2 NUMBERS AND VECTORS

PREVIEW

Generally, the study of vectors in analytic geometry, calculus, and elementary physics takes the geometric point of view when vectors are spoken of as having magnitude and direction in two or three dimensions. However, an algebraic approach is necessary for computer representation and generalization beyond three dimensions. In many problems in physics and engineering, as well as in mathematics, the variables describing a physical situation are treated as functions of vectors. These vectors have components that may represent the values of the variable in different directions or at different times.

The chapter begins with a review of the properties of real and complex numbers, including their representation in MATLAB. Then, we present the basic properties of vectors in two and three dimensions. Many properties of three-dimensional vectors are easily generalized to a vector with $n > 3$ components. Also, collections of functions and other mathematical entities can be described as vectors in a vector space. That such generalizations are useful is an indication of the great power of mathematics when applied to physical problems. Just as the use of a three-dimensional vector to

represent force is an abstraction from the physical world, these further abstractions will prove valuable in analyzing many physical systems.

The remainder of the chapter considers vectors and vector spaces for applications to problems in engineering and physics presented in later chapters. The important topics include linear independence for vectors, abstract vector spaces, and vector spaces of functions.

Table 2.1 lists various scalar and vector topics covered in this textbook.

TABLE 2.1 *Topics in vector analysis*

Chapter	Symbol	Topic
Chapter 2	$\alpha + \beta$	Sum of scalars
	$\alpha\beta$	Product of scalars
	$\alpha\mathbf{x}$	Scalar times vector
	$\mathbf{x} + \mathbf{y}$	Sum of vectors
	$\mathbf{x} \cdot \mathbf{y}$	Dot product
	$\mathbf{x} \times \mathbf{y}$	Cross product
	$||\mathbf{x}||$	Norm
	$\mathbf{R}^n$	Real vector space
	V	Abstract vector space
Chapter 3	$A\mathbf{x} = \mathbf{b}$	Linear equations
Chapter 4	$\mathcal{L}(\alpha\mathbf{x} + \beta\mathbf{y})$	Linearity
	$A\mathbf{x} = \lambda\mathbf{x}$	Eigenvectors
Chapter 5, 6	$\dfrac{d\mathbf{x}}{dt} = A\mathbf{x}$	Differential equations
Chapter 13	$\dfrac{df(t)}{dt}, \ \dfrac{\partial f(t)}{\partial t}$	Derivative of a function
	$\dfrac{d\mathbf{x}(t)}{dt}$	Derivative of a vector
	∇f	Gradient of a function
	$\nabla \cdot \mathbf{x}$	Divergence of a vector
	$\nabla \times \mathbf{x}$	Curl of a vector

In the table, α and β are real numbers (scalars); $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{b}$ are vectors; and A is a matrix.

2.1 PROPERTIES OF REAL NUMBERS

■

Since real numbers are used extensively to describe physical phenomena, a review of the fundamental operations on real numbers is presented first in this chapter. Then, we consider numbers as represented by MATLAB.

REAL NUMBERS The set of real numbers consists of *integers*, *rational* numbers, and *irrational* numbers. The integers are $1, 2, 3, \ldots, -1, -2, -3, \ldots$, and 0. Fractions in the form p/q, where p and q are integers, are rational numbers. *Irrational* numbers can be represented by an infinite decimal expansion, such as $3.14159\ldots$ for π. However, irrational numbers cannot be represented exactly as a ratio of integers.

Real numbers that are not integers can be represented by *nonterminating* decimal expansions, as in the example of π just given. The sequence of decimal digits of π is both *nonterminating* and *nonrepeating*. In contrast, rational numbers have a repeating decimal expansion, such as $\frac{1}{4} = 0.25000000...$, in which the repeated digit is zero. In most cases, the repeated digits will not be zero. For example, the fraction $\frac{7}{22} = 0.3181818\ldots$ has a nonterminating but repeating decimal expansion.

Since the representation of most real numbers requires an infinite series of digits, we expect that only certain numbers can be represented exactly for hand or computer calculations because the number of decimal places is limited in any such calculations. The importance of these observations will be explored later when MATLAB computer numbers are described.

Axioms for Real Numbers In mathematics, certain operations and basic properties involving the quantities being considered such as numbers, vectors or matrices are assumed to be true. These assumptions are called *axioms* or *postulates* and no proof is necessary. Most of the other properties are presented by *theorems*, which are proven using the postulates and perhaps other theorems. The theorems allow further generalizations and applications to be developed.

The basic axioms for real numbers define how numbers are combined by addition and multiplication. These axioms define the associative, commutative, and distributive laws as well as the identity and inverse elements for any real numbers $x, y,$ and z.

Associative laws. For real numbers, the grouping of the operations for addition or multiplication is not important, since

1. $(x + y) + z = x + (y + z)$,

2. $(x \times y) \times z = x \times (y \times z)$.

