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Signal Processing Ch1

Analog Signals – s(t); t is time

- Analogous to the actual physical signal
- Typically continuous – temperature, etc. Values are real numbers

Discrete-time signals s[n]=s(nTs); n= 1,2,…

- Ts is the sampling period in seconds
- Amplitude of s(nTs) is a real number
- Signal is Quantized in Time!

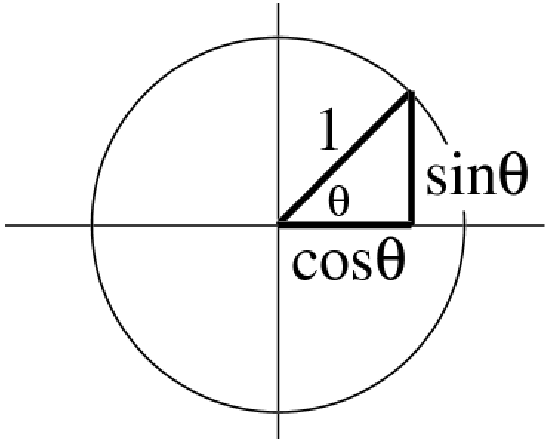
LECTURE OBJECTIVES

- Introduce more tools for manipulating complex numbers   Euler Eq.
  - Conjugate
  - Multiplication & Division
  - Powers
  - N-th Roots of unity

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

Euler's FORMULA

- **<u>Complex Exponential</u>**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



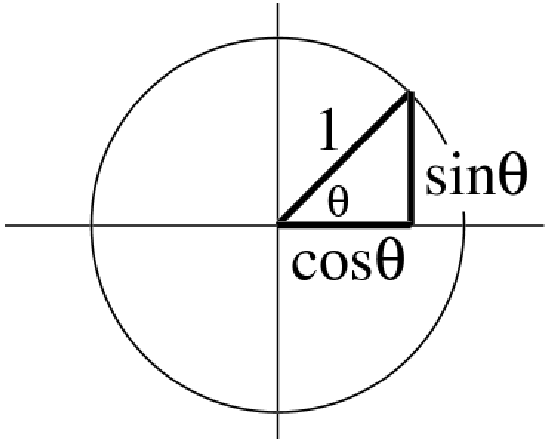
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\boldsymbol{\theta = \omega t}$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

TIME-SHIFT

- In a mathematical formula we can replace t with $t\text{-}t_m$

$$x(t-t_m) = A\cos(\omega(t-t_m))$$

- Thus the t=0 point moves to $t=t_m$

- Peak value of $\cos(\omega(t\text{-}t_m))$ is now at $t=t_m$

Figure 2-9: Plotting the 40-h z Sampled Cosine 2.8(b) for

(A) $T_s = 0 \cdot 005\ S$; **(B)** $T_s = 0 \cdot 0025\ S$; **(C)** $T_s = 0 \cdot 0005\ S$

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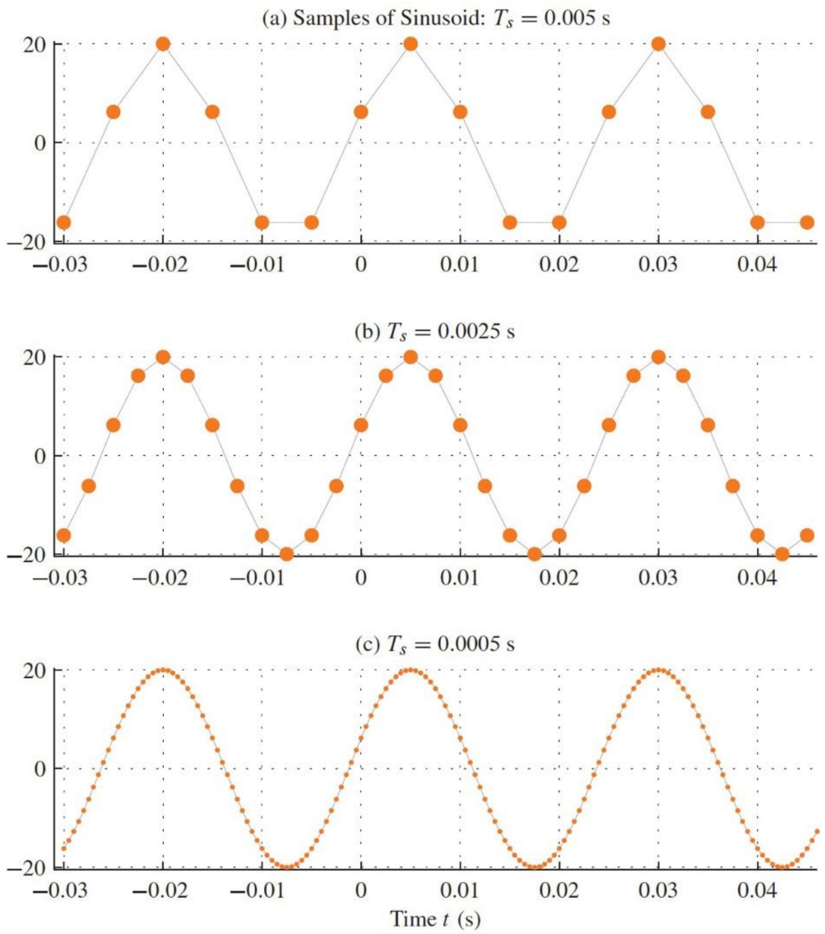

