DSP First, 2/e

Lecture 6

Periodic Signals, Harmonics & Time-Varying Sinusoids

Section 3-4

READING ASSIGNMENTS

- This Lecture:
  - Chapter 3, Sections 3-2 and 3-4
  - Chapter 3, Sections 3-6 and 3-7

- Next Lectures:
  - **Fourier Series ANALYSIS**
  - Sections 3-4 and 3-5

LECTURE OBJECTIVES

- Signals with <u>HARMONIC</u> Frequencies
  - Add Sinusoids with $f_k = kf_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

<u>**Second Topic**</u>: FREQUENCY can change vs. TIME

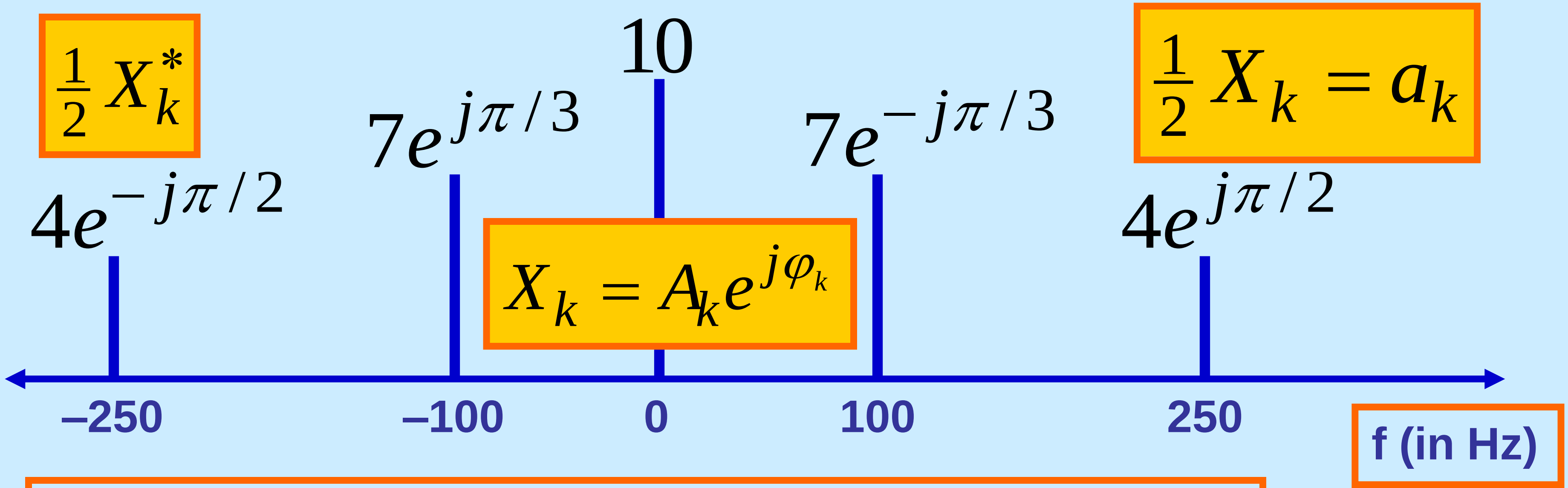
Introduce Spectrogram Visualization

(spectrogram.m) **(plotspec.m)**

Chirps: $x(t) = \cos(\alpha t^2)$

SPECTRUM DIAGRAM

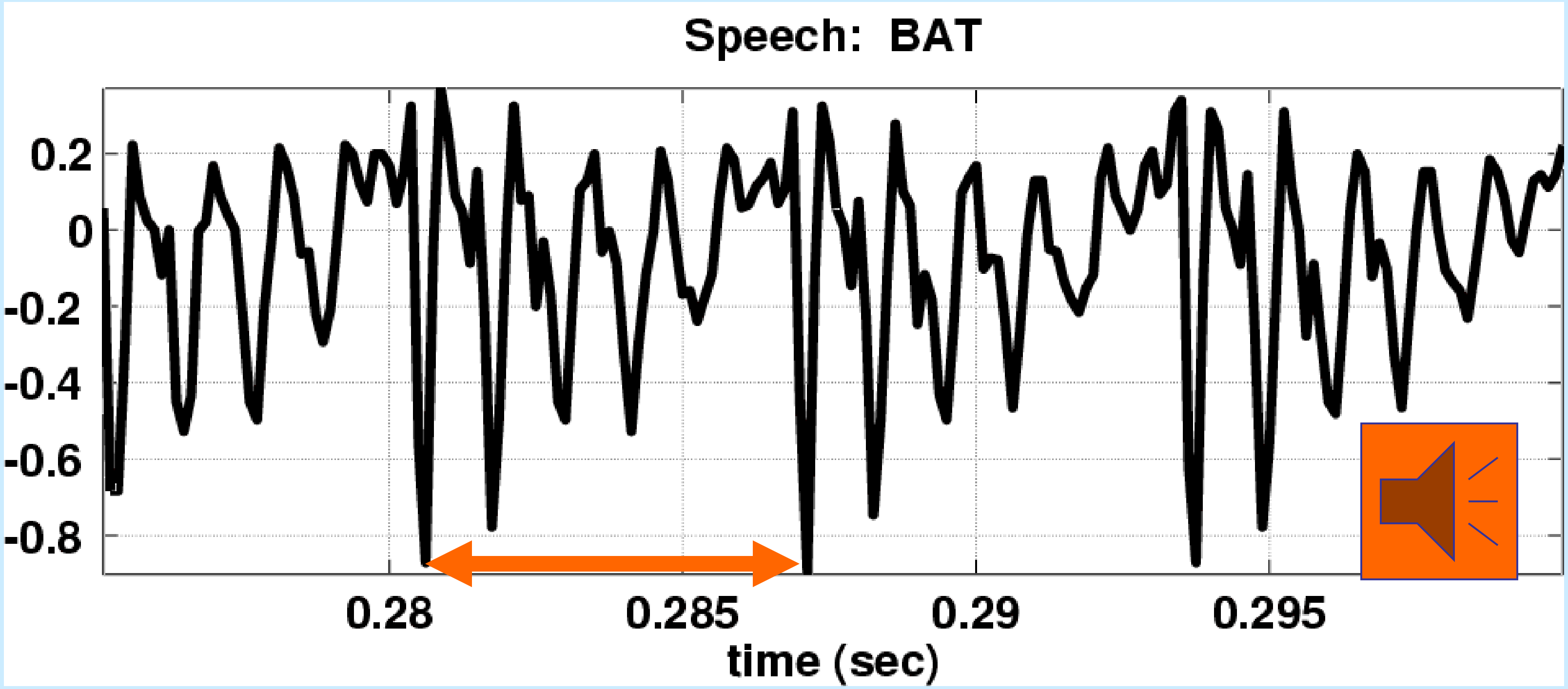
- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

SPECTRUM for PERIODIC ?

- Nearly Periodic in the Vowel Region
  - Period is (Approximately) T = 0.0065 sec



Harmonic Signal

Periodic signal : $x(t) = x(t+T)$

Can only have ***harmonic*** freqs : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$x(t)$ is periodic if $f_0 T = 1$

$$\cos(2\pi k f_0 (t+T) + \varphi_k) = \cos(2\pi k f_0 t + 2\pi k f_0 T + \varphi_k)$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T_0}$$

Largest f_0 such that

$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$

$f_0 = \text{fundamental Frequency}$

$f_k / f_0 = \text{integer, for all } k$

$T_0 = \text{fundamental Period}$

Main point:
for periodic signals, all spectral lines have frequencies that are *integer* multiples of the fundamental frequency

Harmonic Signal Spectrum

$$\textbf{\textit{Harmonic}}\ \text{freqs}: f_k = k f_0$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