A number of basic operations are not associative. For example, subtraction is not associative, since

$$(x - y) - z \neq x - (y - z).$$

Thus, if $x = 5, y = 3, z = 1$, the left grouping yields 1, but 3 results from the operations on the right. For nonassociative operations, careful grouping is important.

Commutative laws. The order of addition or multiplication of real numbers does not matter:

1. $x + y = y + x$,

2. $x \times y = y \times x$.

Certain operations of importance in vector and matrix algebra that we will study later are not commutative with respect to multiplication.

Distributive laws. Addition and multiplication are connected by the distributive law:

$$x \times (y + z) = x \times y + x \times z.$$

Identity elements. The numbers 0 and 1 are called the *identity elements* for addition and multiplication, respectively:

1. $x + 0 = x$,

2. $1 \times x = x$.

Inverse elements. For each real number x there is an inverse element for addition, and if the real number is not zero there is an inverse for multiplication:

1. $x + (-x) = 0$,

2. if $x \neq 0$, $x \times (1/x) = 1$.

The inverse for addition is called the *negative* of the number and the inverse for multiplication is called the *reciprocal*. The negative can be used to define the operation of subtraction as addition of the negative, writing $x - y = x + (-y)$. The reciprocal can be used to define the operation of division as multiplication by the reciprocal, $x \div y = x \times (1/y)$ where $y \neq 0$.

It is interesting to note that the properties just listed do not uniquely define the real numbers. For example, the complex numbers and the binary numbers also satisfy these properties.

Generalization   Part of our mathematical education is the generalization of the concept of *number* from the positive integers (counting numbers) to fractions (rational numbers) and then to negative and irrational numbers. The generalization continues with the study of transcendental numbers such as π and finally the complex numbers. All these numbers satisfy the axioms just presented.

ABSOLUTE
VALUE

The real numbers can be represented as points on the *real number* line shown in Figure 2.1. For convenience, the entire real number line will be referred to as **R**. Thus, a statement about **R** applies to all the real numbers.

On **R**, the distance of any real number from the origin is the magnitude, or *absolute value*, of the number. The absolute value is defined as

$$| x | = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases} \tag{2.1}$$

Applying the formula, $+5$ has the absolute value 5, as expected. The number -5 also has the absolute value $-(-5) = 5$.
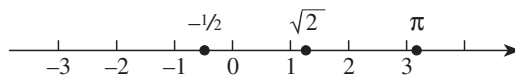

FIGURE 2.1 *The number line* **R**

The *distance* between the real numbers x and y is defined as $| x - y |$. This distance is the length of the line segment of **R** with the endpoints x and y. The distance on **R** between the numbers -5 and 5 is then $| -5 - (+5) | = 10$. Notice that the absolute value defines the positive distance of the number x from the origin where $x = 0$. These concepts will be generalized when vectors are studied.

2.2 MATLAB COMPUTER NUMBERS (OPTIONAL)

In modern computer systems, numbers are represented in the *binary* number system. The digits are 0 and 1 and the computer can store a finite number of binary digits to represent any number. For integers, the decimal value of a binary number is

$$N = \sum_{i=0}^{m-1} d_i \, 2^i = d_0 + d_1 \times 2 + d_2 \times 2^2 + \cdots + d_{m-1} \times 2^{m-1}, \tag{2.2}$$

where m is the number of digits and the d_i are binary digits.

The number range for a binary number with m digits is from

$$(000\ldots0000)_2 \quad \text{to} \quad (11111\ldots1111)_2,$$

where the notation $(N)_2$ means that N is a number in base 2.[1] The most positive value sums to the decimal value $2^m - 1$. An 8-digit positive binary

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[1] In the text, numbers other than decimal numbers will have the base specified explicitly.

number thus has the maximum value of $2^8 - 1$, or 255. The number 1011_2 has the decimal value 11 using the expansion of Equation 2.2.

If a fraction is to be represented, a *binary point* is used to separate the integer from the fractional part of the number, just as the decimal point is used for decimal numbers. The binary number $(0.11111 \ldots 1111)_2$ has the decimal value

$$
\begin{aligned}
N &= \sum_{i=1}^{m} d_{-i}\, 2^{-i} = d_{-1} \times 2^{-1} + d_{-2} \times 2^{-2} + \cdots + d_{-m} \times 2^{-m} \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + d_{-m} \times 2^{-m},
\end{aligned}
\tag{2.3}
$$

since $d_{-i} = 1$ for $i = 1, \ldots, m$. This is a geometric series whose sum converges to 1 as m goes to infinity, as you are asked to show in Problem 2.3. For a finite value of m digits, the sum is $1 - 2^{-m}$. With 16 binary digits, the value is

$$
1 - 2^{-16} = 0.99998
$$

to five places after the decimal.

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FIXED-POINT NOTATION

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The binary number representation just discussed assumes that the binary point was located in a fixed position, yielding either an integer or a fraction as the interpretation of the internal machine representation. Unfortunately, the binary point is not actually stored with the number, but its position must be remembered by the programmer (or the program) to display the results in the usual form. This method of representation is called *fixed point*.