PHASOR ADDITION RULE

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \varphi_k)$$

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$$= A\cos(\omega_0 t + \varphi)$$

Get the new complex amplitude by complex addition

Find Amplitude and
Phase – ω is Known!

$$\sum_{k=1}^{N} A_k e^{j\varphi_k} = Ae^{j\varphi}$$

Example 3-1: To determine the spectrum of the following signal,

$$x(t) = 10 + 14\cos(200\pi t - \pi/3) + 8\cos(500\pi t + \pi/2)$$

which is the sum of a constant and two sinusoids, we must convert from the general form in (3.2) to the two-sided form in (**3.4**). After we apply the inverse Euler formula, we get the following five terms:

$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t} \quad (3.1)$$

Note that the constant component of the signal, often called the ***DC component***, can be expressed as a complex exponential signal with zero frequency (i.e., $10e^{j0t} = 10$). Therefore, in the list form suggested in (3.5), the spectrum of this signal is the set of five rotating phasors represented by the frequency/complex amplitude pairs

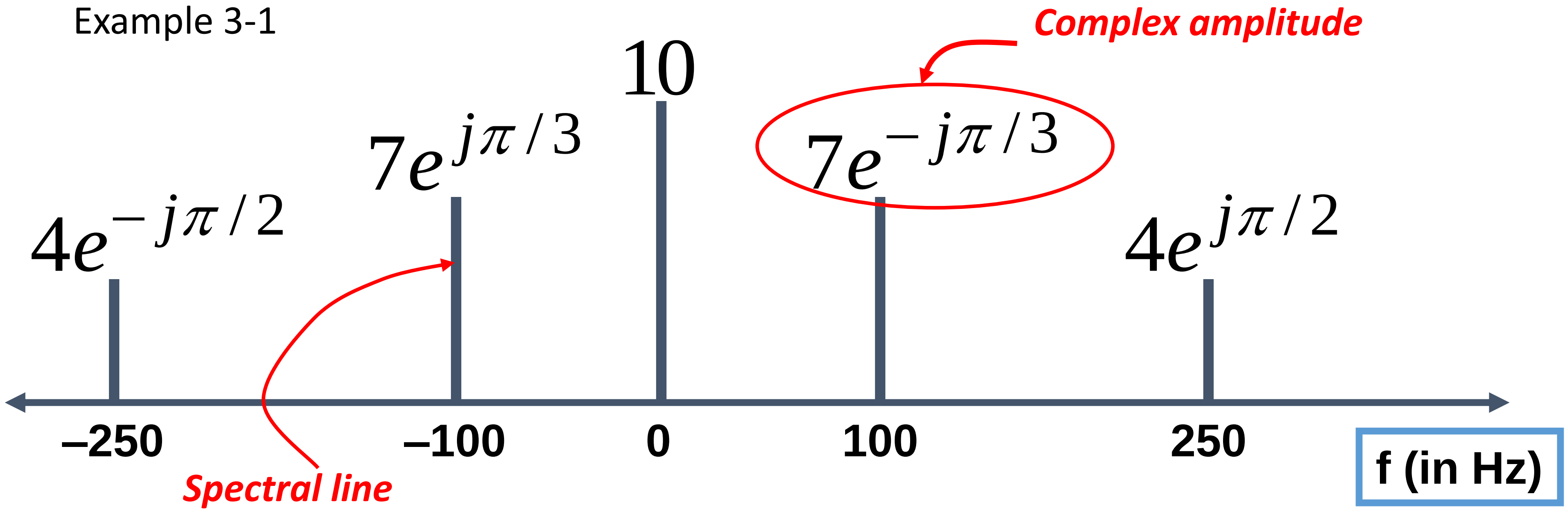
$$\{(0, 10),\ (100, 7e^{-j\pi/3}),\ (-100, 7e^{j\pi/3}),\ (250, 4e^{j\pi/2}),\ (-250, 4e^{-j\pi/2})\}$$

Note: The terminology "DC" comes from electric circuits, where a constant value of current is called direct current, or DC. It is common to call $X_0 = A_0$ the DC component of the spectrum. Since the DC component is constant, its frequency is $f = 0$.

