Fourier Series Analysis

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Trig and Exponential Series

Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

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On the interval $[-T/2, T/2]$, the limits of integration for the Fourie can be changed from $[-\pi, \pi]$ by assigning to the integration var the value $2\pi t/T$. The period of the function is thus T.

Assuming that $f(t)$ is continuous on the interval $-T/2 \leq t \leq$ the coefficients a_n and b_n can be computed by the formulas

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt, \\
a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2n\pi t}{T}\right)\,dt, \\
b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2n\pi t}{T}\right)\,dt,
\end{aligned}
$$

where $n = 1, 2, \ldots$ is any positive integer.

The Fourier series on the interval $[-T/2, T/2]$ is thus

$$
f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{2n\pi t}{T}\right) + b_n\sin\left(\frac{2n\pi t}{T}\right)\right].
$$

☐ **EXAMPLE 8.4** ***Fourier series square wave example***

A square wave of amplitude A and period T shown in Figure 8.4 can be defined as

$$f(t) = \begin{cases} A, & 0 < t < \dfrac{T}{2}, \\ -A, & -\dfrac{T}{2} < t < 0, \end{cases}$$

with $f(t) = f(t+T)$, since the function is periodic.

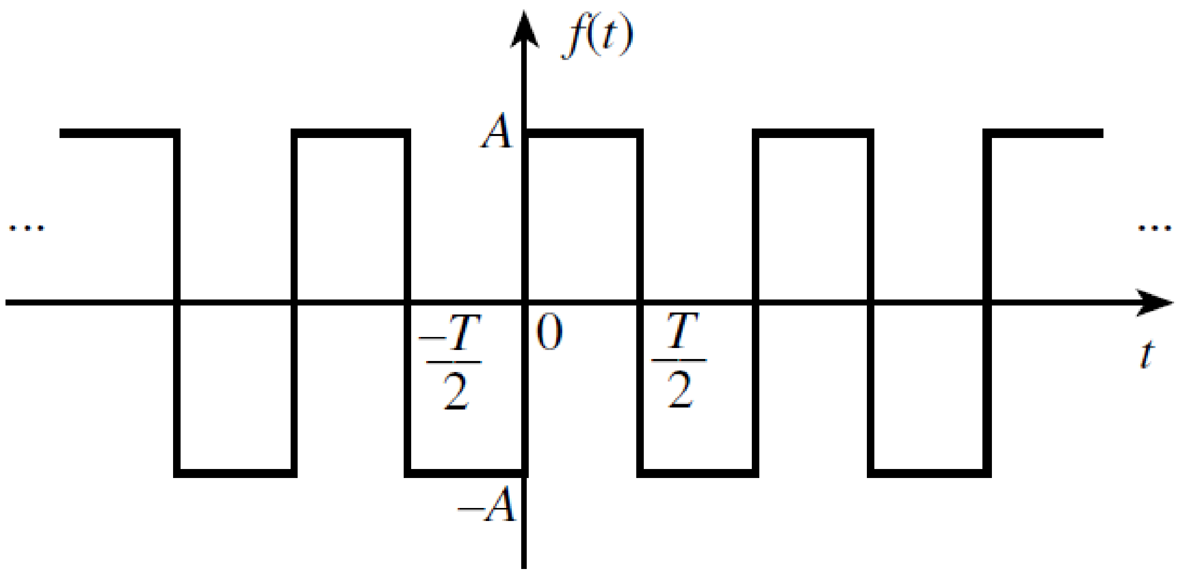


FIGURE 8.4 *Square wave of Example 8.4*

The first observation is that $f(t)$ is odd, which yields the result that $a_0 = 0$ and $a_i = 0$ for every coefficient of the cosine terms. Letting $\omega_0 = 2\pi/T$, the coefficients b_n are

$$b_n = 2\left(\frac{2}{T}\right)\int_0^{T/2} A\sin(n\omega_0 t)\,dt.$$

The result is

$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

where $(2n-1)$ is introduced to assure that only odd terms are included in the summation. The sine waves that make up the Fourier series for the odd square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right],$$

Harmonic Signal->Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k \underline{F_0} t}$$

Sums of ***Harmonic*** *complex exponentials are* ***Periodic*** *signals*

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

☐ EXAMPLE 8.5 ***Complex Series Square Wave Example***

Consider the odd square wave of Example 8.4 and the complex Fourier coefficients

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{0}(-A)e^{-in\omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2}(A)e^{-in\omega_0 t}\,dt, \tag{8.29}$$

which leads to the series

$$f(t) = \frac{2A}{i\pi}\sum_{n=-\infty}^{\infty}\frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}, \tag{8.30}$$

as defined in Equation 8.23.