Periodic Signal: Example

$$\omega_0 = 2\pi / T$$
$$\Rightarrow \omega_0 T = 2\pi$$

Fundamental frequency

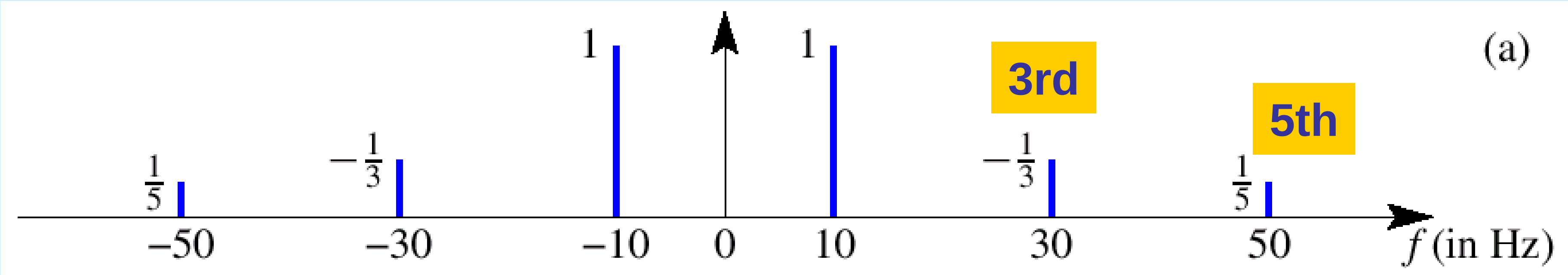
$$e^{j\omega_0(t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} = e^{j\omega_0 t} e^{j2\pi} = e^{j\omega_0 t}$$

$$e^{j7\omega_0(t+T)} = e^{j7\omega_0 t} e^{j14\pi} = e^{j7\omega_0 t}$$

$$x(t+T) = e^{j\omega_0(t+T)} + e^{j7\omega_0(t+T)} + e^{j10\omega_0(t+T)}$$

$$= e^{j\omega_0 t} + e^{j7\omega_0 t} + e^{j10\omega_0 t} = x(t)$$

Harmonic Spectrum (3 Freqs)

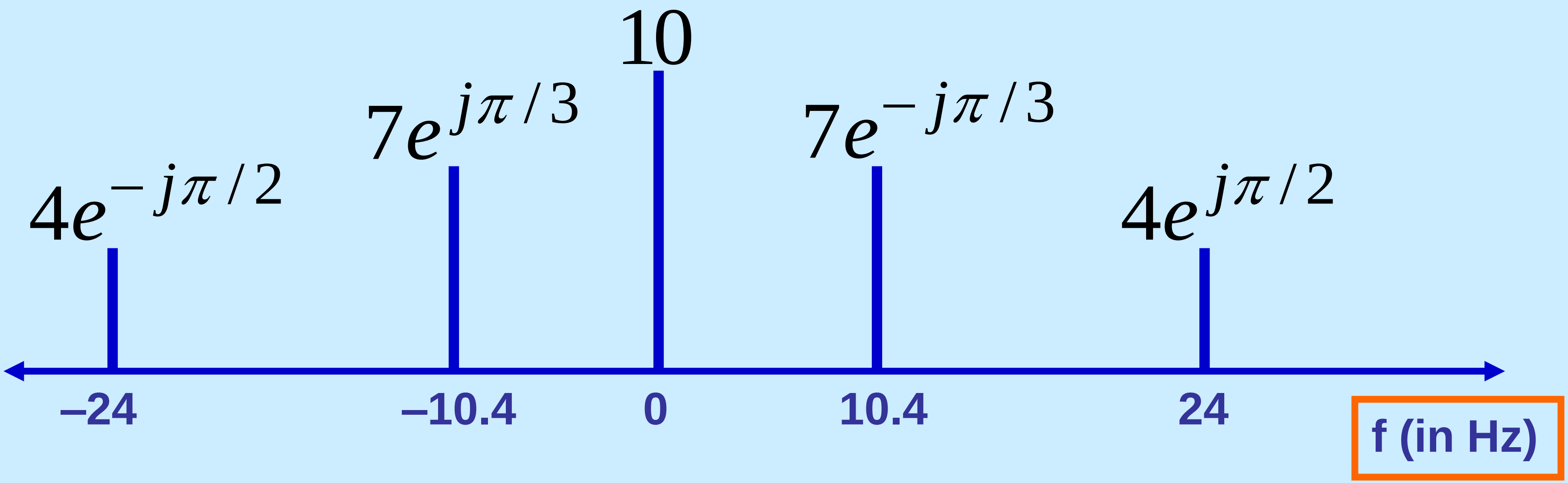


What is the fundamental frequency?

10 Hz

POP QUIZ: FUNDAMENTAL

- Here's another spectrum:



What is the fundamental frequency?