It is a theorem that any real number x such that $0 < x \leq 1$ can be represented in a unique manner as a nonterminating binary fraction. In the computer, the binary series must be terminated after m digits to represent a decimal number. Except in special cases such as $1/2$, the binary representation will be an approximation. The error can be shown to be less than 2^{-m}.

In practice, the machine number also is limited to a finite range determined by the number of binary digits used in the representation. For a 32-digit binary number, the range of positive fixed-point integers is about $+2^{32}$ or $+10^{11}$, obtained by solving the equation $2^{32} = 10^y$ for the exponent $y = 32 \log_{10} 2$. This limited range of the fixed-point numbers is a drawback for certain applications.

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FLOATING-POINT NOTATION

To overcome many of the limitations of fixed-point notation, a method that is the counterpart of scientific notation is used for number representation in computers. *Floating-point* notation represents a number as a fractional part times a selected base raised to a power. The choice for the base is usually 2, although base 16 is sometimes used. In the machine

representation, only the fractional part and the value of the exponent are stored. The decimal equivalent is written as

$$N.n = f \times 2^e \tag{2.4}$$

using base 2 in this case. Thus, 1.5 could be written as $(11)_2 \times 2^{-1}$. The fraction f is called the *mantissa* and e is the *exponent*.

In floating-point notation, the number of binary digits devoted to the fraction determines the precision and the length of the exponent in digits determines the range. The actual arithmetic operations may be carried out by a computer's Central Processing Unit (CPU) or by a separate hardware chip called a co-processor. In some cases, the arithmetic is performed by a software package containing routines for floating-point arithmetic.

MATLAB Floating-point Notation MATLAB and many other programs for mathematics may employ the IEEE floating-point standard representation for machine numbers.[2] The results of a calculation for this format can contain as many as 16 significant decimal digits and have a range of from 10^{-308} to 10^{308}. However, the results may be displayed in a number of different ways.

Comment: Unless floating-point operations are implemented in software, the machine floating-point hardware actually determines the number format used. In fact, some machines do not conform to the IEEE standard. Try the MATLAB command **isieee** to determine if your computer uses IEEE arithmetic.

☐ EXAMPLE 2.1   ***MATLAB Range of Numbers***

Although MATLAB's calculations are carried out to the maximum precision in floating point internally, the results may be displayed as fixed-point decimal numbers or as numbers in scientific notation. Table 2.2 lists the choices.

Although physical measurements rarely justify the precision yielded by the long display formats in the table, MATLAB computes values to the highest precision to reduce roundoff errors. Roundoff error is the difference between the perfectly accurate number used in mathematical analysis and the computed number with only a fixed number of significant digits. The hexadecimal format listed in the table is frequently used to read binary numbers in base 16 notation for convenience.

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[2]IEEE, usually pronounced I-triple-E, is the Institute of Electrical and Electronics Engineers.

TABLE 2.2 *MATLAB formats for numbers*

Command	Format
format	Default setting (short)
format short	Fixed-point format with 5 digits
format long	Fixed-point format with 15 digits
format short e	Floating-point format with 5 digits
format long e	Floating-point format with 15 digits
format hex	Hexadecimal format

The accompanying MATLAB script is taken from a diary file to show various numerical values. The diary file was edited to display the results of each numerical input on the same line. The smallest fraction representable in the author's machine is shown in the MATLAB script as machine epsilon (`eps`). Using `eps` can sometimes avoid a divide by zero. For example, consider the limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

As computed by MATLAB, $\sin x / x$ with $x = 0$ would result in a divide-by-zero error. However, the statements

```
x=x+eps
y=sin(x)/x
```

would lead to the correct result. Adding `eps` to each element in an array should not affect the results of arithmetic operations unless the values are so small that `eps` is significant.

Various long and short formats are also shown in the MATLAB script. Each number was entered by typing the value at the MATLAB prompt, and the MATLAB representation is shown on the same line. Notice that the program warns the user if a division by zero occurs. Remember that machine zero is simply a number smaller than the smallest possible computer number. Mathematically, this number is not necessarily actually zero.

The answer `NaN` (Not a Number) is standard IEEE floating-point notation for an indeterminate result. Thus, if the IEEE standard is being used, MATLAB does not stop after calculations such as $0/0$ or ∞/∞, although the results are not meaningful. The purpose of introducing `NaN` in this way is so that a program can determine that a calculation was invalid and proceed accordingly.

Next in the script, the value `y` is entered in exponential form representing 10^{306}. When multiplied by 100, the number is still within MATLAB's range. If the new `y` is then multiplied by 5, the value overflows and MATLAB indicates that the product `z` is infinity. Other values, such as 10^{-307} and $\log_{10}(2)$, are also shown.

