comes from electric circuits, where a constant value of current is called direct current, or DC. It is common to call $X_0 = A_0$ the DC component of the spectrum. Since the DC component is constant, its frequency is $f = 0$.

FREQUENCY DIAGRAM

- Want to visualize relationship between frequencies, amplitudes and phases
- Plot Complex Amplitude vs. Frequency



Another FREQ. Diagram

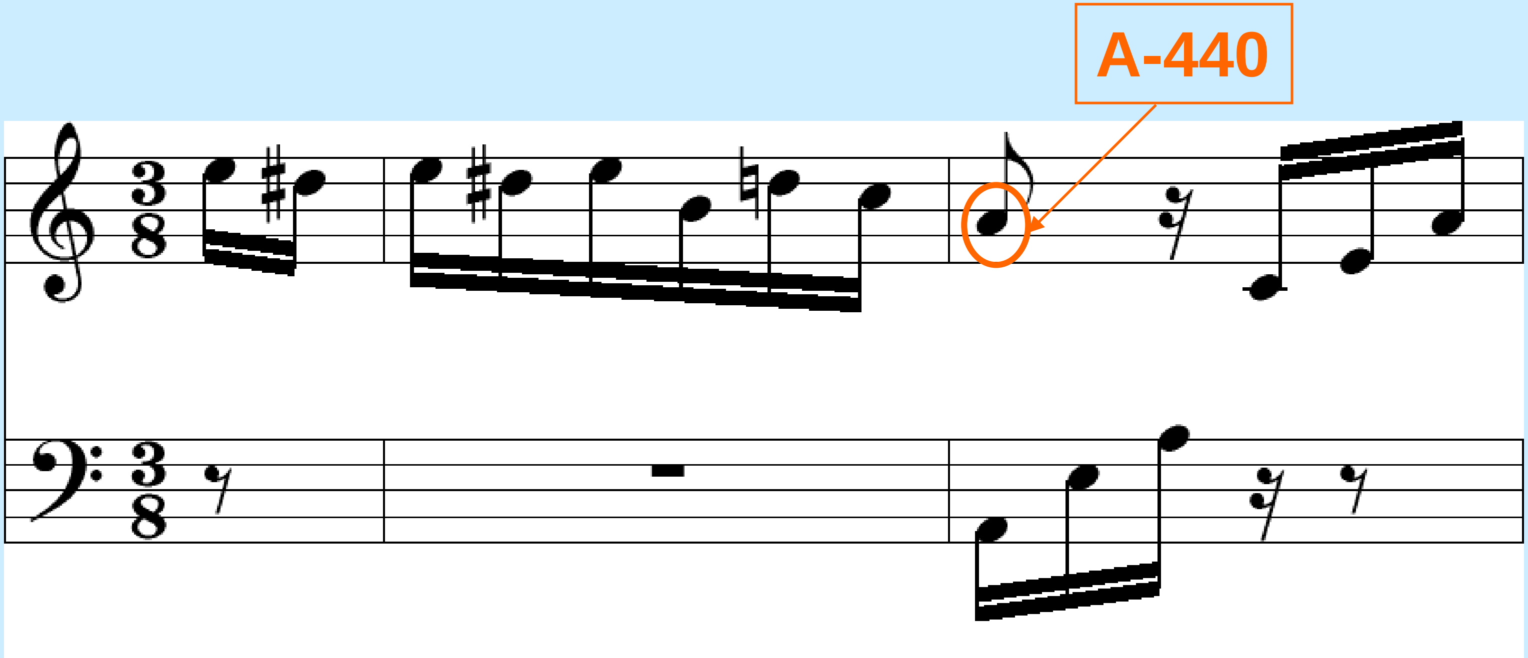


Figure 3.18 Sheet-music notation is a time–frequency diagram.

Frequency is the vertical axis

Time is the horizontal axis

A musical scale consists of a discrete set of frequencies.

MOTIVATION

- Synthesize Complicated Signals
  - Musical Notes
    - Piano uses 3 strings for many notes
    - Chords: play several notes simultaneously
  - Human Speech
    - Vowels have dominant frequencies
    - Application: computer generated speech
  - Can all signals be generated this way?
    - Sum of sinusoids?