(0.1)GCD(104,240) = (0.1)(8)=0.8 Hz

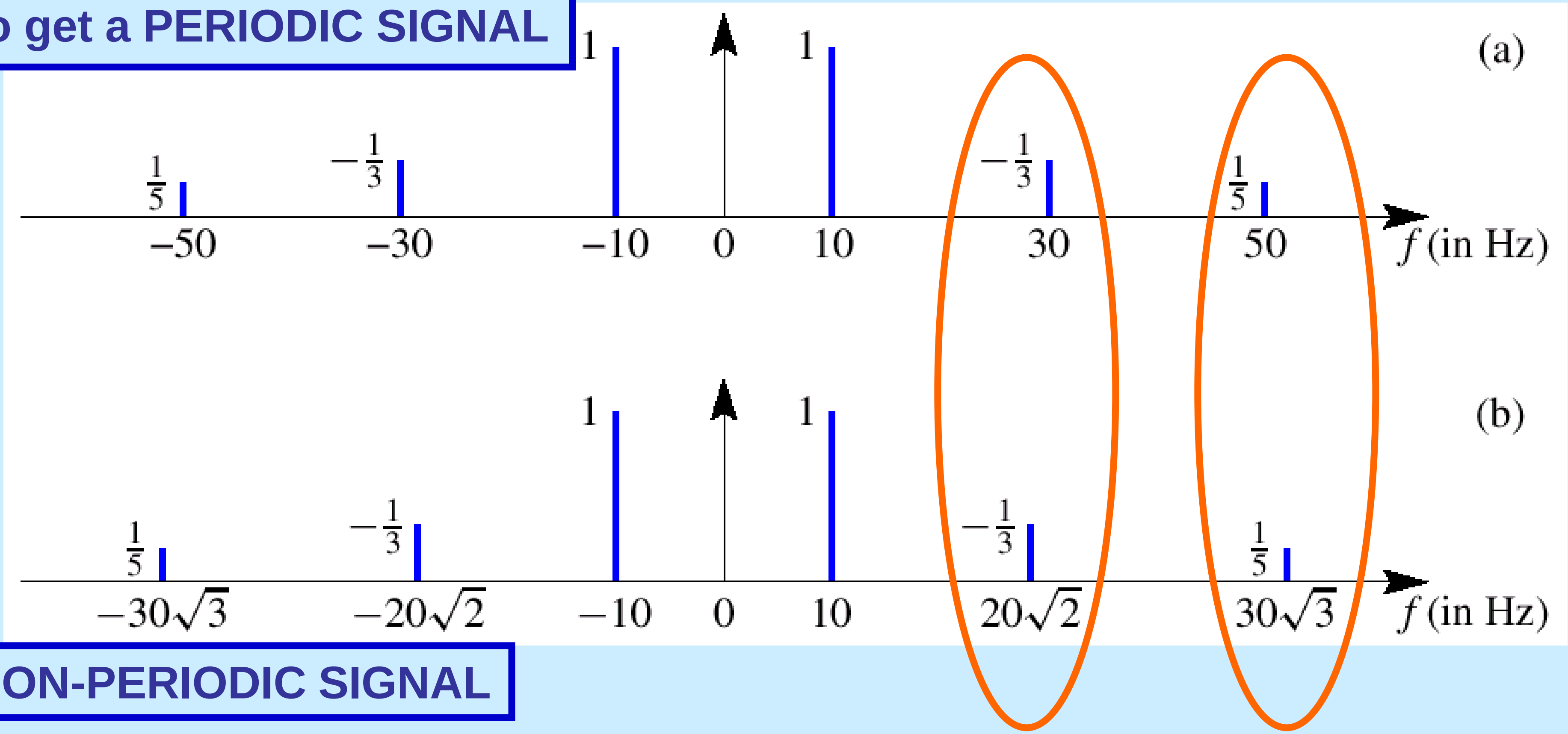
Fundamental Frequency

- Multiply and divide by 10
- 104, 240   8 divides 13, 30 –
- Now divide by 10 > 0.8