MATLAB Script _____

```
Example 2.1
>>% (Machine epsilon)
% EX1_2.M
>>eps
eps =  2.2204e-016
>>format short
1/3
ans =  0.3333
>>format long
>>1/3
ans =  0.33333333333333
>>1/0
Warning: Divide by zero
ans =  Inf
>>0/0
Warning: Divide by zero
ans =  NaN
>>y=1.0e306
y =    1.000000000000000e+306
>>y=100*y
y =    1.000000000000001e+308
>>z=5*y
z =    Inf
log10(2)
ans = 0.30102999566398
>>x=1e-307
  x = 9.999999999999999e-308
>>log10(2)
  ans =  0.30102999566398
>>quit
```

□

WHAT IF? Suppose the MATLAB representation of a number is not accurate enough for an application. Investigate the variable precision arithmetic command **vpa** of the *Symbolic Math Toolbox*. Compare MATLAB's numerical value of π with the symbolic representation to more than 15 digits.

Problem 2.18 and Problem 2.24 treat MATLAB roundoff errors. Use the command **format long** to see the 15 digit fixed-point representation of the numbers or **format long e** for the floating-point representation.

MATLAB COMMANDS FOR REAL NUMBERS

MATLAB has various commands that are useful for manipulating real numbers. Some commands are listed in Table 2.3 and are illustrated in Example 2.2. The **abs** command computes the absolute value. The commands for rounding numbers produce the results listed in the table. The **sign** evaluates the *Signum* function. The result from **sign** is zero if the value being tested is zero, $+1$ if the value is positive, and -1 if the value is negative.

TABLE 2.3 *MATLAB commands for real numbers*

Command	Format
abs	Absolute value
ceil	Round toward infinity
fix	Round toward zero
floor	Round toward minus infinity
sign	Signum function

☐ **EXAMPLE 2.2**   ***MATLAB Commands for Real Numbers***

Let $x = 9.2$. The accompanying script illustrates the edited results using the commands from Table 2.3.

MATLAB Script

```
Example 2.2
>> x = 9.2
x = 9.2000
>> abs(x)
ans =  9.2000
>> ceil(x)
ans =    10
>> floor(x)
ans =     9
>> fix(x)
ans =     9
>> sign(x)
ans =     1
```

☐

2.3 COMPLEX NUMBERS

A *complex number* can be written $z = x + iy$ or $z = x + yi$, where x and y are real numbers and i is the imaginary unit with the property $i^2 = -1$. The number x is called the *real part* of z and y is called the *imaginary part*. In functional notation it is sometimes convenient to write

$$x = \operatorname{Re} z \quad \text{and} \quad y = \operatorname{Im} z. \tag{2.5}$$

A complex number can also be represented as a point $P(x, y)$ in the xy-plane (Cartesian plane) as shown in Figure 2.2. This representation is called an *Argand* diagram using the xy-plane to plot the complex number. When used this way, the xy-plane is often referred to as the complex plane or the z-plane. Another notation for a complex number is formed by writing $z = x + iy$ as the ordered pair

$$z = (x, y)$$

which gives the correspondence between an ordered pair of real numbers and each complex number. Thus, $x + iy$ is a single value in the xy-plane.

Polar coordinates can be associated with a complex number in the complex plane. From the figure, the radial distance r and the polar angle θ with respect to the positive real x-axis are

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \tan \theta &= \frac{y}{x} \end{aligned} \tag{2.6}$$

for $x \neq 0$. If $x = 0$, $\theta = \pi/2$ when $y > 0$ and $\theta = -\pi/2$ when $y < 0$.
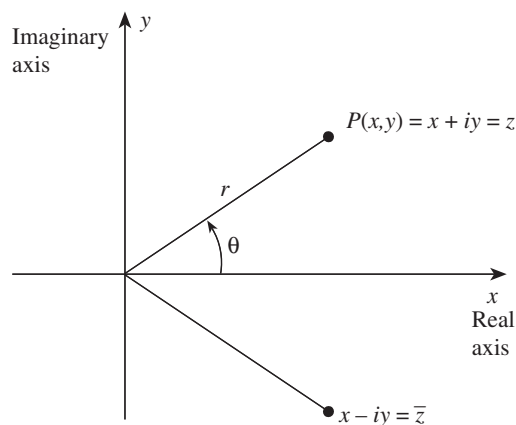

FIGURE 2.2 *Complex numbers*

The radial distance of z is written as $r = |z|$ and is called the *magnitude* of z. The magnitude is also called the *modulus* or length of z. The *complex conjugate* of z is the number $\bar{z} = x - iy$ in Figure 2.2 with polar angle $-\theta$. From the figure, it is evident that $|\bar{z}| = |z|$. The polar angle is called the *argument* of z, written arg z. We see that

$$\arg \bar{z} = -\arg z.$$

The value of arg z is usually taken to be in the range from $-\pi$ to π, and this value is called the *principal value* of the argument. In calculations, it is frequently useful to take advantage of the fact that angle measure is a multi-valued function, with values of the form $\theta + n2\pi$, where n is an integer and θ is a particular angle measure.

