

DSP First, 2/e



Lecture 7C

Fourier Series Examples:

Common Periodic Signals

READING ASSIGNMENTS



- This Lecture:
 - Appendix C, Section C-2
 - Various Fourier Series
 - Pulse Waves
 - Triangular Wave
 - Rectified Sinusoids (also in Ch. 3, Sect. 3-5)

LECTURE OBJECTIVES

- Use the Fourier Series Integral

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi k / T_0)t} dt$$

- Derive Fourier Series coeffs for common periodic signals
- Draw spectrum from the Fourier Series coeffs
 - a_k is Complex Amplitude for k-th Harmonic

Harmonic Signal is Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k \underline{F_0} t}$$

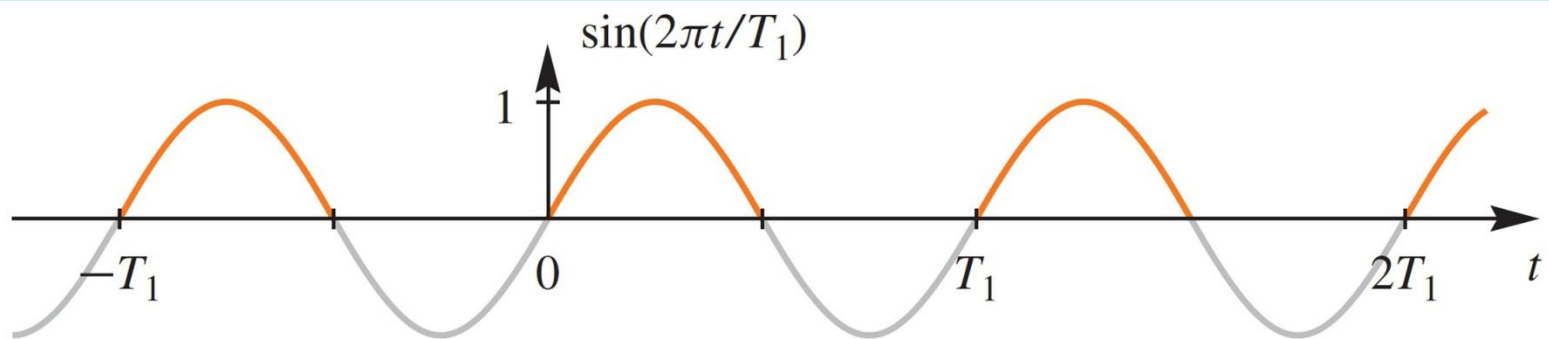
Sums of Harmonic
complex exponentials
are Periodic signals

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

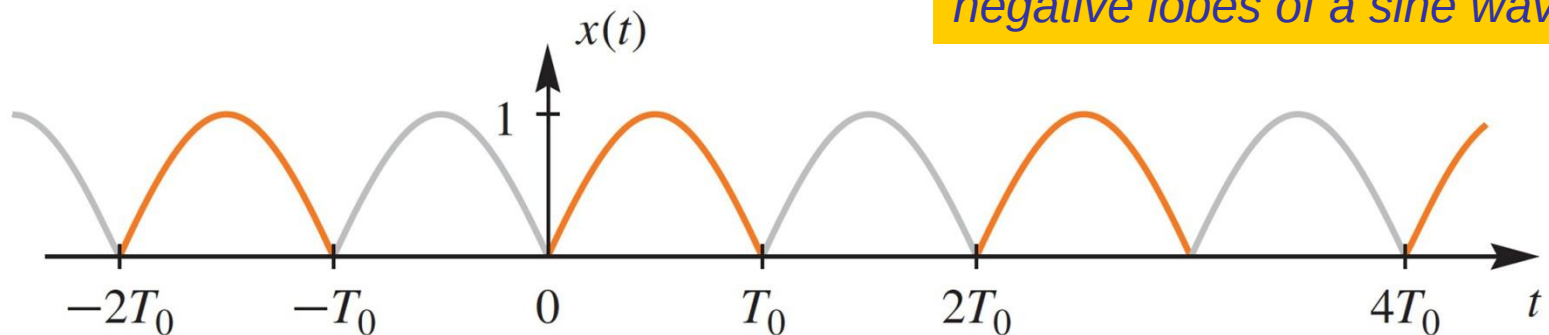
Recall FWRS

$$x(t) = |\sin(2\pi t / T_1)| \quad \text{Period is } T_0 = \frac{1}{2} T_1$$



(a)

Absolute value flips the negative lobes of a sine wave



(b)

FWRS Fourier Integral $\rightarrow \{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} \sin\left(\frac{\pi}{T_0} t\right) e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{1}{T_0} \int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))} \Bigg|_0^{T_0} - \frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))} \Bigg|_0^{T_0}$$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

Full-Wave Rectified Sine

$$x(t) = |\sin(2\pi t / T_1)|$$

$$\text{Period : } T_0 = \frac{1}{2} T_1$$

$$\Rightarrow x(t) = |\sin(\pi t / T_0)|$$

FWRS Fourier Coeffs: a_k

- a_k is a function of k
 - Complex Amplitude for k -th Harmonic

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

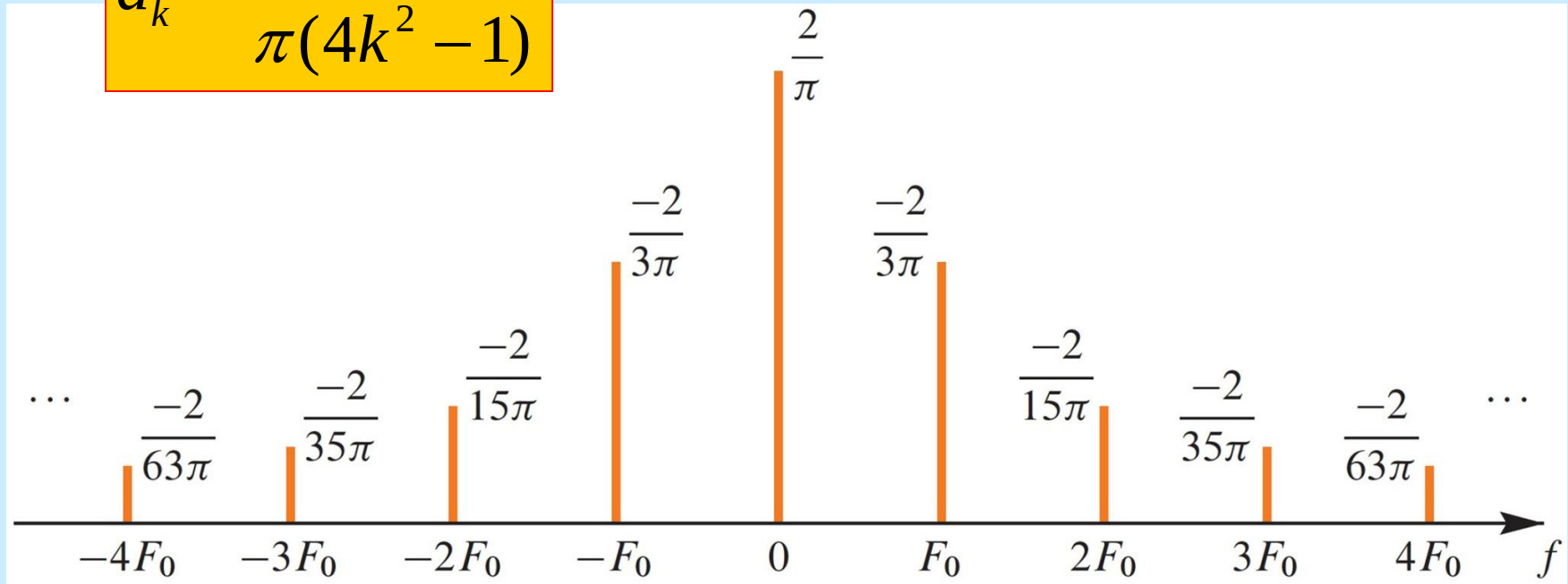
- Does not depend on the period, T_0
- DC value is $a_0 = 2 / \pi = 0.6336$