- 0.8, 1.6, 2.4,…8, 8.8, 9.6, **10.4**,… 16, …
-     **24**,24.8, …

Harmaonic vs IRRATIONAL SPECTRUM

SPECIAL RELATIONSHIP to get a PERIODIC SIGNAL

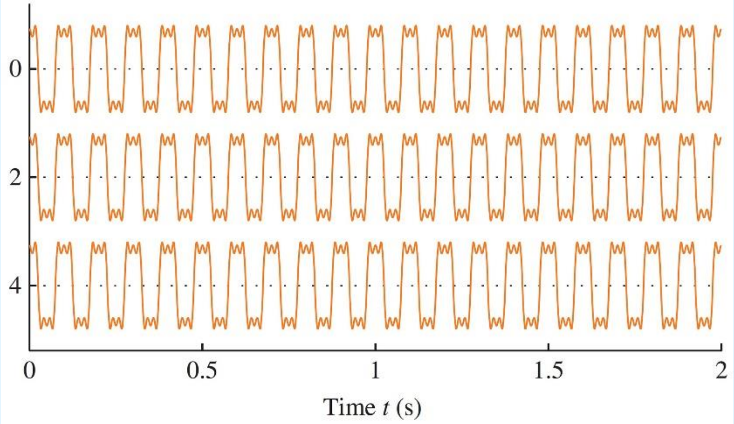


NON-PERIODIC SIGNAL

Example of a Periodic Signal (1 of 3)

Figure 3-16: Sum of three cosine waves with harmonic frequencies. The spectrum is shown in Figure 3-18(a), and the fundamental frequency of $x_h(t)$ is 10 Hz.

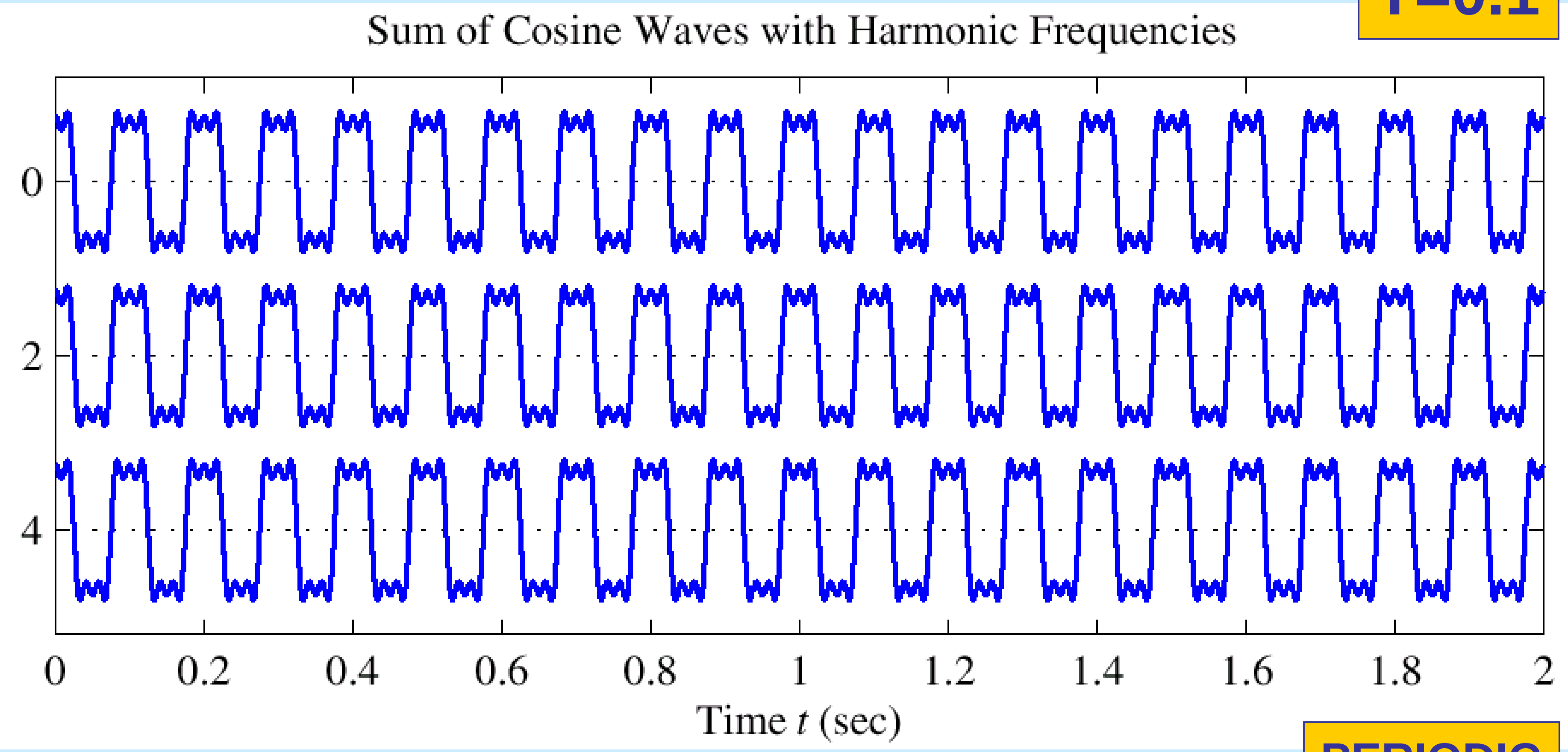
$$X(t) = 2\cos(20\pi t) - 2/3\cos(20\pi(3)t) + 2/5\cos(20\pi(5)t)$$