Example 3-2: For the special case of a signal formed as the product of two sinusoids with frequencies $\frac{1}{2}$ Hz and 5 Hz

$$x(t) = \cos(\pi t)\sin(10\pi t) \tag{3.2}$$

it is necessary to rewrite $x(t)$ as a sum before its spectrum can be defined. One approach is to use the inverse Euler formula as follows:

$$x(t) = \left(\frac{e^{j\pi t} + e^{-j\pi t}}{2}\right)\left(\frac{e^{j10\pi t} - e^{-j10\pi t}}{2j}\right) \tag{3.3a}$$

$$= \tfrac{1}{4}e^{-j\pi/2}e^{j11\pi t} + \tfrac{1}{4}e^{-j\pi/2}e^{j9\pi t} + \tfrac{1}{4}e^{j\pi/2}e^{-j9\pi t} + \tfrac{1}{4}e^{j\pi/2}e^{-j11\pi t} \tag{3.3b}$$

$$= \tfrac{1}{2}\cos(11\pi t - \pi/2) + \tfrac{1}{2}\cos(9\pi t - \pi/2) \tag{3.3c}$$

In this derivation, we see four terms in the additive combination (3.10b), so there are four spectrum components at frequencies $\pm 11\pi$ and $\pm 9\pi$ rad/s, which convert to hertz as 5.5, 4.5, −4.5, and −5.5 Hz. The magnitude is the same ($\frac{1}{4}$) for all four components. It is also worth noting that neither of the originalfrequencies (5 Hz and $\frac{1}{2}$ Hz) used to define $x(t)$ in (3.9) appear in the spectrum.

Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

INVERSE Euler's Formula

- What is the "spectrum" representation for a single sinusoid?
- Solve Euler's formula for cosine (or sine)

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

SPECTRUM Interpretation

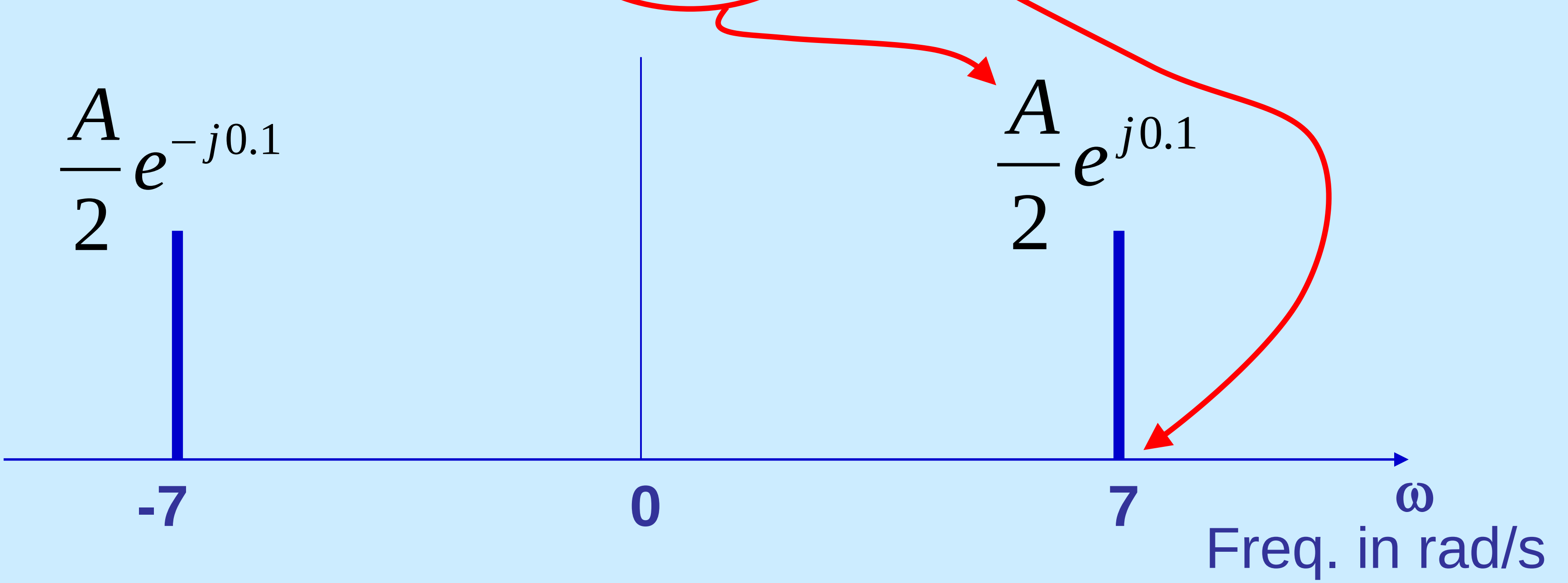
- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