Spectrum from Fourier Series

Plot a_k for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$$F_0 = 1/T_0 \quad \text{and} \quad \omega_0 = 2\pi F_0$$



Fourier Series Synthesis

- HOW do you APPROXIMATE $x(t)$?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^N a_k e^{j2\pi k F_0 t}$$

$$a_{-k} = a_k^* \text{ when } x(t) \text{ is real}$$

Reconstruct From Finite Number of Harmonic Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

$$T_0 = 10 \text{ ms}$$

$$\Rightarrow F_0 = 100 \text{ Hz}$$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$$a_0 = 2 / \pi = 0.6336$$

$$x_N(t) = a_0 + \sum_{k=1}^N \left\{ a_k e^{j2\pi k F_0 t} + a_k^* e^{-j2\pi k F_0 t} \right\}$$

How close is $x_N(t)$ to $x(t) = |\sin(\pi t / T_0)|$?

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{-2}{\pi(4k^2-1)} = \frac{-2}{\pi(4k^2-1)} \quad \text{is real - valued}$$

$$\begin{aligned} x_N(t) &= \sum_{k=-N}^N \frac{-2}{\pi(4k^2-1)} e^{jk\omega_0 t} \\ &= \frac{-2}{-\pi} + \frac{-2}{\pi(4-1)} e^{j\omega_0 t} + \frac{-2}{\pi(4-1)} e^{-j\omega_0 t} + \frac{-2}{\pi(16-1)} e^{j2\omega_0 t} + \frac{-2}{\pi(16-1)} e^{-j2\omega_0 t} \dots \\ &= \frac{2}{\pi} - \frac{2}{3\pi} e^{j\omega_0 t} - \frac{2}{3\pi} e^{-j\omega_0 t} - \frac{2}{15\pi} e^{j2\omega_0 t} - \frac{2}{15\pi} e^{-j2\omega_0 t} + \dots \\ &= \frac{2}{\pi} - \frac{4}{3\pi} \cos(\omega_0 t) - \frac{4}{15\pi} \cos(2\omega_0 t) - \dots - \frac{4}{(4N^2-1)\pi} \cos(N\omega_0 t) \end{aligned}$$

- Plots for N=4 and N=9 are shown next
- Excellent Approximation for N=9

Reconstruct From Finite Number of Spectrum Components

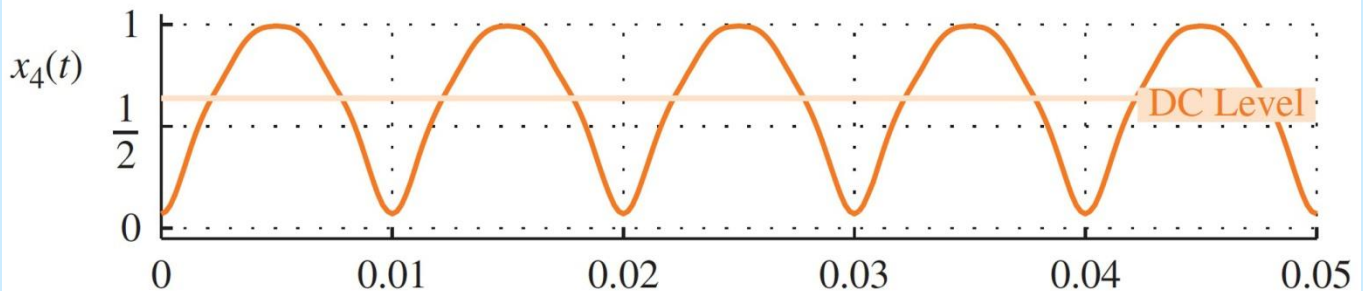
Full-Wave Rectified Sinusoid $x(t) = \left| \sin(\pi t / T_0) \right|$