Harmonic Signal (3 Freqs)

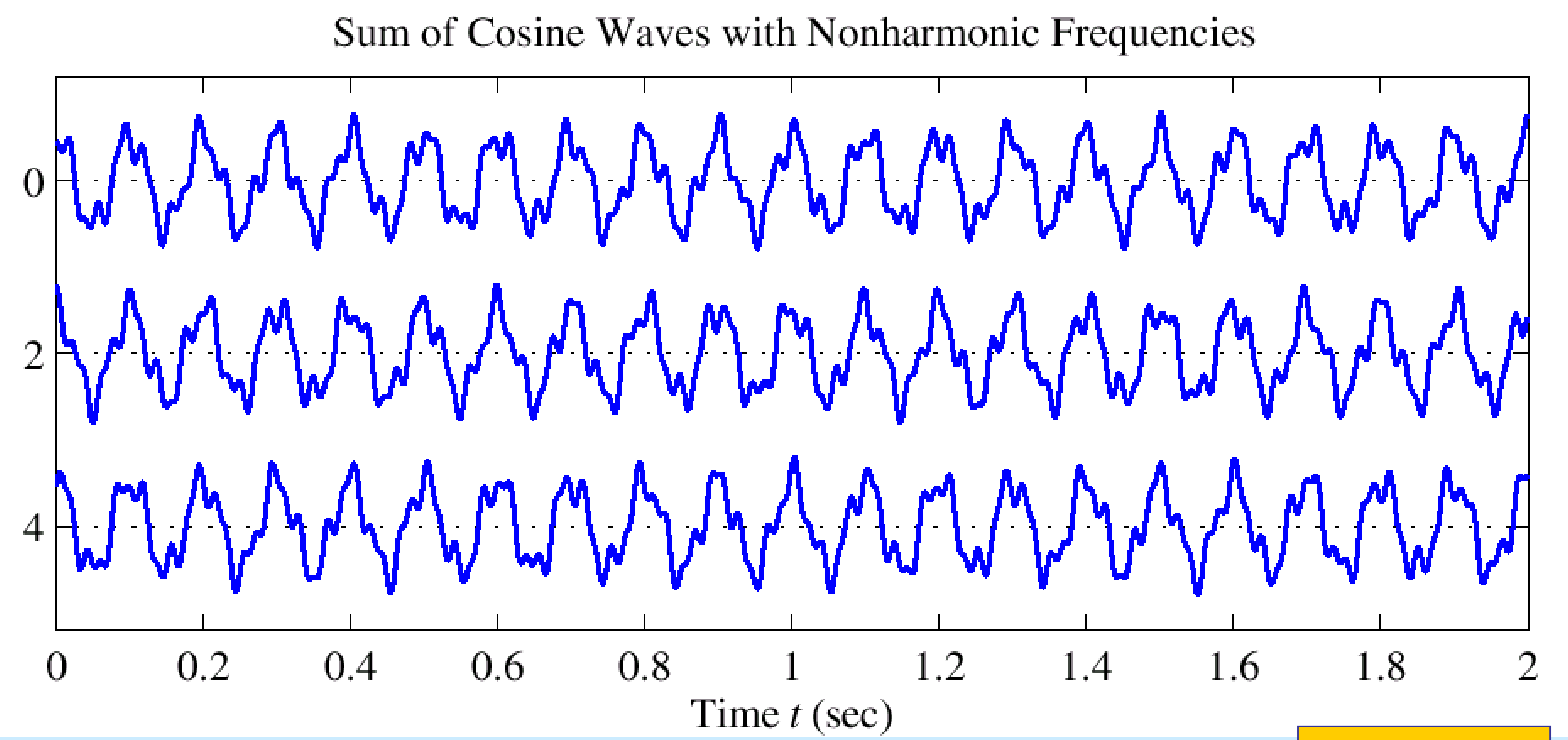
Sum of f_0, 3 f_0 , 5 f_0 $f_0 = 10$ Hz

T=0.1



PERIODIC

NON-Harmonic Signal



NOT PERIODIC

FREQUENCY ANALYSIS

- **<u>Now, a much HARDER problem</u>**
- Given a recording of a song, have the computer write the music
- Can a machine extract frequencies?