  - One has a positive frequency
  - The other has negative freq.
  - Amplitude of each is half as big

GRAPHICAL SPECTRUM

$$A\cos(7t+0.1) = \frac{A}{2}e^{j0.1}e^{j7t} + \frac{A}{2}e^{-j0.1}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are labels

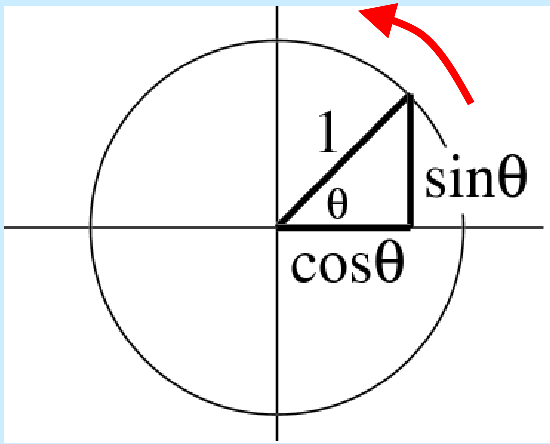
NEGATIVE FREQUENCY

- Is negative frequency real?
- Doppler Radar provides intuition
  - Police radar measures speed by using the Doppler shift principle
  - Let's assume 400Hz ←→60 mph
  - +400Hz means towards the radar
  - -400Hz means away (opposite direction)
  - Think of a train whistle

Negative Frequency is still a rotating phasor

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- View as vector rotating counterclockwise
  - $\theta = \omega t$
  - Angle changes vs. time



$$e^{-j\omega t} = e^{j(-\omega)t}$$

Negative frequency ➔ clockwise rotation

General form of sinusoid spectrum

- General form:

$$A\cos(\omega t + \varphi)$$

$$= \frac{A}{2} e^{j\varphi} e^{j\omega t} + \frac{A}{2} e^{-j\varphi} e^{-j\omega t}$$

  - Amplitudes are multiplied by ½
  - Complex amplitudes are complex conjugates
    - Called ***conjugate symmetry***

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

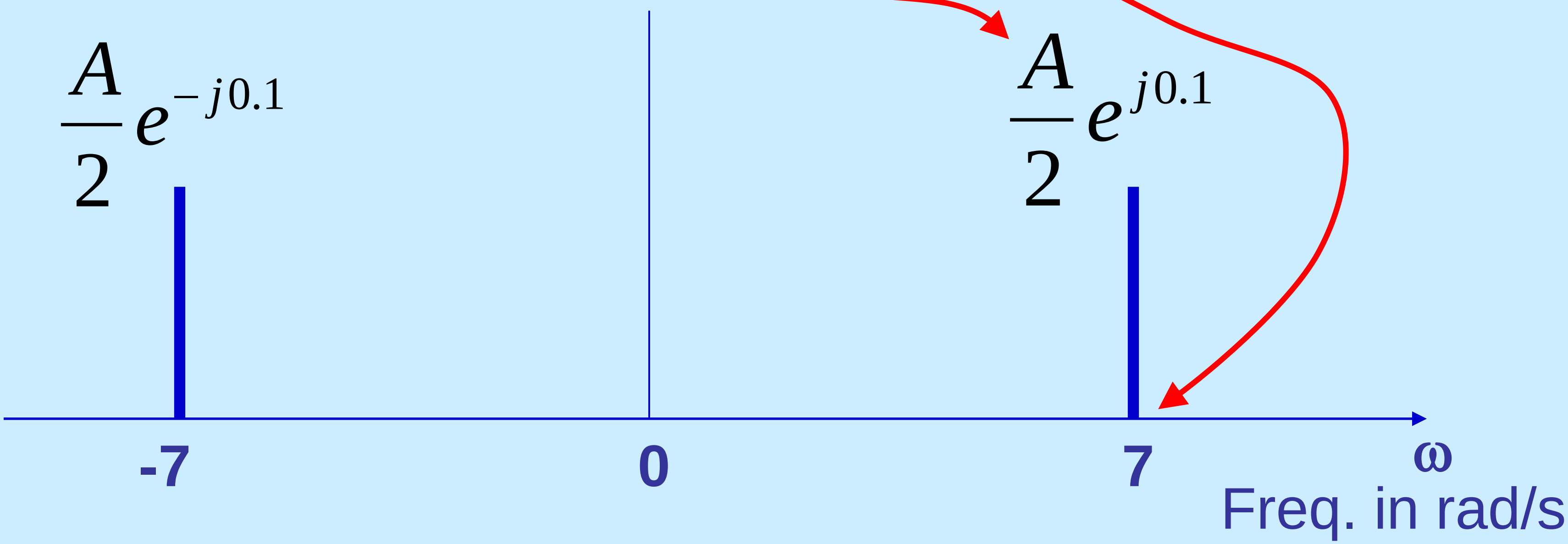
$$A\cos(7t+0.1) = \Re\left\{Ae^{j0.1}e^{j7t}\right\}$$