$$T_0 = 10 \text{ ms}$$

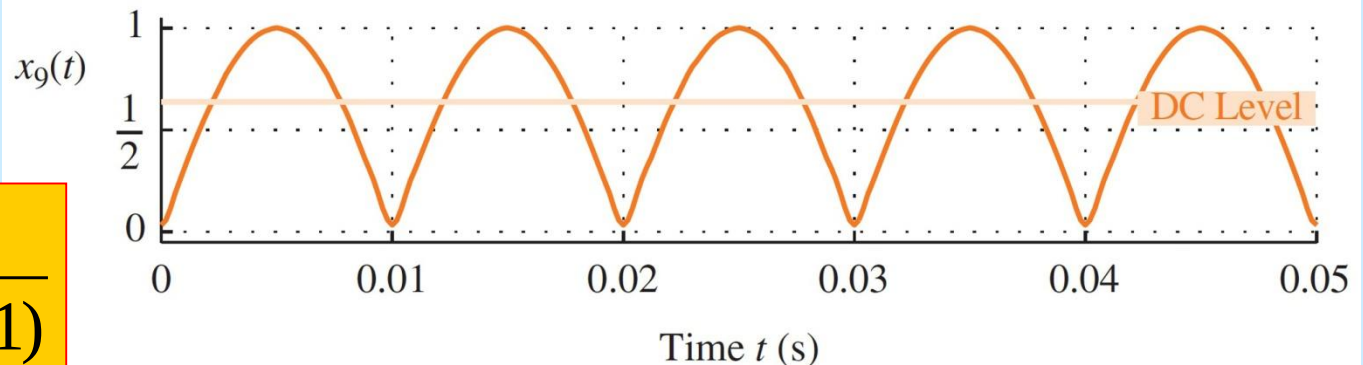
$$\Rightarrow F_0 = 100 \text{ Hz}$$

$$a_0 = 2 / \pi = 0.6336$$

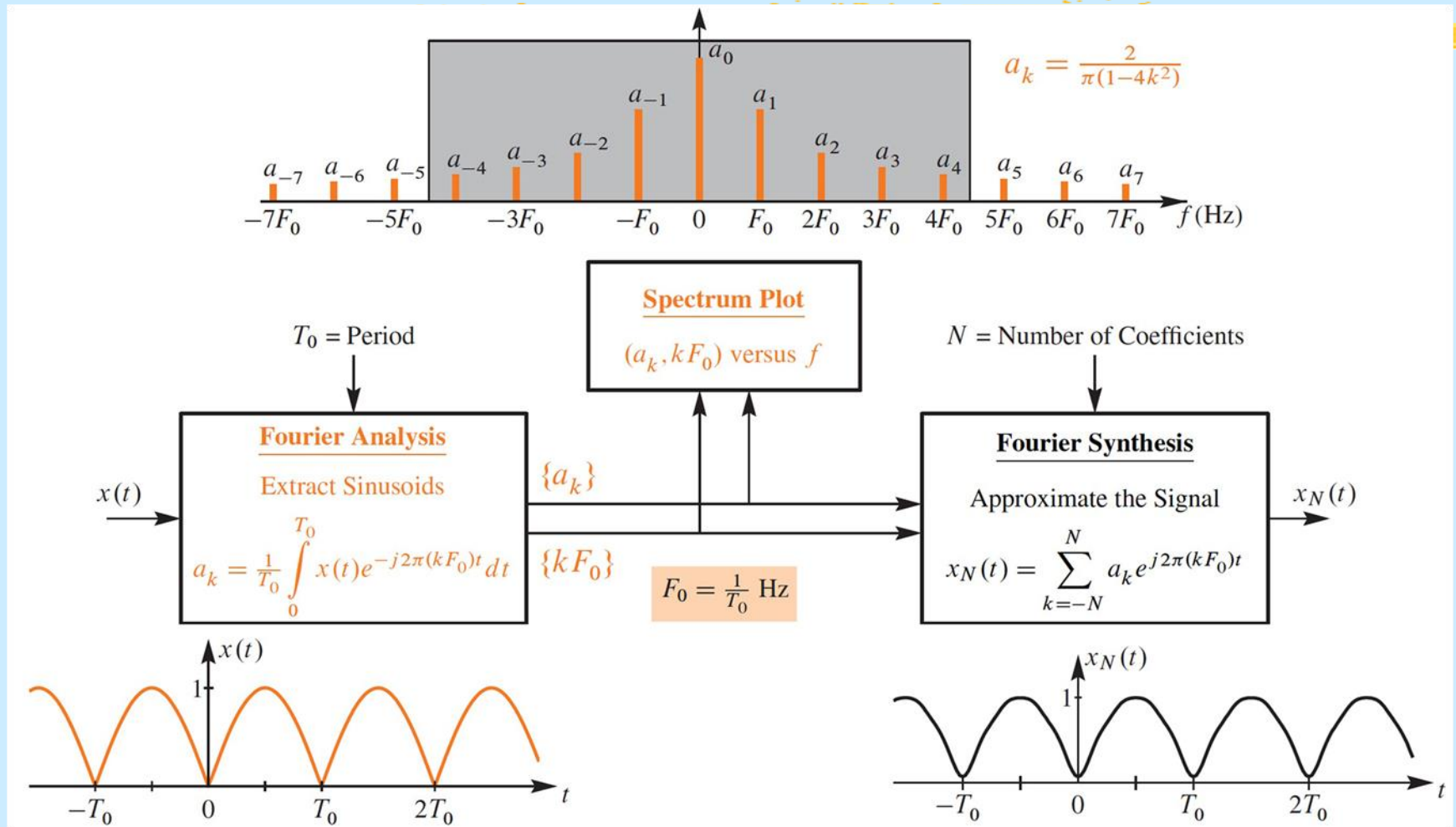
(a) Sum of DC and 1st through 4th Harmonics



(b) Sum of DC and 1st through 9th Harmonics



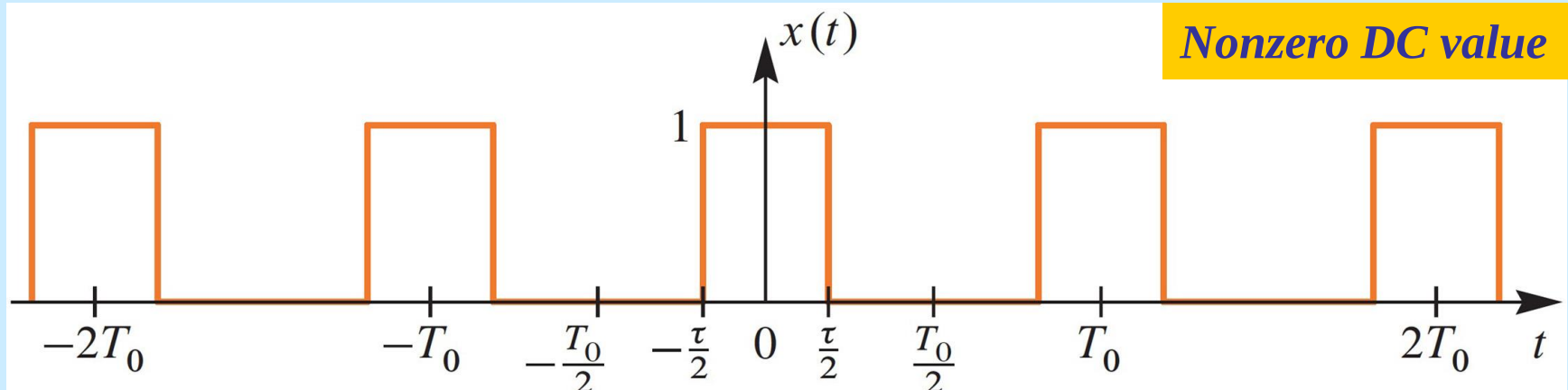
Fourier Series Synthesis



PULSE WAVE SIGNAL GENERAL FORM

Defined over one period

$$x(t) = \begin{cases} 1 & 0 \leq |t| < \tau/2 \\ 0 & \tau/2 \leq |t| \leq T_0/2 \end{cases}$$



Pulse Wave $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j(2\pi/T_0)(k)t} dt$$

$$a_k = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} 1 e^{-j(2\pi/T_0)kt} dt$$

$$= \left(\frac{1}{T_0} \right) \frac{e^{-j(2\pi/T_0)kt} \Big|_{-\tau/2}^{\tau/2}}{-j(2\pi/T_0)k} = \frac{e^{-j(2\pi/T_0)k(\tau/2)} - e^{-j(2\pi/T_0)k(-\tau/2)}}{-j(2\pi)k}$$

$$= \frac{e^{j(\pi/T_0)k(\tau)} - e^{-j(\pi/T_0)k(\tau)}}{(j2)\pi k} = \frac{\sin(\pi k \tau / T_0)}{\pi k}$$

General Pulse Wave

$$x(t) = \begin{cases} 1 & 0 \leq |t| < \tau/2 \\ 0 & \tau/2 \leq |t| \leq T_0/2 \end{cases}$$

Pulse Wave $\{a_k\} = \text{sinc}$

Pulse Wave

$$a_k = \frac{\sin(\pi k \tau / T_0)}{\pi k} \quad k = 0, \pm 1, \pm 2, \dots$$

Double check the DC coefficient:

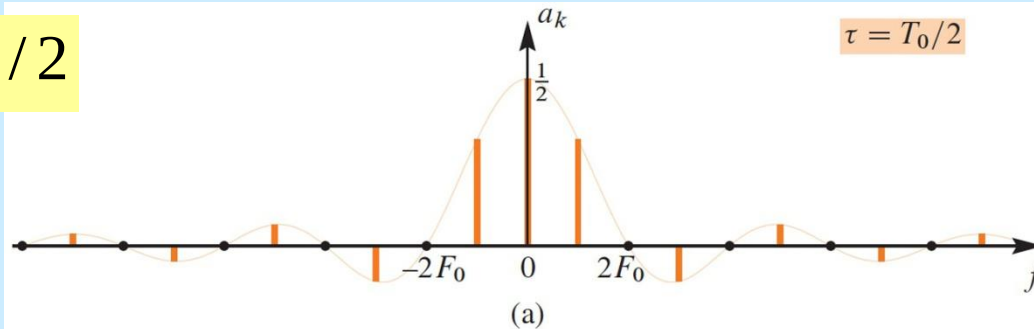
$$\begin{aligned} a_0 &= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} 1 e^{-j(2\pi/T_0)(0)t} dt \\ &= \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} 1 dt = \frac{1}{T_0} \left[\frac{\tau}{2} - \frac{-\tau}{2} \right] = \frac{\tau}{T_0} \end{aligned}$$

$$\text{Note, } \lim_{k \rightarrow 0} \frac{\sin(\pi k \tau / T_0)}{\pi k} \rightarrow \frac{\tau}{T_0}$$

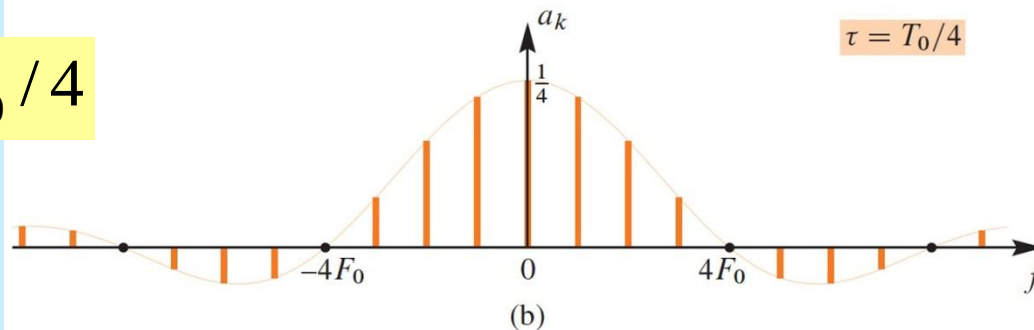
Where do you go if sick?

PULSE WAVE SPECTRA

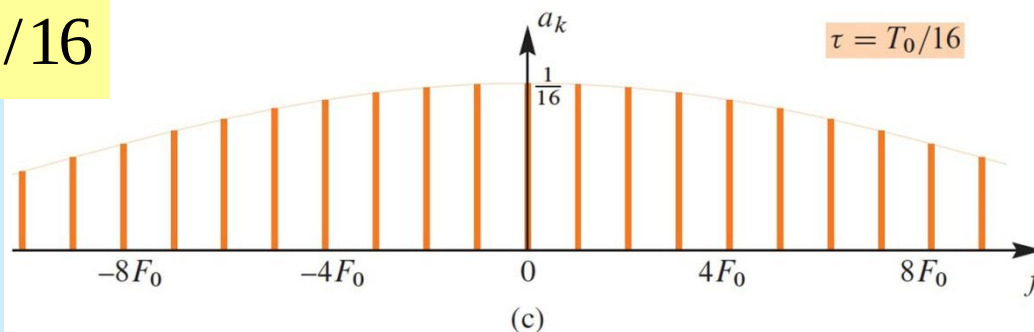
$$\tau = T_0 / 2$$



$$\tau = T_0 / 4$$



$$\tau = T_0 / 16$$



50% duty-cycle (Square) Wave

$$\tau = T_0 / 2 \Rightarrow a_k = \frac{\sin(\pi k (T_0 / 2) / T_0)}{\pi k} = \frac{\sin(\pi k / 2)}{\pi k} \quad k = 0, \pm 1, \pm 2, \dots$$

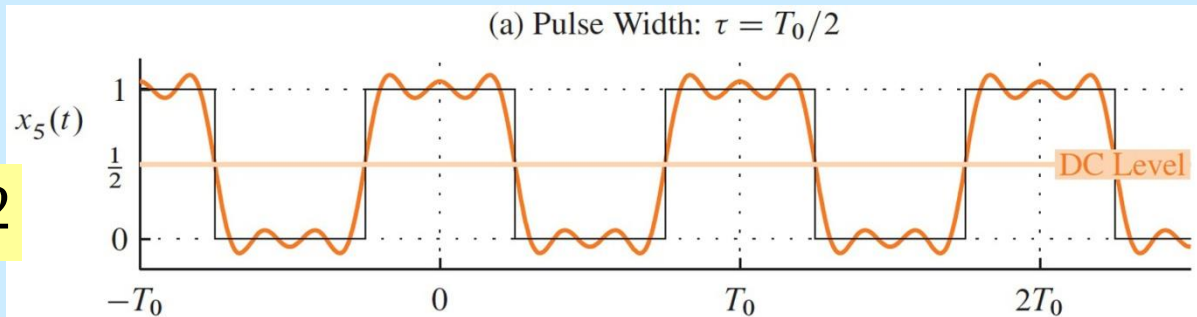
- Thus, $a_k=0$ when k is odd
 - Phase is zero because $x(t)$ is centered at $t=0$
 - different from a previous case

PulseWave starting at $t=0$

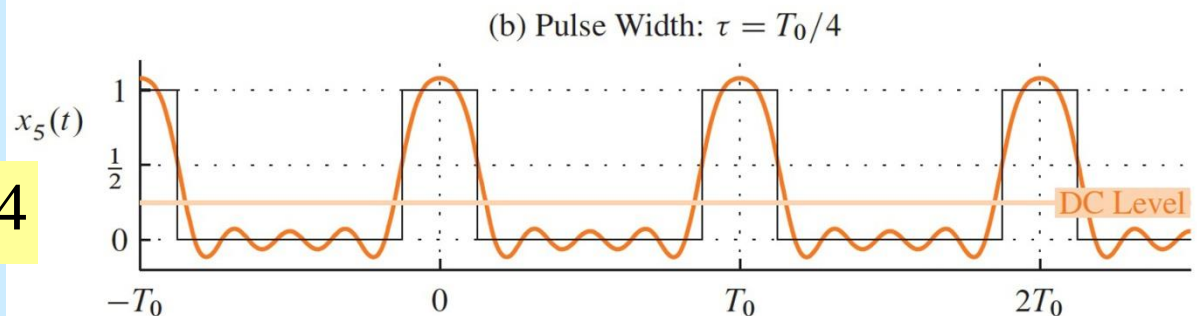
$$x(t) = \begin{cases} 1 & 0 \leq |t| < \tau \\ 0 & \tau \leq |t| \leq T_0 \end{cases} \leftrightarrow a_k = \frac{1 - (-1)^k}{j2\pi k} = \begin{cases} \frac{1}{j\pi k} & k = \pm 1, \pm 3, \dots \\ 0 & k = \pm 2, \pm 4, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

