DSP First, 2/e

Lecture 17

DFT: Discrete Fourier Transform

READING ASSIGNMENTS

- This Lecture:
  - Chapter 8, Sections 8-1, 8-2 and 8-4

LECTURE OBJECTIVES

- **D**iscrete **F**ourier **T**ransform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

- DFT from DTFT by **frequency sampling**
- DFT computation (FFT)
- DFT pairs and properties
  - Periodicity in DFT (time & frequency)

Sample the DTFT → DFT

- Want **computable** Fourier transform
  - Finite signal length (L)
  - Finite number of frequencies

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\hat{\omega}n} \quad \Rightarrow \quad X(e^{j\hat{\omega}_k}) = \sum_{n=0}^{L-1} x[n]e^{-j\hat{\omega}_k n}$$

$$\hat{\omega}_k = (2\pi / N)k, \quad k = 0,1,2,\ldots N-1$$

k is the frequency index

$$\boxed{\text{Periodic} : X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}}) \quad \Rightarrow \quad X[k+N] = X[k]}$$

4-pt DFT: Numerical Example

- Take the 4-pt DFT of the following signal

$$x[n] = \delta[n] + \delta[n-1] \qquad \{x[n]\} = [1, 1, 0, 0]$$

$$X[0] = x[0]e^{-j0} + x[1]e^{-j0} + x[2]e^{-j0} + x[3]e^{-j0} = 1 + 1 + 0 + 0 = 2$$

$$X[1] = x[0]e^{-j0} + x[1]e^{-j\pi/2} + x[2]e^{-j2\pi/2} + x[3]e^{-j3\pi/2}$$

$$= 1 - j = \sqrt{2}e^{-j\pi/4}$$

$$X[2] = x[0]e^{-j0} + x[1]e^{-j\pi} + x[2]e^{-j2\pi} + x[3]e^{-j3\pi} = 1 - 1 + 0 + 0 = 0$$

$$X[3] = x[0]e^{-j0} + x[1]e^{-j3\pi/2} + x[2]e^{-j3\pi} + x[3]e^{-j9\pi/2}$$

$$= 1 + j = \sqrt{2}e^{j\pi/4}$$

N-pt DFT: Numerical Example

- Take the N-pt DFT of the impulse

$$x[n] = \delta[n] \qquad \{x[n]\} = [1, 0, 0, \ldots 0]$$

$$X[k] = \sum_{n=0}^{N-1} \delta[n] e^{-j(2\pi/N)kn}$$

$$= \sum_{n=0}^{0} \delta[n] e^{-j(2\pi/N)kn} = 1$$

$$\{X[k]\} = [1, 1, 1, \ldots, 1]$$

4-pt iDFT: Numerical Example

Example 66-8: Short-Length IDFT

The 4-point DFT in Example 66-7 is $X[k] = \{2,\ \sqrt{2}e^{-j\pi/4},\ 0,\ \sqrt{2}e^{j\pi/4}\}$. If we compute the 4-point IDFT of the sequence $X[k]$, we should recover $x[n]$ when we apply the IDFT summation (66.52) for each value of $n = 0, 1, 2, 3$. As before, the exponents in (66.52) will all be integer multiples of $\pi/2$ when $N = 4$.

$$
\begin{aligned}
x[0] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j0} + X[2]e^{j0} + X[3]e^{j0}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{-j\pi/4} + 0 + \sqrt{2}e^{j\pi/4}\right) = 1\\
x[1] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j\pi/2} + X[2]e^{j\pi} + X[3]e^{j3\pi/2}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+\pi/2)} + 0 + \sqrt{2}e^{j(\pi/4+3\pi/2)}\right) = \tfrac{1}{4}(2 + (1+j) + (1-j)) = 1\\
x[2] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j\pi} + X[2]e^{j2\pi} + X[3]e^{j3\pi}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+\pi)} + 0 + \sqrt{2}e^{j(\pi/4+3\pi)}\right) = \tfrac{1}{4}(2 + (-1+j) + (-1-j)) = 0\\
x[3] &= \tfrac{1}{4}\left(X[0]e^{j0} + X[1]e^{j3\pi/2} + X[2]e^{j3\pi} + X[3]e^{j9\pi/2}\right)\\
&= \tfrac{1}{4}\left(2 + \sqrt{2}e^{j(-\pi/4+3\pi/2)} + 0 + \sqrt{2}e^{j(\pi/4+9\pi/2)}\right) = \tfrac{1}{4}(2 + (-1-j) + (-1+j)) = 0
\end{aligned}
$$

Thus we recover the signal $x[n] = \{1,\ 1,\ 0,\ 0\}$ from its DFT coefficients, $X[k] = \{2,\ \sqrt{2}e^{-j\pi/4},\ 0,\ \sqrt{2}e^{j\pi/4}\}$.

FFT: Fast Fourier Transform

- FFT is an **algorithm** for computing the DFT
- $N \log_2 N$ versus N^2 operations
  - Count multiplications (and additions)
  - For example, when $N = 1024 = 2^{10}$
  - ≈10,000 ops vs. ≈1,000,000 operations
  - ≈1000 times faster
- What about N=256, how much faster?