$$= \tfrac{A}{2}e^{j0.1}e^{j7t} + \tfrac{A}{2}e^{-j0.1}e^{-j7t}$$

  - One has a positive frequency
  - The other has negative freq.
  - Amplitude of each is half as big

Recall SPECTRUM of cosine

$$A\cos(7t+0.1) = \frac{A}{2}e^{j0.1}e^{j7t} + \frac{A}{2}e^{-j0.1}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are labels

REPRESENTATION of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

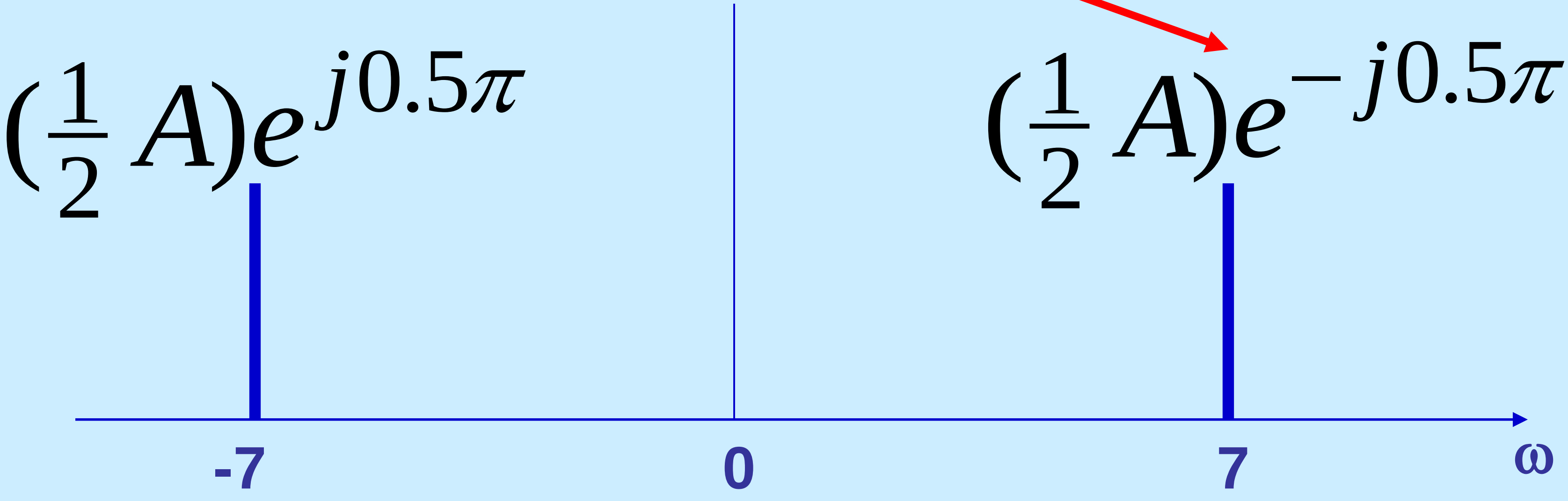
$$\frac{-1}{j} = j = e^{j0.5\pi}$$

- Positive freq. has phase = -0.5π
- Negative freq. has phase = $+0.5\pi$

GRAPHICAL Spectrum of sine

EXAMPLE of SINE **(has Phase of –π/2)**