This form contains complex coefficients, but the series can be written in terms of sine waves by combining the corresponding terms for positive and negative arguments. To determine the coefficients, the amount of difficulty is about the same for the trigonometric series and the complex series. However, the complex series perhaps has an advantage when the magnitude of the coefficients are of interest.

Each coefficient has the form

$$\alpha_n = \frac{2A}{in\pi} = \frac{2A}{n\pi}e^{-i\pi/2}, \qquad n = \pm 1, \pm 3, \ldots,$$

and the coefficients for even values, $n = 0, \pm 2, \ldots$, are zero. Notice that the

The trigonometric series is derived from the complex series by expanding the complex series of Equation 8.30 as

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t} \\
&= \cdots - \frac{2A}{3\pi i} e^{-i3\omega_0 t} - \frac{2A}{\pi i} e^{-i\omega_0 t} + \frac{2A}{\pi i} e^{i\omega_0 t} + \frac{2A}{3\pi i} e^{i3\omega_0 t} + \cdots
\end{aligned}
$$

and recognizing the sum of negative and positive terms for each n as $2\sin(n\omega_0 t)$. The trigonometric series becomes

$$
f(t) = \frac{4A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right) = \frac{4A}{\pi}\sum_{n=1}^{\infty} \frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},
$$

which is the result of Example 8.4.

□

LECTURE OBJECTIVES

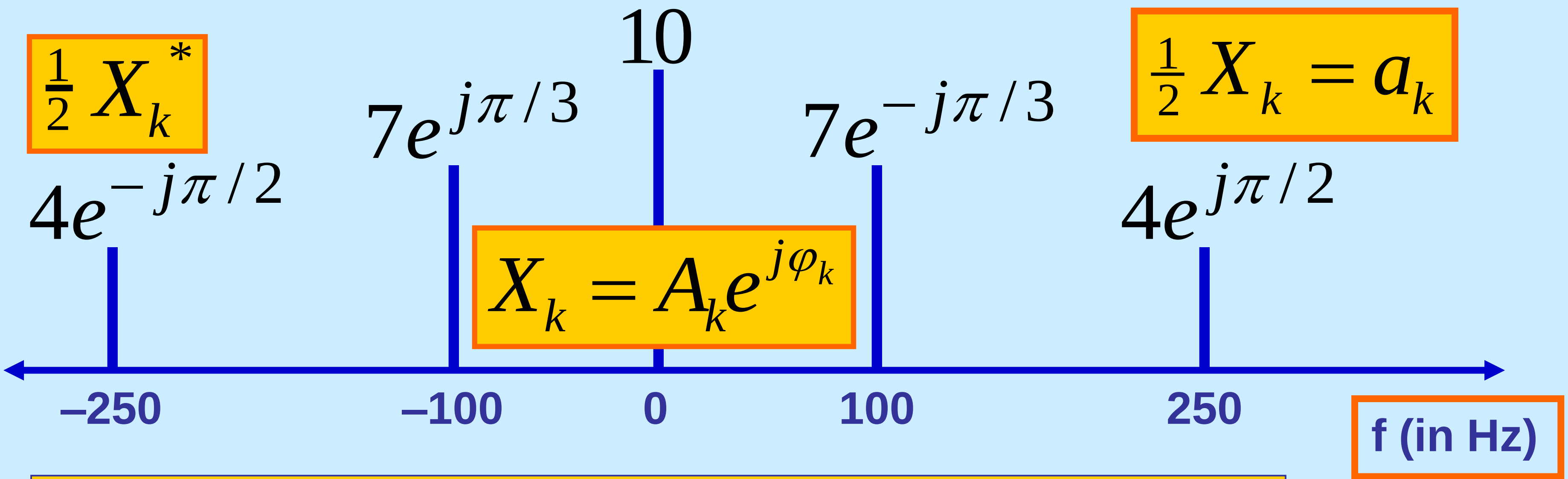
- Work with the Fourier Series Integral

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi k/T_0)t}dt$$

- **ANALYSIS** via Fourier Series
  - For **PERIODIC** signals: $\mathbf{x(t+T_0) = x(t)}$
  - Draw spectrum from the Fourier Series coeffs