FREQUENCY DIAGRAM

- Want to visualize relationship between frequencies, amplitudes and phases
- Plot Complex Amplitude vs. Frequency

Example 3-1



Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

INVERSE Euler's Formula

- What is the "spectrum" representation for a single sinusoid?
- Solve Euler's formula for cosine (or sine)

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

SPECTRUM Interpretation

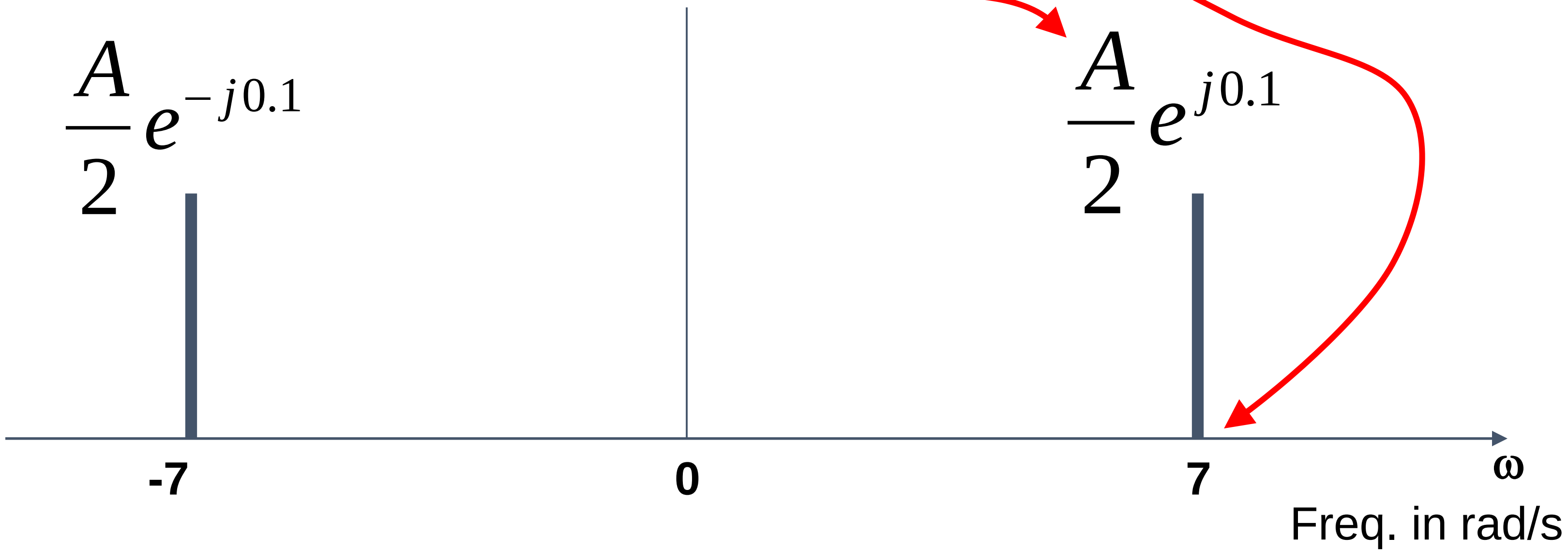
- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

- One has a positive frequency
- The other has negative freq.
- Amplitude of each is half as big

GRAPHICAL SPECTRUM

$$A\cos(7t+0.1) = \frac{A}{2}e^{j0.1}e^{j7t} + \frac{A}{2}e^{-j0.1}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are labels

REPRESENTATION of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

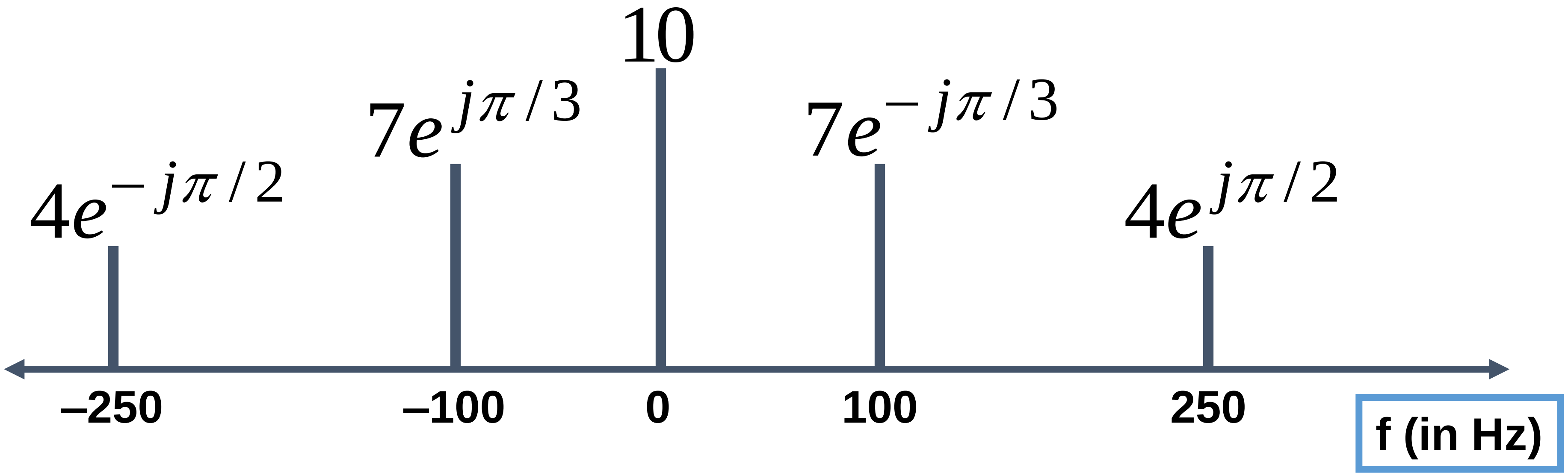
$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