PULSE WAVE SYNTHESIS with first 5 Harmonics

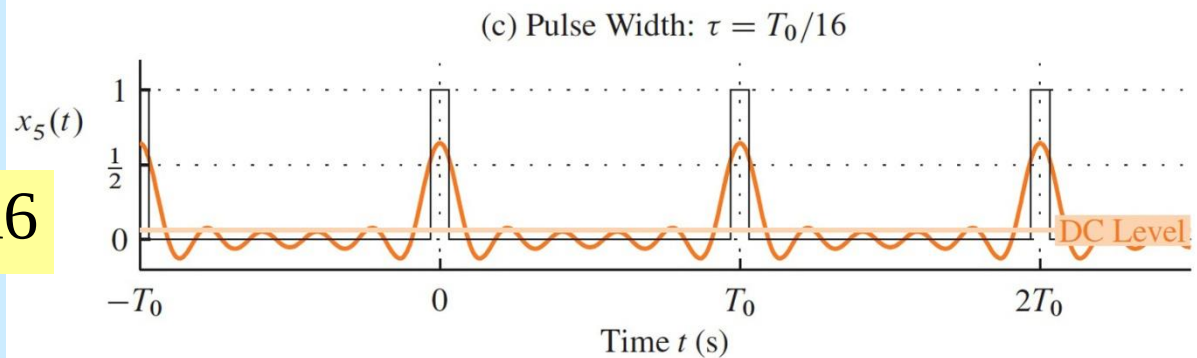
$$\tau = T_0 / 2$$



$$\tau = T_0 / 4$$



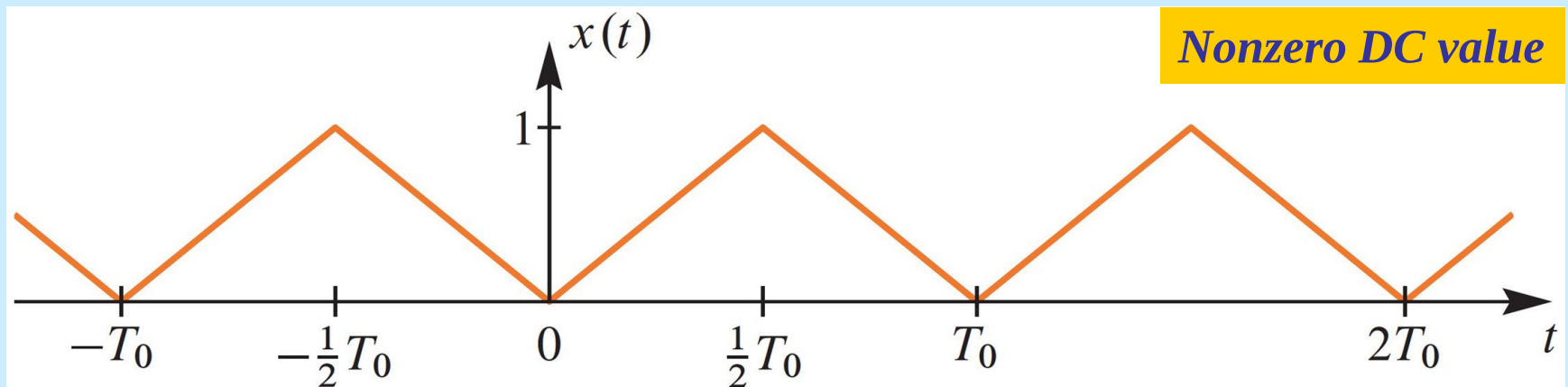
$$\tau = T_0 / 16$$



Triangular Wave: Time Domain

Defined over one period

$$x(t) = |2t / T_0| \quad \text{for } -T_0 / 2 < t \leq T_0 / 2$$



Triangular Wave $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j(2\pi/T_0)kt} dt$$

Triangular Wave

$$x(t) = |2t / T_0|$$

$$\omega_0 = 2\pi / T_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |2t / T_0| e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{2}{T_0^2} \int_0^{T_0/2} t e^{-j\omega_0 kt} dt + \frac{2}{T_0^2} \int_{-T_0/2}^0 (-t) e^{-j\omega_0 kt} dt$$

$$\begin{aligned} a_k &= \frac{2}{T_0^2} e^{-j\omega_0 kt} \left(\frac{j\omega_0 kt + 1}{\omega_0^2 k^2} \right) \Big|_0^{T_0/2} - \frac{2}{T_0^2} e^{-j\omega_0 kt} \left(\frac{j\omega_0 kt + 1}{\omega_0^2 k^2} \right) \Big|_{-T_0/2}^0 \\ &= \frac{2}{T_0^2} \left(e^{-j\omega_0 k T_0/2} \left(\frac{j\omega_0 k T_0/2 + 1}{\omega_0^2 k^2} \right) - \frac{1}{\omega_0^2 k^2} \right) - \frac{2}{T_0^2} \left(\frac{1}{\omega_0^2 k^2} - e^{j\omega_0 k T_0/2} \left(\frac{-j\omega_0 k T_0/2 + 1}{\omega_0^2 k^2} \right) \right) \end{aligned}$$

use the indefinite integral $\int t e^{-j\omega_0 kt} dt = e^{-j\omega_0 kt} \left(\frac{j\omega_0 kt + 1}{\omega_0^2 k^2} \right)$

Triangular Wave $\{a_k\}$

$$a_k = \frac{2}{T_0^2} \left[e^{-j\omega_0 k T_0 / 2} \left(\frac{j\omega_0 k T_0 / 2 + 1}{\omega_0^2 k^2} \right) - \frac{1}{\omega_0^2 k^2} \right] - \frac{2}{T_0^2} \left[\frac{1}{\omega_0^2 k^2} - e^{j\omega_0 k T_0 / 2} \left(\frac{-j\omega_0 k T_0 / 2 + 1}{\omega_0^2 k^2} \right) \right]$$

$$= \frac{-4}{T_0^2 \omega_0^2 k^2} + \frac{2}{T_0^2} e^{-j\pi k} \left(\frac{j\pi k + 1}{\omega_0^2 k^2} \right) + \frac{2}{T_0^2} e^{j\pi k} \left(\frac{-j\pi k + 1}{\omega_0^2 k^2} \right)$$

$$= \frac{-1}{\pi^2 k^2} + \frac{2(-1)^k}{T_0^2} \left(\frac{j\pi k + 1}{T_0^2 \omega_0^2 k^2} + \frac{-j\pi k + 1}{T_0^2 \omega_0^2 k^2} \right) = \frac{-1}{\pi^2 k^2} + \left(\frac{4(-1)^k}{T_0^2 \omega_0^2 k^2} \right)$$

$$\omega_0 T_0 = 2\pi$$

$$\Rightarrow \frac{4}{T_0^2 \omega_0^2} = \frac{1}{\pi^2}$$

$$= \frac{-1}{\pi^2 k^2} + \left(\frac{(-1)^k}{\pi^2 k^2} \right) = \begin{cases} \frac{-1}{\pi^2 k^2} & k = \pm 2, \pm 4, \dots \\ 0 & k = \pm 1, \pm 3, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$