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = \boxed{a_0} + \sum_{k=1}^{N} \left\{ \boxed{a_k} e^{j2\pi f_k t} + \boxed{a_k^*} e^{-j2\pi f_k t} \right\}$$

Harmonic Signal->Periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k \underline{F_0} t}$$

*Sums of **Harmonic** complex exponentials are **Periodic** signals*

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(F_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{F_0}$$

Notation for Fundamental Frequency in Fourier Series

- The k-th frequency is $f_k = kF_0$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi kF_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi f_k t}$$

- Thus, $f_0 = 0$ is DC
- This is why we use upper case F_0 for the Fundamental Frequency

STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
  - Measure similarity between signal & harmonic
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

CALCULUS for complex exp

$$\boxed{\frac{d}{dt}e^{\alpha t} = \alpha e^{\alpha t} \quad \rightarrow \quad \frac{d}{dt}e^{j\alpha t} = j\alpha e^{\alpha t}}$$

$$\int_a^b e^{\beta t}dt = \frac{1}{\beta}e^{\beta t}\Big|_a^b = \frac{1}{\beta}\left(e^{\beta b} - e^{\beta a}\right)$$

$$\int_a^b e^{j\beta t}dt = \frac{1}{j\beta}e^{j\beta t}\Big|_a^b = \frac{1}{j\beta}\left(e^{j\beta b} - e^{j\beta a}\right)$$

Fourier Series Integral

- Use orthogonality to determine a_k from x(t)

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Freq.

$$F_0 = 1/T_0$$

$$a_{-k} = a_k^* \quad \text{when } x(t) \text{ is real}$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

- THIS IS THE AVERAGE!

(DC component)

Fourier Series: x(t) → a_k

- **ANALYSIS**
  - Given a **PERIODIC Signal**
  - Fourier Series coefficients are obtained via an **INTEGRAL over one period**

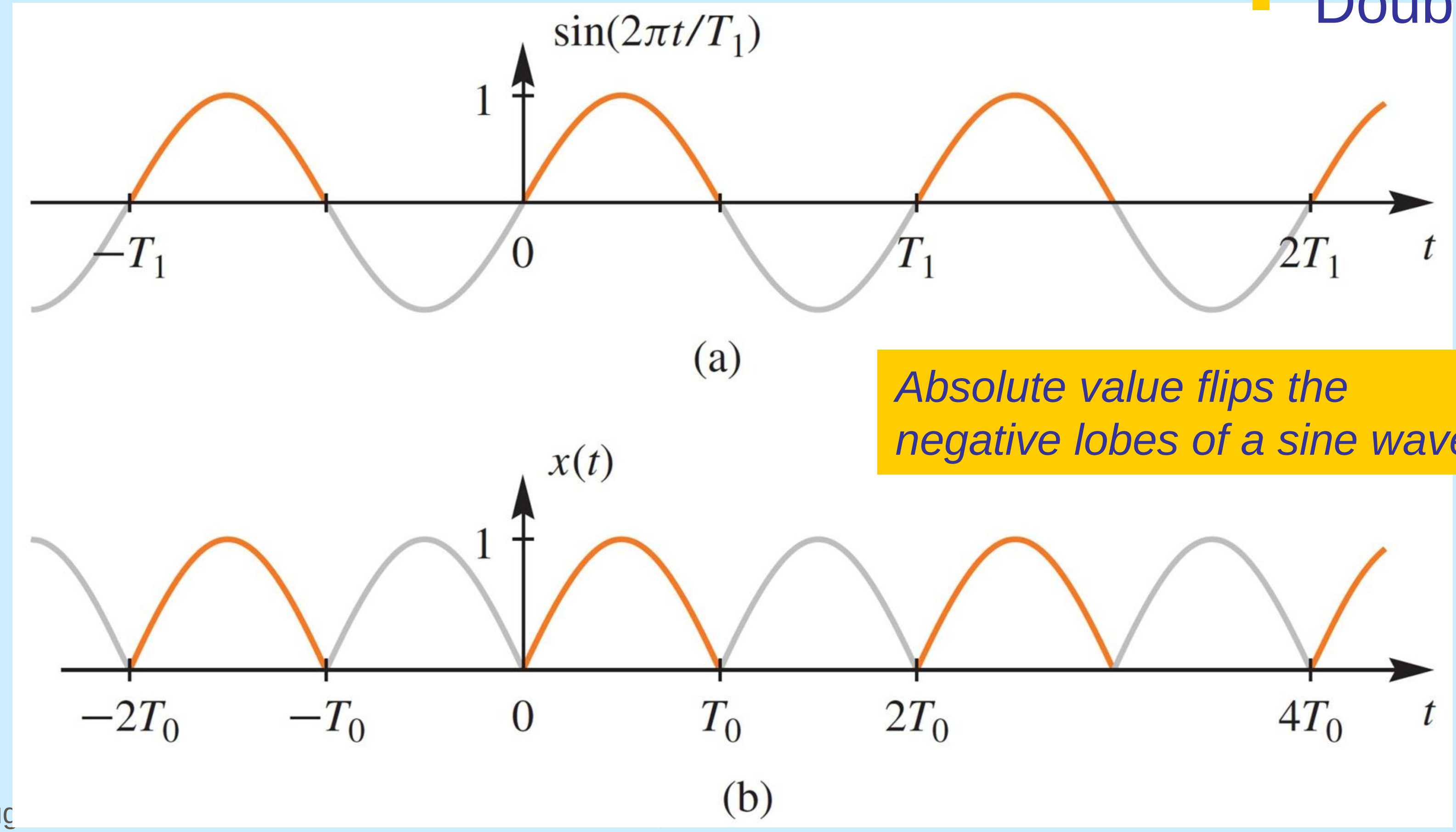
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