$$\frac{-1}{j} = j = e^{j0.5\pi}$$

- Positive freq. has phase = -0.5π
- Negative freq. has phase = +0.5π

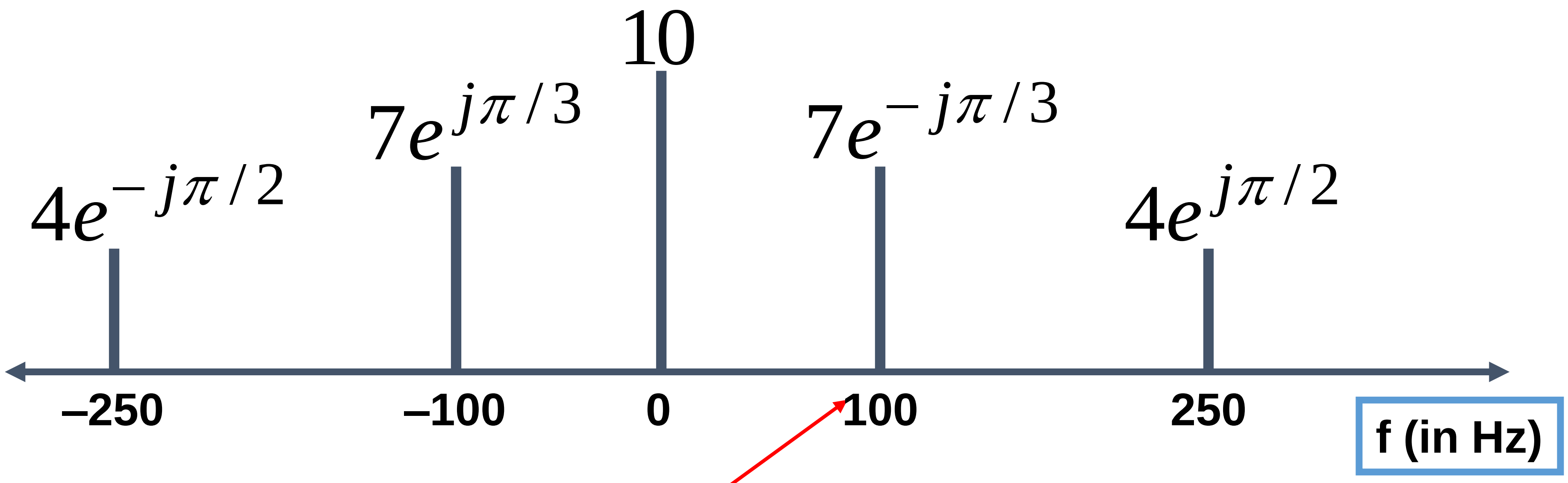
SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal x(t)?

Add Spectrum Components-2



$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

FINAL ANSWER

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

NOTE: - sine(x) = cos(x)cos(pi/2) – sin(x)sin(pi/2)

Example 3-1

So, we get the general form:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Harmonic Signal

Periodic signal : $x(t) = x(t+T)$

Can only have ***harmonic*** freqs : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$f_0 T = 1$

$x(t)$ is periodic if

$$\cos(2\pi k f_0 (t+T) + \varphi_k) = \cos(2\pi k f_0 t + 2\pi k f_0 T + \varphi_k)$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

Largest f_0 such that

$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$

$$f_0 = \frac{1}{T_0}$$

$f_0 =$ fundamental Frequency

$f_k / f_0 =$ integer, for all k

$T_0 =$ fundamental Period

Main point:
for periodic signals, all spectral lines have frequencies that are *integer* multiples of the fundamental frequency

LECTURE OBJECTIVES

- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\mathbf{x(t+T_0) = x(t)}$
  - Draw spectrum from the Fourier Series coeffs

NEXT SLIDE SHOW

Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.