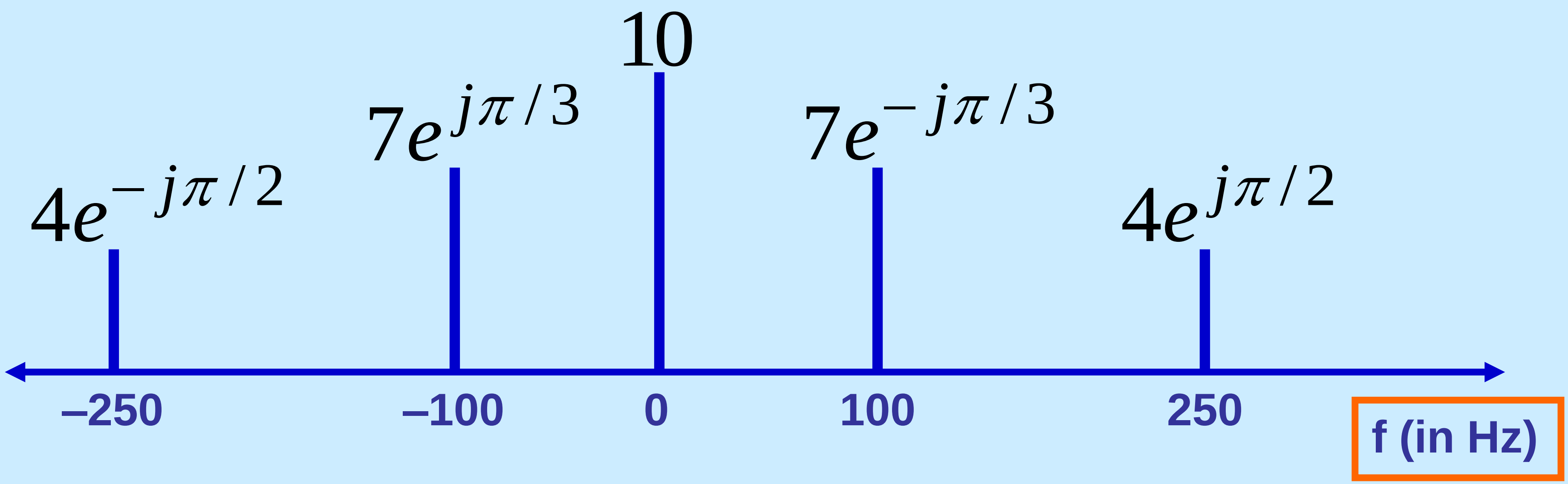
$$A\sin(7t) = \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are labels

SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal x(t)?

Gather (A,ω,ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0** Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 **0**
  - 7 $-\pi/3$
  - 4 $+\pi/2$

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has $\phi=0$ or π (for real ***x(t)***)

Add Spectrum Components-1

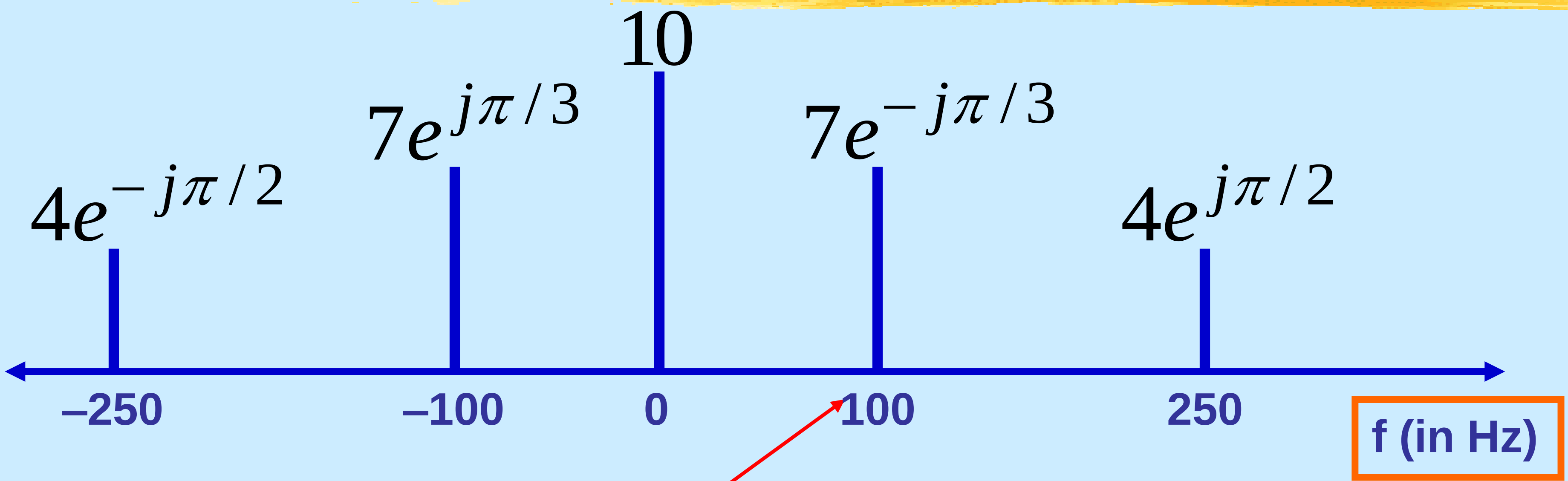
- Frequencies:
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 0
  - 7 $-\pi/3$
  - 4 $+\pi/2$