The complex numbers have operations of addition and multiplication that obey the properties of real numbers previously listed. The *sum* of two complex numbers is formed by adding the real and imaginary parts. Thus, if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum is

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \tag{2.7}$$

The *negative* of $z_2 = x_2 + iy_2$ is $-z_2 = -x_2 - iy_2$. Subtraction can be accomplished by adding $-z_2$ to z_1 to form the difference $z_1 - z_2$. Thus

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2). \tag{2.8}$$

The *product* of two complex numbers z_1 and z_2 is

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \tag{2.9}$$

From this formula, it follows that $z\bar{z} = |z|^2$.

Using the definition of multiplication of complex numbers, division of complex numbers can be shown to be given by the formula

$$\frac{z_1}{z_2} = \left(\frac{z_1}{z_2}\right)\left(\frac{\bar{z}_2}{\bar{z}_2}\right) = \frac{z_1 \bar{z}_2}{|z_2|^2} = \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 + y_2^2}. \tag{2.10}$$

By definition, two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. Thus,

$$z_1 = z_2 \quad \text{if and only if} \quad \operatorname{Re} z_1 = \operatorname{Re} z_2 \text{ and } \operatorname{Im} z_1 = \operatorname{Im} z_2 \tag{2.11}$$

or

$$x_1 + iy_1 = x_2 + iy_2 \quad \text{if and only if} \quad x_1 = x_2 \text{ and } y_1 = y_2. \tag{2.12}$$

The real numbers can be viewed as a subset of the complex numbers, namely the complex numbers of the form $z = x + i0 = (x, 0)$. In the plane, this set is the x-axis. Moreover, the definitions of addition, subtraction, multiplication and division for these complex numbers correspond exactly

to the usual definitions for real numbers. It is traditional to speak of a complex number as real (or a real scalar) in this case. Since $z = \bar{z}$ if and only if $\operatorname{Im} z = 0$, we have that z is real if and only if $z = \bar{z}$. Multiplication of $z = x + iy$ by a real scalar value k yields the results

$$kz = k(x + iy) = (k + i0)(x + iy) = kx + iky. \tag{2.13}$$

It is straightforward to show that the complex modulus function follows exactly the same rules relative to arithmetic as the real absolute value function. Namely,

$$\begin{aligned} |z_1 z_2| &= |z_1||z_2| \\ \left|\frac{z_1}{z_2}\right| &= \frac{|z_1|}{|z_2|}. \end{aligned} \tag{2.14}$$

POLAR FORM OF COMPLEX NUMBERS

Using the magnitude and argument of z in Figure 2.2, a complex number can be expressed in the form

$$z = r(\cos\theta + i\sin\theta). \tag{2.15}$$

Using Equation 2.15 and some basic trigonometry it can be shown that for $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_1(\cos\theta_2 + i\sin\theta_2)$,

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right] \tag{2.16}$$

which means that multiplying complex numbers not only corresponds to multiplying their absolute values, as observed in Equation 2.14, but also corresponds to adding their arguments, a quite useful fact in applications involving rotations. Strictly speaking, the addition of arguments must be understood modulo 2π, since the principal value of the argument of $(1 + i)(1 + i) = 2i$ is not $3\pi/2$ ($= 3\pi/4 + 3\pi/4$), but rather $-\pi/2$.

Complex analysis extends the basic concepts and many of the familiar functions of calculus to the set of complex numbers. For example, there is a *complex exponential function* $\exp z$, or e^z, which has the values of the real exponential function when z is real. In complex analysis it is shown that the complex exponential function satisfies the relationship

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{2.17}$$

which is known as *Euler's formula*. In Equation 2.17, the sin and cos functions on the right are the familiar real valued functions.[3] Thus, the polar form of a complex number can also be written as

$$z = re^{i\theta}. \tag{2.18}$$

---

[3]For our purposes, we will consider θ to be real, although actually Equation 2.17 holds for any complex θ, with the extensions of the trigonometric functions to the complex numbers.

This is a convenient form for certain applications, as discussed in Chapter 5. Since the complex exponential function satisfies the same multiplication rule as the real exponential function e^x, the expression e^z has the representation

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y). \qquad (2.19)$$

ROOTS OF COMPLEX NUMBERS

Let a be a nonzero complex number. Then if a complex number z satisfies the equation

$$z^N = a,$$

z is called an Nth *root* of a. The determination of the Nth roots of a uses the theorem of De Moivre

$$(\cos \theta + i \sin \theta)^N = \cos N\theta + i \sin N\theta \qquad (2.20)$$

which follows from Equation 2.16. Writing a in polar coordinates as

$$a = r(\cos \theta + i \sin \theta),$$

the roots of a are

$$a^{1/N} = \sqrt[N]{r}\left[\cos\left(\frac{\theta}{N} + p\frac{2\pi}{N}\right) + i\sin\left(\frac{\theta}{N} + p\frac{2\pi}{N}\right)\right] \qquad (2.21)$$

where p takes the values $0, 1, \ldots, N-1$. These are N distinct values, and it can be shown that there are no others. The term $\sqrt[N]{r}$ is the positive real Nth root of r and for the specific case $|a| = 1$ this value is 1.