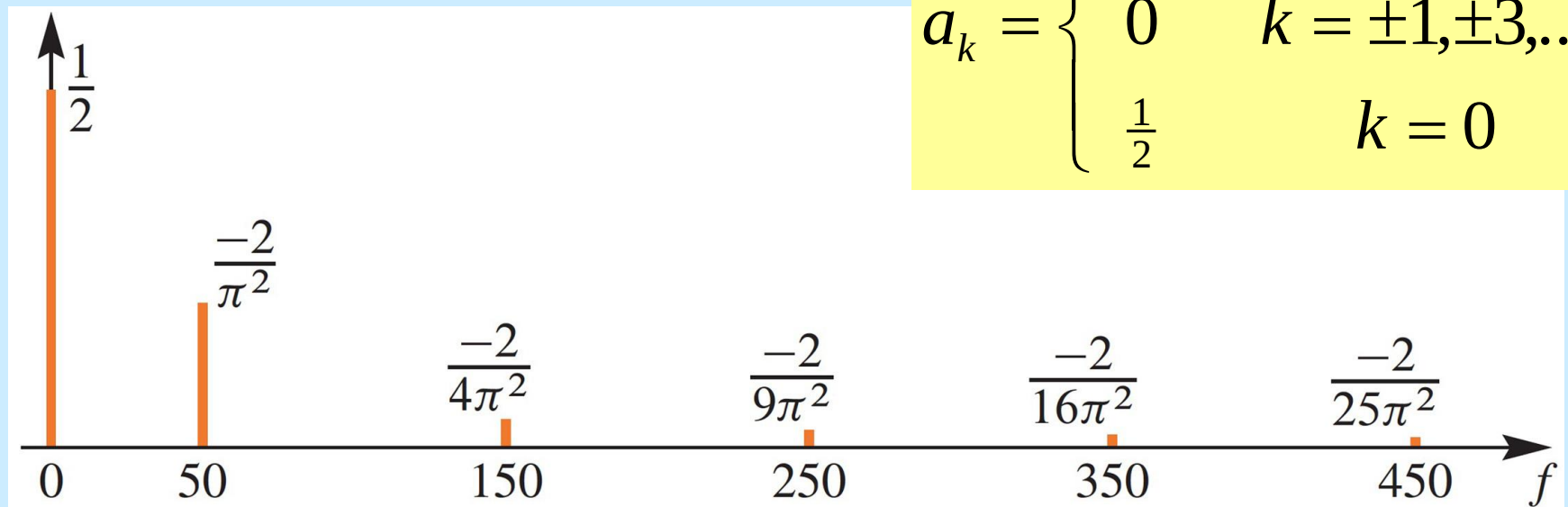
$$\text{DC} = \frac{\text{Area}}{\text{Period}}$$

$$\Rightarrow a_0 = \frac{T_0/2}{T_0} = \frac{1}{2}$$

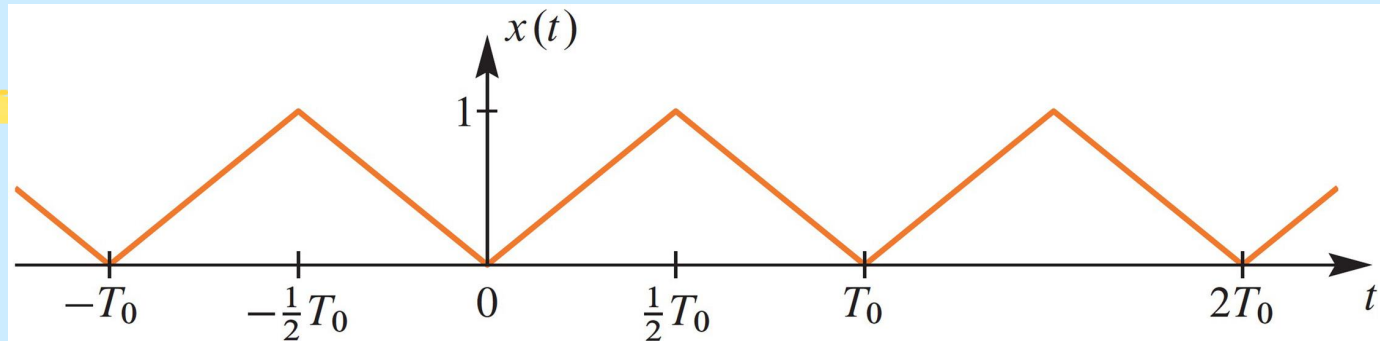
Triangular Wave $\{a_k\}$

- Spectrum, assuming 50 Hz is the fundamental frequency

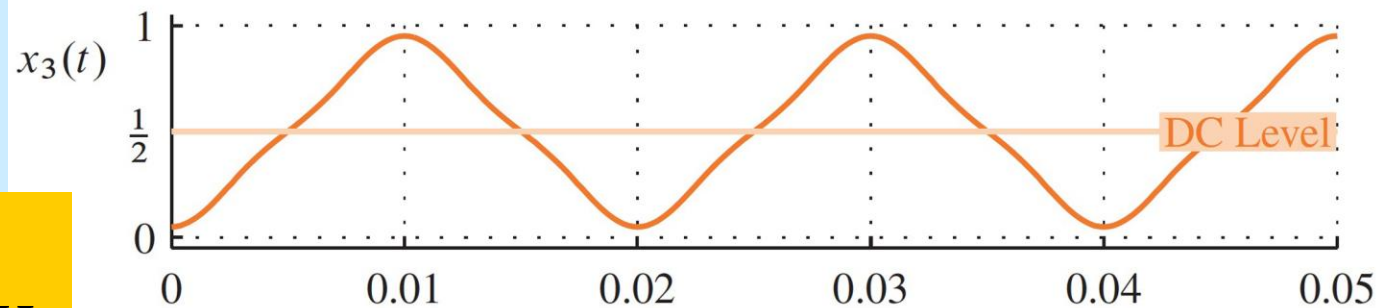
$$a_k = \begin{cases} \frac{-1}{\pi^2 k^2} & k = \pm 2, \pm 4, \dots \\ 0 & k = \pm 1, \pm 3, \dots \\ \frac{1}{2} & k = 0 \end{cases}$$