  - Next, consider a specific signal, the FWRS
    - Full Wave Rectified Sine

Full-Wave Rectified Sine

$$x(t) = \left|\sin(2\pi t / T_1)\right| \qquad \text{Period is } T_0 = \tfrac{1}{2} T_1$$

- Frequency
- Doubles



Absolute value flips the negative lobes of a sine wave

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt}\,dt$$

Full-Wave Rectified Sine

$$x(t) = \left|\sin(2\pi t/T_1)\right|$$

$$Period: T_0 = \tfrac{1}{2}T_1$$

$$\Rightarrow x(t) = \left|\sin(\pi t/T_0)\right|$$

$$a_k = \tfrac{1}{T_0}\int_0^{T_0} \sin(\tfrac{\pi}{T_0}t) e^{-j(2\pi/T_0)kt}\,dt$$

$$= \tfrac{1}{T_0}\int_0^{T_0} \frac{e^{j(\pi/T_0)t} - e^{-j(\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}\,dt$$

$$= \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k-1)t}\,dt - \tfrac{1}{j2T_0}\int_0^{T_0} e^{-j(\pi/T_0)(2k+1)t}\,dt$$

$$= \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

Full-Wave Rectified Sine $\{a_k\}$

$$a_k = \left.\frac{e^{-j(\pi/T_0)(2k-1)t}}{j2T_0(-j(\pi/T_0)(2k-1))}\right|_0^{T_0} - \left.\frac{e^{-j(\pi/T_0)(2k+1)t}}{j2T_0(-j(\pi/T_0)(2k+1))}\right|_0^{T_0}$$

$$= \tfrac{1}{2\pi(2k-1)}\left(e^{-j(\pi/T_0)(2k-1)T_0}-1\right) - \tfrac{1}{2\pi(2k+1)}\left(e^{-j(\pi/T_0)(2k+1)T_0}-1\right)$$

$$= \tfrac{1}{\pi(2k-1)}\left(e^{-j\pi(2k-1)}-1\right) - \tfrac{1}{\pi(2k+1)}\left(e^{-j\pi(2k+1)}-1\right)$$

$$= \left(\tfrac{2k+1-(2k-1)}{\pi(4k^2-1)}\right)\left(-(-1)^{2k}-1\right) = \frac{-2}{\pi(4k^2-1)}$$