  - Yes, if we COMPUTE the spectrum for x(t)
    - During short intervals

Time-Varying FREQUENCIES Diagram

Frequency is the vertical axis

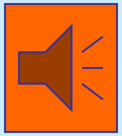
A-440

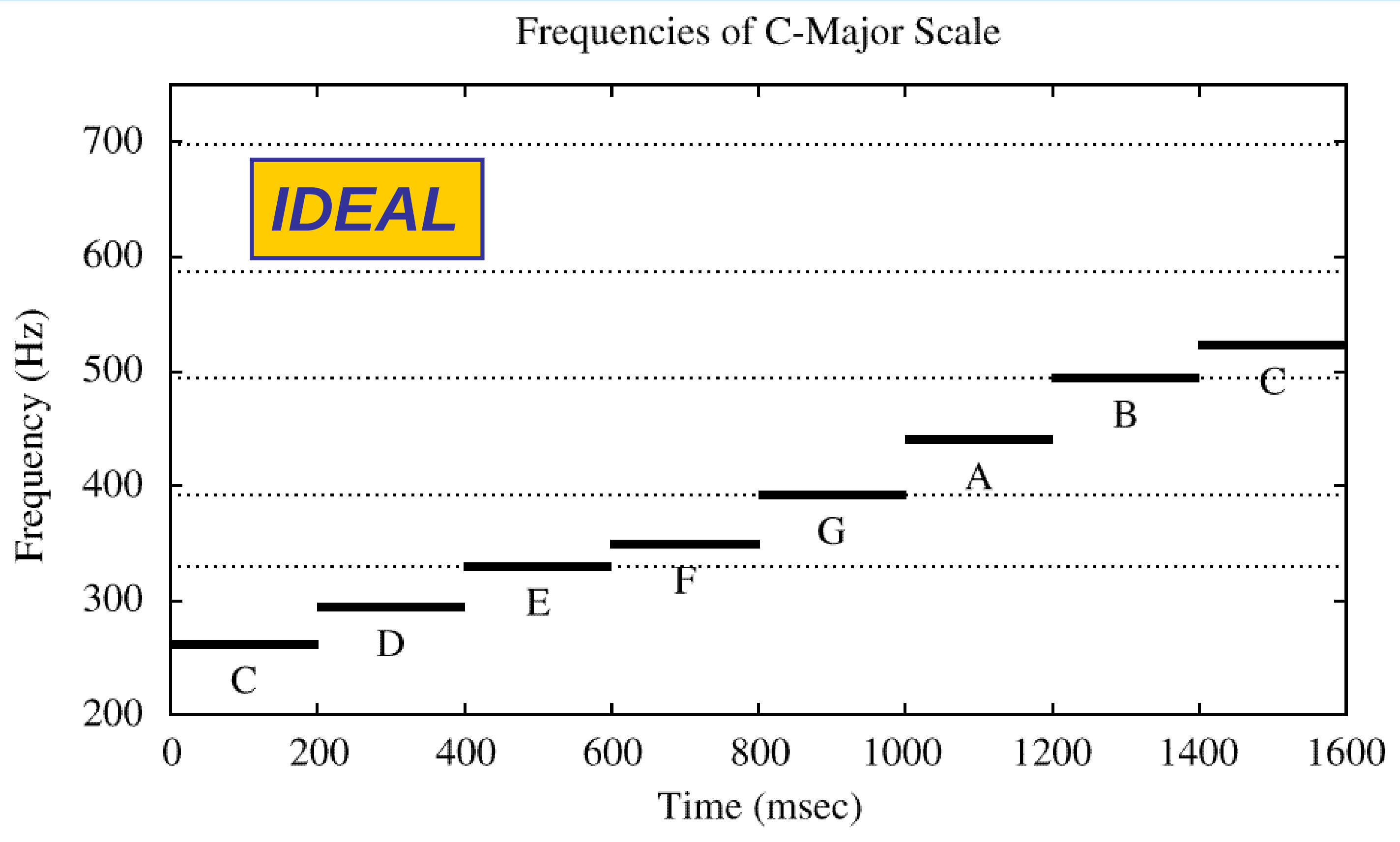
Figure 3.18 Sheet-music notation is a time–frequency diagram.