☐ **EXAMPLE 2.3**   *Complex Numbers*

    a. Various operations on complex numbers are illustrated as follows:

        (i). $(3 + 6i) + (2 - 3i) = 5 + 3i$;

        (ii). $(7 + 5i) - (1 + 2i) = 6 + 3i$;

        (iii). $(5 + 7i)(3 + 4i) = (15 - 28) + i(20 + 21) = -13 + 41i$;

        (iv).

$$\frac{4+i}{2-3i} = \frac{4+i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{(8-3) + i(2+12)}{13} = \frac{5 + 14i}{13} = \frac{5}{13} + \frac{14}{13}i.$$

    b. To solve the equation

$$(x + iy)(2 - i3) = 4 + i$$

for x and y, equate the real and imaginary parts to yield the simultaneous set of equations

$$2x + 3y = 4$$
$$-3x + 2y = 1.$$

The solutions are $x = 5/13$ and $y = 14/13$.

c. In Equation 2.21, let $a = 1$. For this complex number, the argument θ is zero and the magnitude is 1 so we can write

$$z = a^{1/N} = e^{ip\ 2\pi/N} = \cos\frac{p\ 2\pi}{N} + i\sin\frac{p\ 2\pi}{N}$$

with the property that $|a| = 1$ so that every one of these complex numbers lies on the unit circle in the complex plane.

Consider the values $(p/N)2\pi$ for integers $p = 0, 1, 2, 3$ when $N = 4$. Writing $z = e^{\alpha}$ where $\alpha = 2\pi p/4$, these values correspond to the angles

$$\alpha = 0\ (2\pi),\ \alpha = \pi/2,\ \alpha = \pi \text{ and } \alpha = 3\pi/2$$

in the complex plane, respectively. The complex values allowed for z are then

$$
\begin{aligned}
z_0 &= e^0 &= 1 \\
z_1 &= e^{i\pi/2} &= i \\
z_2 &= e^{i\pi} &= -1 \\
z_3 &= e^{i3\pi/2} &= -i.
\end{aligned}
$$

(2.22)

The square roots of unity $(N = 2)$ are ± 1 and the square roots of these roots are $1, -1, i, -i$ as is easily verified. The sum of the roots $(1 + i - 1 - i)$ is zero.

$\square$

MATLAB COMPLEX NUMBERS

MATLAB accepts complex numbers in both Cartesian and polar form. The Cartesian format for the complex number $z = 3 + 4i$ would be

$\gg$ `z=3+4i`
$\gg$ `z=3+4j`

since either `i` or `j` can be used to specify the imaginary number $\sqrt{-1}$. Because MATLAB would interpret `i4` as a variable name, even though mathematically $4i = i4$, in MATLAB we must write `i*4` in statements such as `z=3+i*4`.

The MATLAB polar form is

$\gg$ `w=r*exp(i*theta)`

where `r` and `theta` must be defined before `w` is used.

Comment: It is important not to use the variables i and j for loop variables or similar purposes in programs that also use MATLAB's complex numbers since the loop variables may override their definition as imaginary numbers.

Table 2.4 lists some of the MATLAB commands that are useful for manipulating complex numbers.

TABLE 2.4 *MATLAB commands for complex numbers*

Command	Format
`z=x+yi, z=x+yj`	Complex number
`z=r*exp(i*theta)`	Polar form
abs	Magnitude $|z| = \sqrt{x^2 + y^2}$
angle	Angle in radians $(-\pi, \pi)$; $\theta = \tan^{-1}(y/x)$
conj	Complex conjugate $x - yi$
imag	Complex imaginary part y
real	Complex real part x
Plotting:	
compass	Draws complex numbers as arrows on polar plot
feather	Draws complex numbers as arrows on linear plot

Ordinary operations $+, -, *, /$ for addition, subtraction, multiplication, and division respectively perform these operations on two complex numbers.

☐ EXAMPLE 2.4   ***MATLAB Complex Numbers***

The accompanying script shows the MATLAB commands to compute Part a and Part c of Example 2.3.

MATLAB Script

```
Example 2.4
>> 3+6i + 2-3i
ans =   5.0000 + 3.0000i
>> 7+5i -(1+2i)
ans =   6.0000 + 3.0000i
>> (5+7i)*(3+4i)
ans = -13.0000 +41.0000i
>> (4+i)/(2-3i)
ans =   0.3846 + 1.0769i
>> % Solve z^4-1=0
>> roots([1 0 0 0 -1])
ans =
  -1.0000
   0.0000 + 1.0000i
   0.0000 - 1.0000i
   1.0000
```

☐

2.4 VECTORS IN TWO DIMENSIONS AND THREE DIMENSIONS

The notion of *vectors* has proven invaluable in physics and mathematics. The vector in space is a combination of a positive number called the *magnitude* and a *direction*. One useful feature of vectors is that the equations of physics can be represented in terms of vectors independent of a particular coordinate system. Newton's second law

$$\mathbf{F} = m\mathbf{a}$$

provides an example. The vector form of the equation is not altered by a change in the coordinate axes. For example, the form shown for Newton's law holds whether $\mathbf{F}$ is defined in rectangular, polar, or spherical coordinate systems. Also, the vector equation is a convenient "shorthand" to represent complicated formulas. The use of vectors can lead to greater understanding of a problem as well as simplification of computations as will be shown in many examples in this text.