Fourier Coefficients: a_k

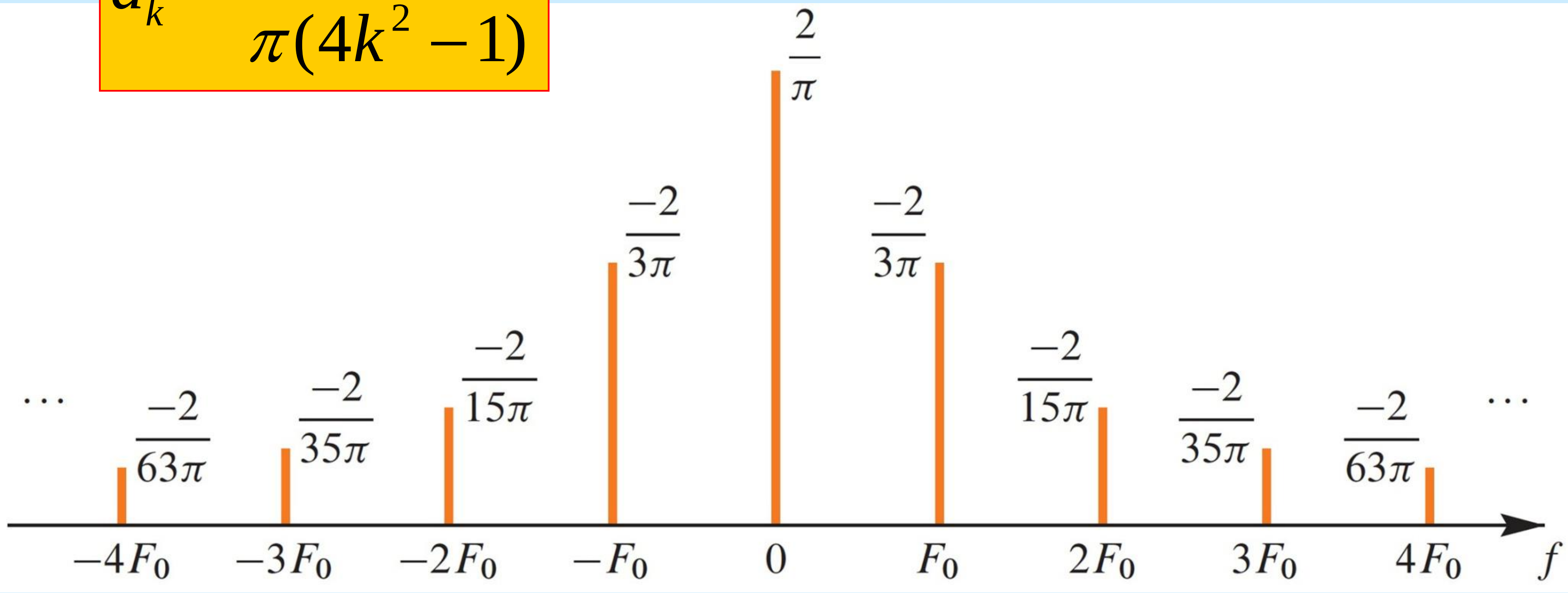
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - $$\boxed{a_k = \frac{-2}{\pi(4k^2-1)}}$$ NOTE: $\frac{1}{k^2}$ *for large k*
  - Does not depend on the period, T_0
  - DC value is $\boxed{a_0 = 2/\pi = 0.6336}$

Spectrum from Fourier Series

Plot a_k for Full-Wave Rectified Sinusoid

$$a_k = \frac{-2}{\pi(4k^2-1)}$$

$$F_0 = 1/T_0 \quad \text{and} \quad \omega_0 = 2\pi F_0$$



Reconstruct From Finite Number of Harmonic Components

Full-Wave Rectified Sinusoid $x(t) = |\sin(\pi t / T_0)|$

$$T_0 = 10 \text{ ms}$$
$$\Rightarrow F_0 = 100 \text{ Hz}$$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

$$a_0 = 2/\pi = 0.6336$$

$$x_N(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi k F_0 t} + a_k^* e^{-j2\pi k F_0 t} \right\}$$