Zero-Padding with the FFT

- Get many samples of DTFT
  - Finite signal length (L)
  - Finite number of frequencies (N)
  - Thus, we need $L < N, \quad N \to \infty, \quad X[k] \to X(e^{j\hat{\omega}})$

In MATLAB

- **Use** `X = fft(x,N)`
- With `L=length(x)` less than N
- Define `xpadtoN = [x,zeros(1,N-L)];`
- Take the N-pt DFT of `xpadtoN`

DFT periodic in k (frequency domain)

- Since DTFT is periodic in frequency, the DFT must also be periodic in k

$$X[k] = X(e^{j(2\pi/N)k})$$

$$X[k+N] = X(e^{j(2\pi/N)(k+N)}) = X(e^{j(2\pi/N)(k)+(2\pi/\cancel{N})\cancel{N}}) = X(e^{j(2\pi/N)k})$$

- What about Negative indices and Conjugate Symmetry?

$$X(e^{-j(2\pi/N)k}) = X^*(e^{j(2\pi/N)k})$$

$$\Rightarrow X[-k] = X^*[k]$$

$$X[N-k] = X^*[k]$$

$$N = 32 \quad \Rightarrow$$

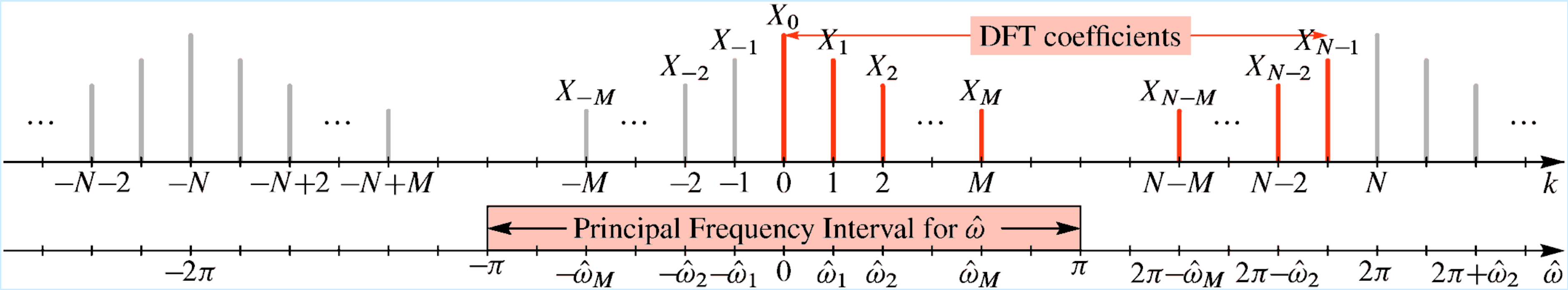
$$X[31] = X^*[1]$$

$$X[30] = X^*[2]$$

$$X[29] = X^*[3]$$

DFT Periodicity in Frequency Index

$$X[k] = X(e^{j\hat{\omega}_k}) = X(e^{j(2\pi/N)k})$$
$$k = 0,1,2,\ldots N-1$$



$$X[k+N] = X[k] \Leftrightarrow X(e^{j(\hat{\omega}+2\pi)}) = X(e^{j\hat{\omega}})$$

$$\Rightarrow X[N-k] = X[-k],$$
e.g., $X[N-2] = X[-2]$

Table 8-1 Basic discrete Fourier transform pairs.

Table of DFT Pairs	
Time-Domain: $x[n]$	***Frequency-Domain:*** $X[k]$
$\delta[n]$	1
$\delta[n-n_d]$	$e^{-j(2\pi k/N)n_d}$
$r_L[n] = u[n] - u[n-L]$	$\underbrace{\frac{\sin(\frac{1}{2}L(2\pi k/N))}{\sin(\frac{1}{2}(2\pi k/N))}}_{=D_L(2\pi k/N)} e^{-j(2\pi k/N)(L-1)/2}$
$r_L[n]\, e^{j(2\pi k_0/N)n}$	$D_L(2\pi(k-k_0)/N)\, e^{-j(2\pi(k-k_0)/N)(L-1)/2}$

Table 8-2 Basic discrete Fourier transform properties.

Table of DFT Properties		
Property Name	***Time-Domain: $x[n]$***	***Frequency-Domain: $X[k]$***
Periodic	$x[n] = x[n+N]$	$X[k] = X[k+N]$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Conjugate Symmetry	$x[n]$ is real	$X[N-k] = X^*[k]$
Conjugation	$x^*[n]$	$X^*[N-k]$
Time-Reversal	$x[((N-n))_N]$	$X[N-k]$
Delay	$x[((n-n_d))_N]$	$e^{-j(2\pi k/N)n_d} X[k]$
Frequency Shift	$x[n]e^{j(2\pi k_0/N)n}$	$X[k-k_0]$
Modulation	$x[n]\cos((2\pi k_0/N)n)$	$\frac{1}{2}X[k-k_0] + \frac{1}{2}X[k+k_0]$
Convolution	$\sum_{m=0}^{N-1} h[m]x[((n-m))_N]$	$H[k]X[k]$
Parseval's Theorem	$\sum_{n=0}^{N-1} \lvert x[n]\rvert^2 = \frac{1}{N}\sum_{k=0}^{N-1} \lvert X[k]\rvert^2$	