INTRODUCTION TO VECTORS

In some cases involving two-dimensional (2D) and three-dimensional (3D) space, visualization of a physical situation is aided by drawing the vectors involved as directed line segments to give the vectors a geometric interpretation. A vector can also be identified using coordinates that represent its origin and endpoint. This latter view is the algebraic view.

As discussed previously, the real numbers can be identified with the points of a straight line, as shown in Figure 2.1. Each number x is represented by a point on the line, so the value can be pictured geometrically with respect to other real numbers. Similarly, the origin and endpoint of vectors in the 2D plane can be represented as pairs of real numbers, and the components can be plotted in the xy-plane, as shown in Figure 2.3. When the origin of the vector is $[0, 0]$ in the xy-plane, the vector is typically defined only by the coordinates of its endpoint.
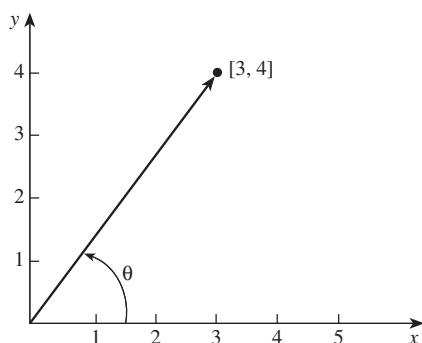

FIGURE 2.3 *Points and vectors*

For example, the 2D vector starting at $[0, 0]$ with endpoint $[3, 4]$ can be drawn as a line segment from the origin $[0, 0]$ to its endpoint 3 units in the x direction and 4 units in the y direction. These values of 3 and 4 represent the *components* of the vector in the x and y directions, respectively. In three dimensions, the vector's endpoints are represented by three real numbers so the vector can be designated as $[x_1, x_2, x_3]$.

A vector can thus be described geometrically by a line from its origin to its endpoint. The geometric view is used to visualize the problem at hand, as when the vectors represent distances or forces in various directions. The algebraic notation is useful when performing calculations such as addition with vectors. The legitimate operations that are allowed for vectors will be defined later in this chapter.

Notation   In this text, individual vectors are usually designated by roman letters printed in boldface type ($\mathbf{a}, \mathbf{b}, \mathbf{A}, \mathbf{x} \ldots$). As an example, the equation $\mathbf{a} = [3, 4]$ defines $\mathbf{a}$ as the vector from the origin to the point $[3, 4]$. The same notation means that the vector $\mathbf{a}$ has *component* 3 in the x direction and *component* 4 in the y direction.

Numbers are often called *scalars* to emphasize the distinction between numbers and vectors. Thus, energy and power in physics are scalar quantities, whereas velocity and acceleration are vector quantities, since these later quantities have a direction associated with them.

The term *scalar multiple of a vector* means multiplication of the vector components by a number. The multiplication operation is denoted as $\alpha \mathbf{x}$ for the scalar α and the vector $\mathbf{x}$.

The notation $\mathbf{R}^2$ is used to denote the set of all 2D vectors, and $\mathbf{R}^3$ indicates all of the 3D vectors. The exponent defines the *dimension* of the vectors being discussed. Thus, $\mathbf{a} = [3, 4]$ is a 2D vector that is a member of $\mathbf{R}^2$.

Addition and Subtraction of Vectors   Consider two arbitrary vectors $\mathbf{x} = [x_1, x_2]$ and $\mathbf{y} = [y_1, y_2]$. The sum and difference of the vectors are the new vectors

$$
\begin{aligned}
\mathbf{x} + \mathbf{y} &= [x_1 + y_1, x_2 + y_2], \\
\mathbf{x} - \mathbf{y} &= [x_1 - y_1, x_2 - y_2],
\end{aligned}
$$

in terms of the components of the original vectors. Notice that the resulting vectors would not generally have the origin as their beginning point. We can also draw the vectors and determine the sum and difference as shown in Figure 2.4.
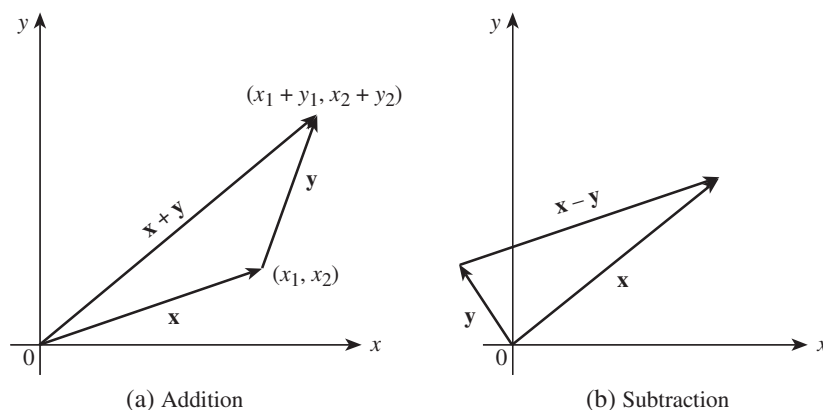
(a) Addition   **(b)** Subtraction