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Add Spectrum Components-2



$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

FINAL ANSWER

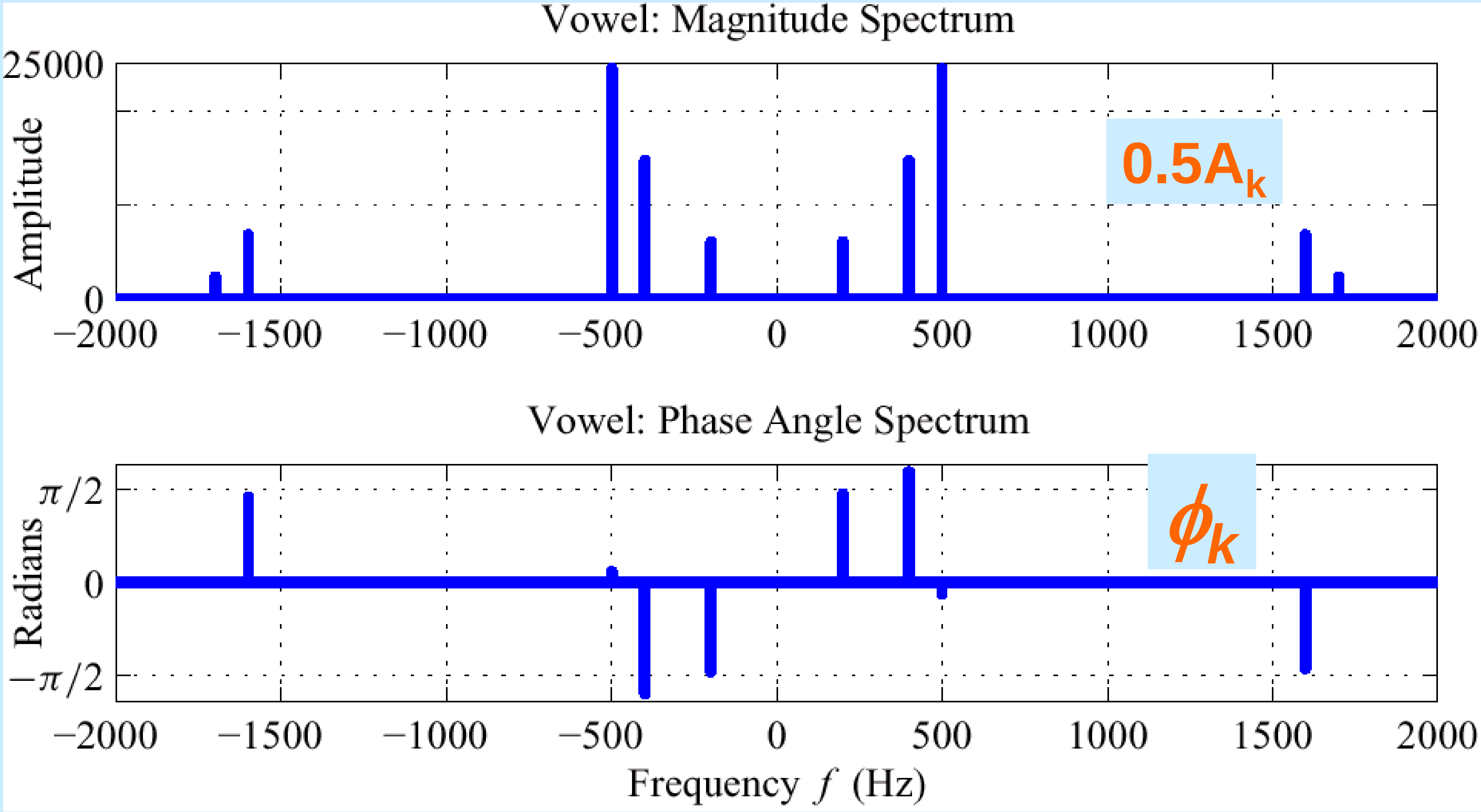
$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