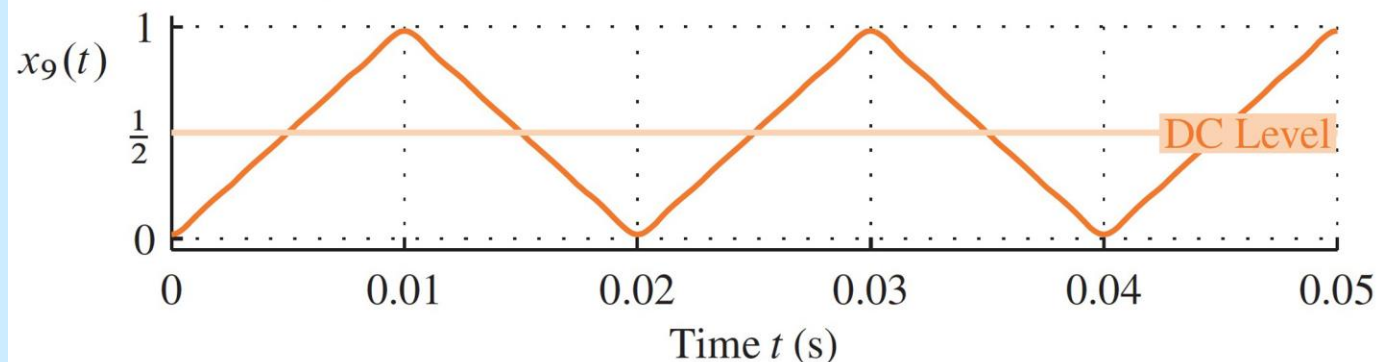
Triangular Wave Synthesis



(a) Sum of DC, 1st and 3rd Harmonics



(b) Sum of DC, 1st, 3rd, 5th, 7th, and 9th Harmonics



$T_0 = 20 \text{ ms}$
 $\Rightarrow F_0 = 50 \text{ Hz}$

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} \sin\left(\frac{\pi}{T_0} t\right) e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{1}{T_0} \int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{1}{j2T_0} \int_0^{T_0} e^{-j(\pi/T_0)(2k-1)t} dt - \frac{1}{j2T_0} \int_0^{T_0} e^{-j(\pi/T_0)(2k+1)t} dt$$

$$= \left. \frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))} \right|_0^{T_0} - \left. \frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))} \right|_0^{T_0}$$

Full-Wave Rectified Sine

$$x(t) = |\sin(2\pi t / T_1)|$$

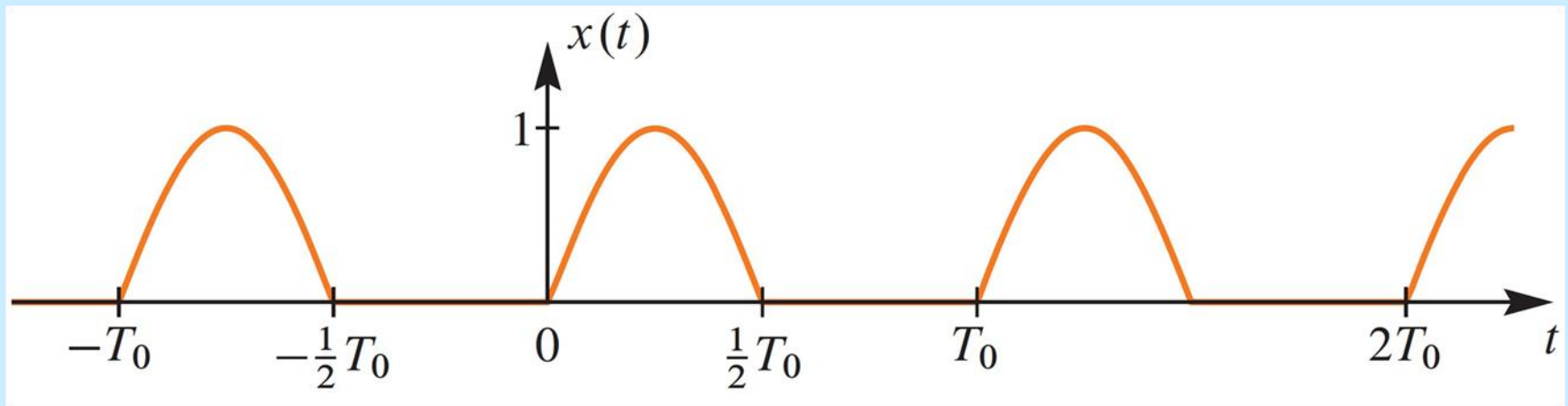
$$\text{Period : } T_0 = \frac{1}{2} T_1$$

Full-Wave Rectified Sine $\{a_k\}$

$$\begin{aligned} a_k &= \frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))} \bigg|_0^{T_0} - \frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))} \bigg|_0^{T_0} \\ &= \frac{1}{2\pi(2k-1)} \left(e^{-j(\pi/T_0)(2k-1)T_0} - 1 \right) - \frac{1}{2\pi(2k+1)} \left(e^{-j(\pi/T_0)(2k+1)T_0} - 1 \right) \\ &= \frac{1}{\pi(2k-1)} \left(e^{-j\pi(2k-1)} - 1 \right) - \frac{1}{\pi(2k+1)} \left(e^{-j\pi(2k+1)} - 1 \right) \\ &= \left(\frac{2k+1-(2k-1)}{\pi(4k^2-1)} \right) \left(-(-1)^{2k} - 1 \right) = \frac{-2}{\pi(4k^2-1)} \end{aligned}$$

Half-Wave Rectified Sine

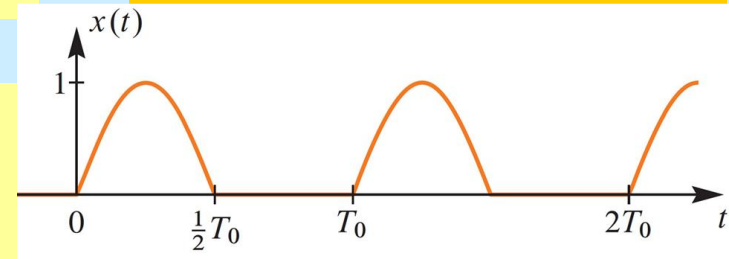
- Signal is positive half cycles of sine wave
- HWRS = Half-Wave Rectified Sine



Half-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt \quad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \frac{1}{T_0} \int_0^{T_0/2} \sin\left(\frac{2\pi}{T_0} t\right) e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{1}{T_0} \int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt} dt$$

$$= \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t} dt - \frac{1}{j2T_0} \int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t} dt$$

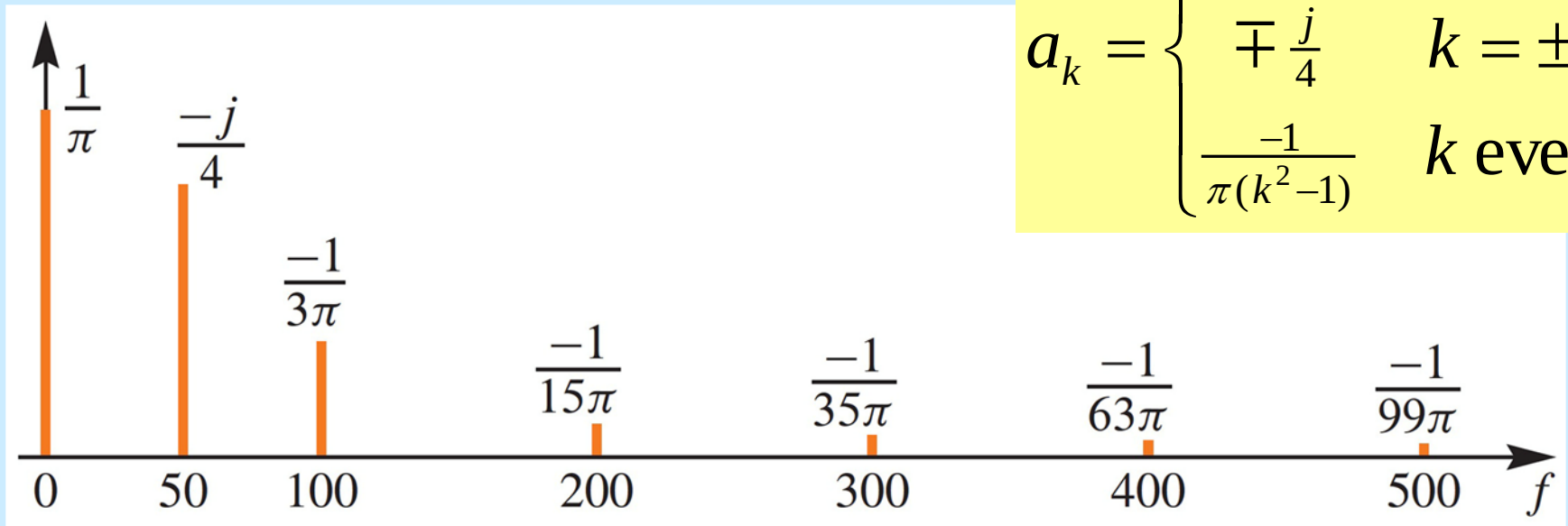
$$= \left. \frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))} \right|_0^{T_0/2} - \left. \frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))} \right|_0^{T_0/2}$$