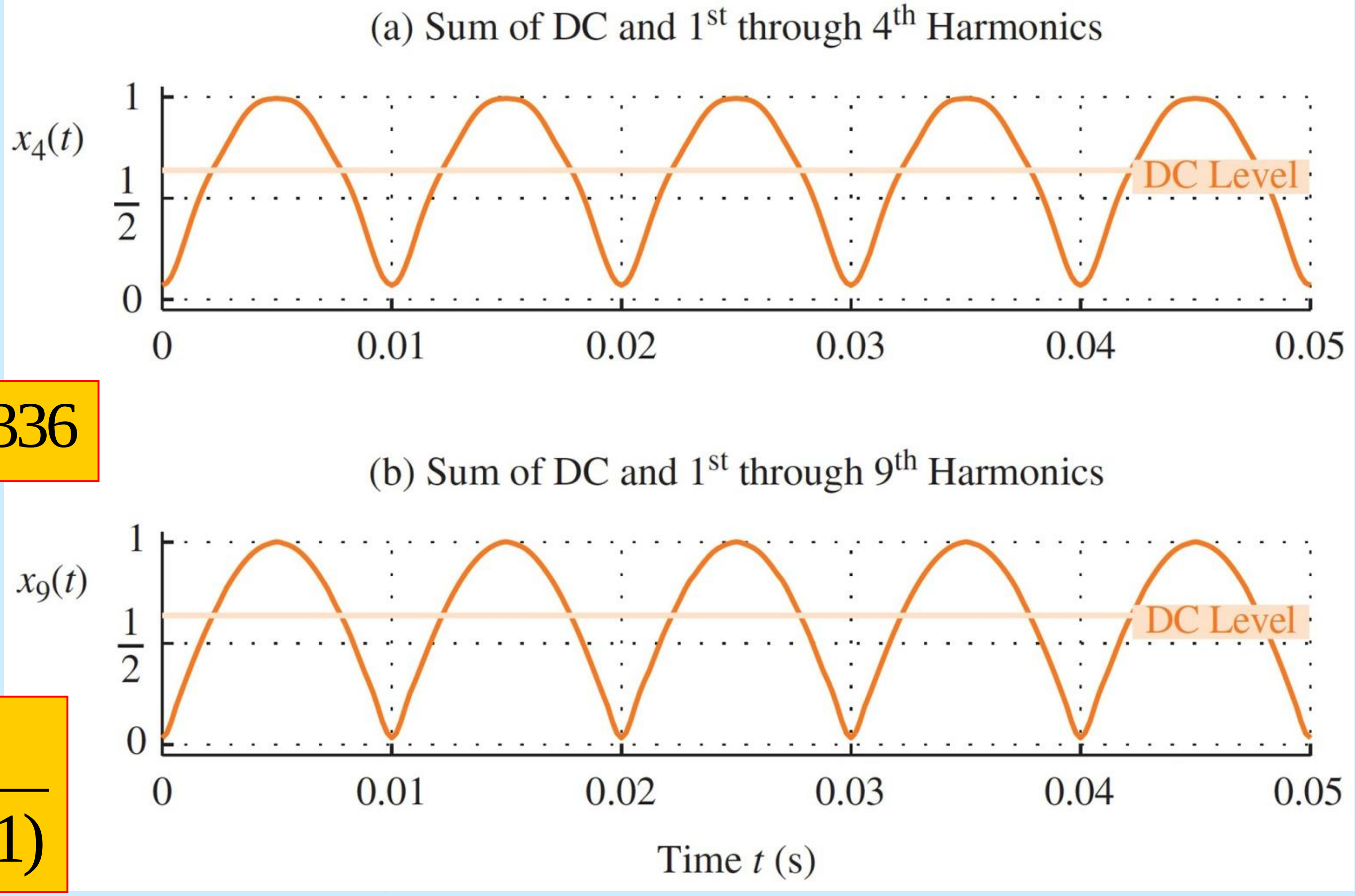
How close is $x_N(t)$ to $x(t) = |\sin(\pi t / T_0)|$?

Reconstruct From Finite Number of Spectrum Components

Full-Wave Rectified Sinusoid $x(t) = \left|\sin(\pi t / T_0)\right|$

$T_0 = 10\text{ ms}$

$\Rightarrow F_0 = 100\text{ Hz}$



$a_0 = 2/\pi = 0.6336$

$$a_k = \frac{-2}{\pi(4k^2 - 1)}$$

Synthesis: up to 7th Harmonic

$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$