So, we get the general form:

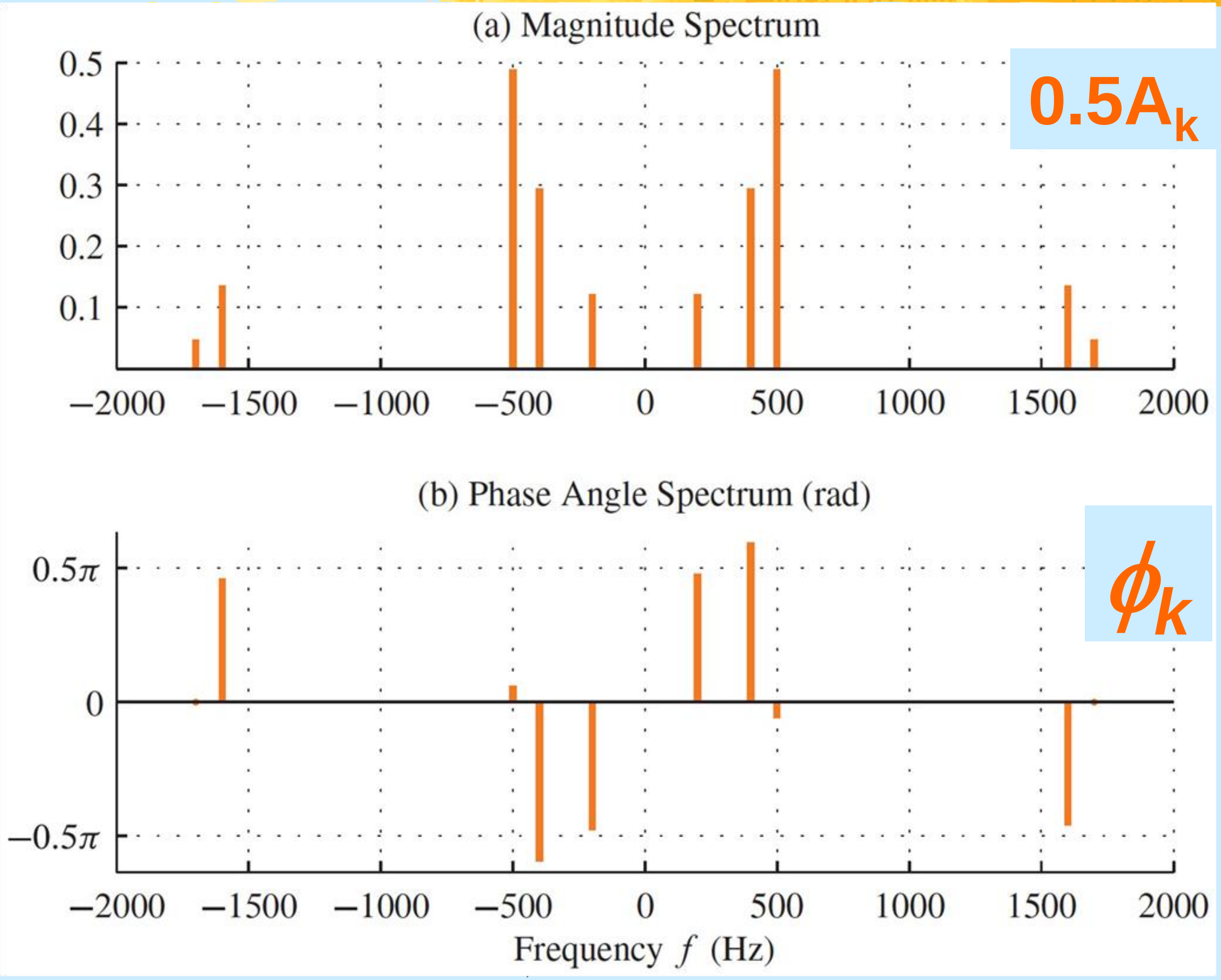
Example 3-1

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

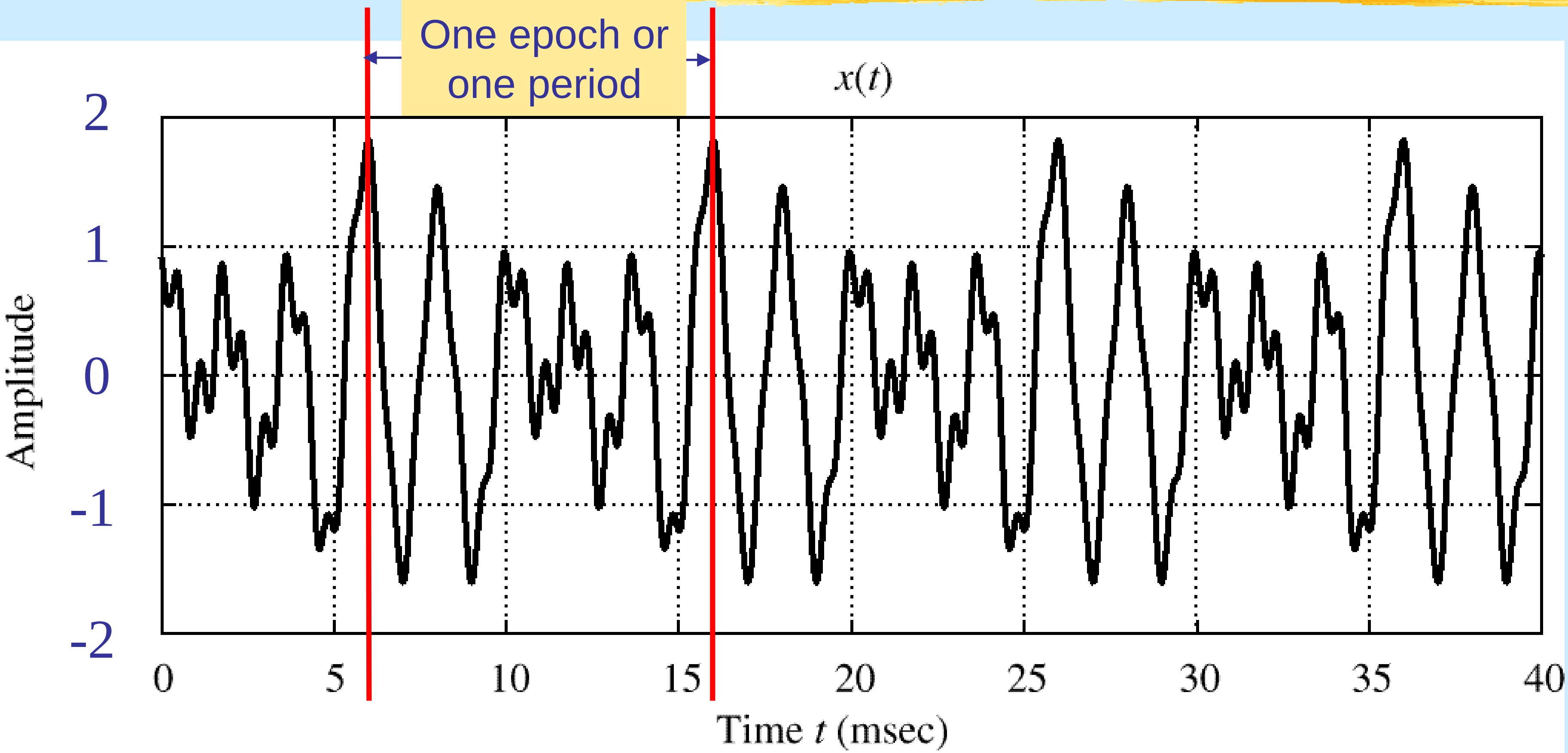
SPECTRUM of VOWEL (Polar Format)



SPECTRUM of VOWEL (Polar Format)



Vowel Waveform (sum of all 5 components)



Note that the period is 10 ms, which equals $1/f_0$