Time is the horizontal axis

SIMPLE TEST SIGNAL

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note





SPECTROGRAM

- SPECTROGRAM Tool
  - MATLAB function is **`spectrogram.m`**
  - SP-First has **`plotspec.m`** & **`spectgr.m`**
- **ANALYSIS** program
  - Takes x(t) as input
  - Produces spectrum values X_k
  - Breaks x(t) into SHORT TIME SEGMENTS
    - Then uses the FFT (Fast Fourier Transform)

AM Radio Signal

- Same form as BEAT Notes, but <u>higher in freq</u>

$$\cos(2\pi(\underline{660})t)\sin(2\pi(12)t)$$

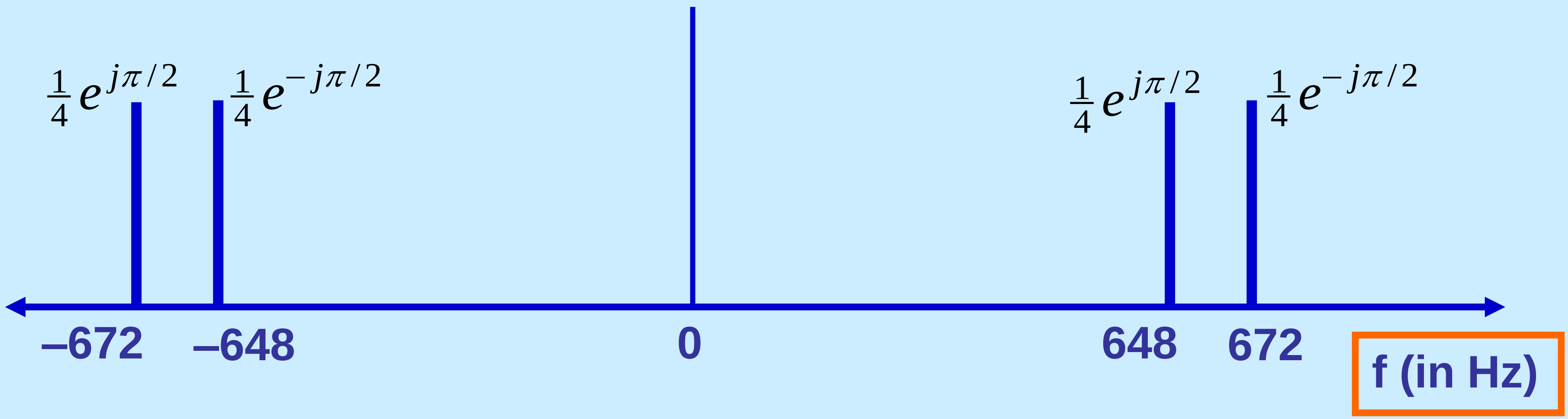
$$\frac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

$$\tfrac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})$$

SPECTRUM of AM (Amplitude Modulation)

- **<u>SUM</u>** of 4 complex exponentials:



What is the fundamental frequency?

648 Hz ? **24 Hz ?**

Time-Varying Frequency

- Frequency can change vs. time
  - **Continuously, not stepped**
- **FREQUENCY MODULATION (FM)**

$$x(t) = \cos(2\pi f_c t + v(t))$$

VOICE

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)

New Signal: Linear FM

- Called Chirp Signals (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\alpha t^2 + 2\pi f_0 t + \varphi)$$

- Freq will change LINEARLY vs. time
  - Example of Frequency Modulation (FM)
  - Define “instantaneous frequency”