Half-Wave Rectified Sine $\{a_k\}$

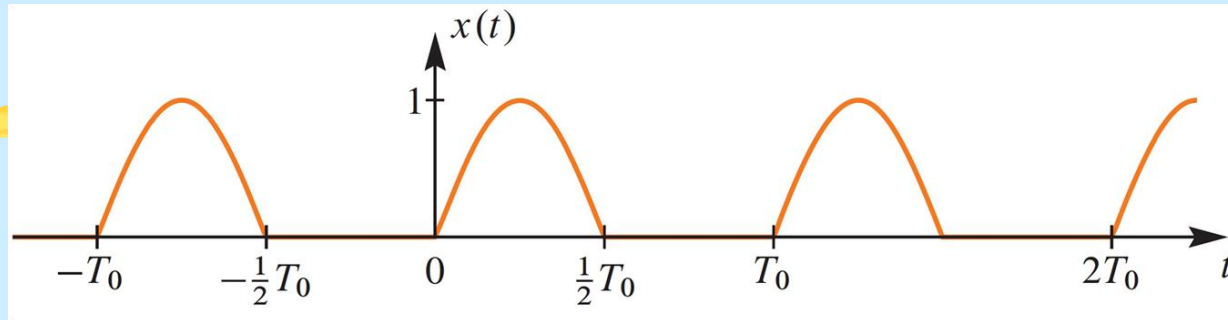
$$\begin{aligned} a_k &= \frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))} \bigg|_0^{T_0/2} - \frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))} \bigg|_0^{T_0/2} \\ &= \frac{1}{4\pi(k-1)} \left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1 \right) \\ &= \frac{1}{4\pi(k-1)} \left(e^{-j\pi(k-1)} - 1 \right) - \frac{1}{4\pi(k+1)} \left(e^{-j\pi(k+1)} - 1 \right) \\ &= \left(\frac{k+1-(k-1)}{4\pi(k^2-1)} \right) \left(-(-1)^k - 1 \right) = \begin{cases} 0 & k \text{ odd} \\ \pm \frac{1}{j4} & k = \pm 1 \\ \frac{-1}{\pi(k^2-1)} & k \text{ even} \end{cases} \end{aligned}$$

Half-Wave Rectified Sine $\{a_k\}$

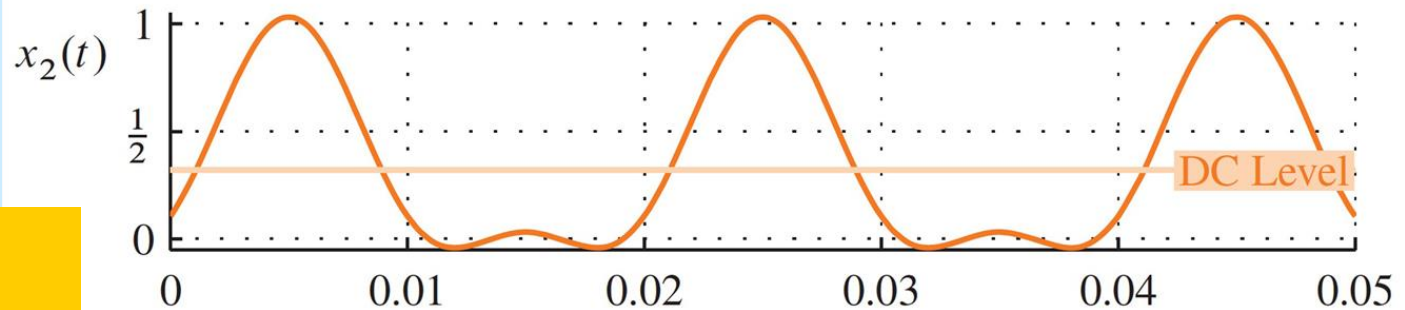
- Spectrum, assuming 50 Hz is the fundamental frequency



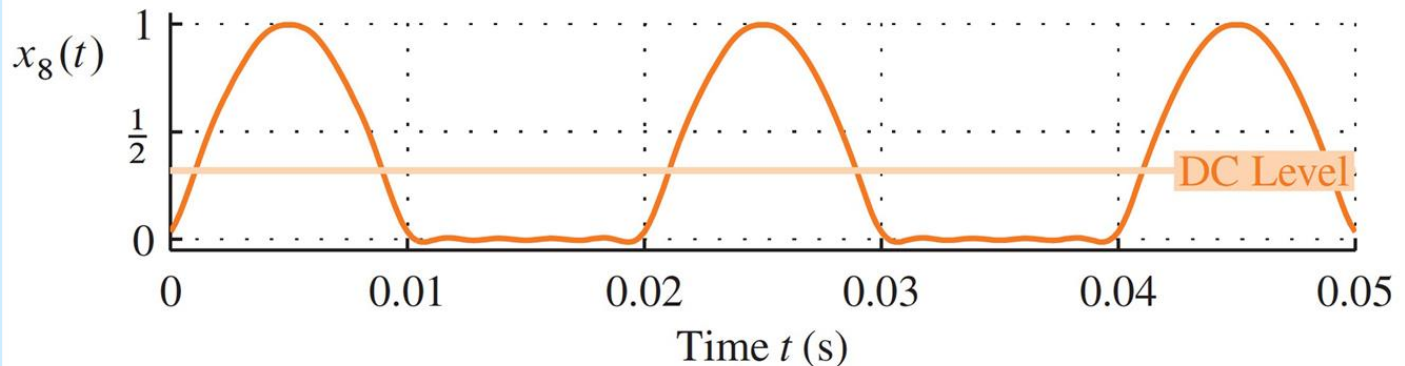
HWRS Synthesis



(a) Sum of DC, 1st and 2nd Harmonics



(b) Sum of DC, 1st, 2nd, 4th, 6th, and 8th Harmonics



$T_0 = 20 \text{ ms}$
 $\Rightarrow F_0 = 50 \text{ Hz}$