FIGURE 2.4 *Addition and subtraction of vectors*

In Figure 2.4(a), the vector $\mathbf{y}$ is shifted so that initial point of $\mathbf{y}$ is placed at the terminal point of $\mathbf{x}$. The vector $\mathbf{x} + \mathbf{y}$ is the vector from the origin to the end point of $\mathbf{y}$ in the figure. By drawing the diagram with $\mathbf{y}$ at the origin, you can show that $\mathbf{y} + \mathbf{x} = \mathbf{x} + \mathbf{y}$. Notice that this definition of addition of 2D vectors corresponds exactly to the addition of complex numbers.

Figure 2.4(b) shows the subtraction of two vectors. As an aid to remembering the direction, note that $\mathbf{x} = \mathbf{x} - \mathbf{y} + \mathbf{y}$, so that $\mathbf{x} - \mathbf{y}$ is the vector that must be added to $\mathbf{y}$ to obtain $\mathbf{x}$. The vector $\mathbf{y} - \mathbf{x}$ has the same length as $\mathbf{x} - \mathbf{y}$ but points in the opposite direction.

Scalar multiplication of vectors   The multiplication of a vector $\mathbf{x} = [x_1, x_2]$ by a scalar r is defined by

$$r\mathbf{x} = [rx_1, rx_2]$$

which, for r positive, has the effect of stretching or shrinking $\mathbf{x}$ by a factor of r. For r negative, it also reverses its direction. Notice that this definition for 2D vectors corresponds exactly to the multiplication of a complex number by a scalar.

Length of a Vector   The *length* of a vector $\mathbf{x}$ is a nonnegative scalar quantity that measures the distance between its initial point and terminal point. In the Cartesian coordinate system, the components of a 3D vector represent the lengths along the three mutually perpendicular axes x, y, and z.

The length of a vector is also called its *magnitude*, especially when the vector represents a physical quantity. Thus, if $\mathbf{a}$ defines the acceleration vector for a body, the magnitude of the acceleration is written a in this textbook.

The length of a 2D vector with components $[x_1, x_2]$ in the xy-plane follows from the Pythagorean theorem as

$$|| \, \mathbf{x} \, || \;\; = \;\; \sqrt{x_1{}^2 + x_2{}^2}. \tag{2.23}$$

The length of the 2D vector $\mathbf{a} = [3, 4]$ previously defined is thus

$$|| \, \mathbf{a} \, || = \sqrt{3^2 + 4^2} = 5.$$

Generalization to three dimensions is straightforward.

The length of the vector $\mathbf{a}$ between points $[x_1, y_1]$ and $[x_2, y_2]$ can be found by forming $x = x_2 - x_1$ and $y = y_2 - y_1$ and then computing

$$|| \, \mathbf{a} \, || \;\; = \;\; \sqrt{x^2 + y^2}.$$

Some authors use the notation $| \, \mathbf{a} \, |$ to indicate the length or magnitude of the vector $\mathbf{a}$. This representation is quite common for vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$. In a later section of this chapter, the length or magnitude of a vector will be called the *norm* of the vector. This is a generalization of the concept of length, which is useful for vectors with more than two or three components. The norm of $\mathbf{a}$ is written $|| \, \mathbf{a} \, ||$.

Unit Vectors A vector $\mathbf{x}$ of length 1 ($|| \, \mathbf{x} \, || = 1$) will be called a *unit vector*. For any vector $\mathbf{x}$, a unit vector $\mathbf{n}$ in the direction of $\mathbf{x}$ can be computed as

$$\mathbf{n} = \frac{\mathbf{x}}{|| \, \mathbf{x} \, ||}$$

assuming that $|| \, \mathbf{x} \, ||$ is not zero. This unit vector defines the direction of vector $\mathbf{x}$.

A nonzero vector that is divided by its magnitude is said to be *normalized*. Such a vector, as for example the vector $\mathbf{n} = \mathbf{x}/||\mathbf{x}||$, has the property that $||\mathbf{n}|| = 1$.

DOT PRODUCT An important operation for two vectors of the same dimension is the *scalar*, or *dot*, product. One definition of the dot product for two nonzero vectors with a common origin point is

$$\mathbf{x} \cdot \mathbf{y} \;\; = \;\; || \, \mathbf{x} \, || \; || \, \mathbf{y} \, || \; \cos \theta. \tag{2.24}$$

In the equation, θ is the angle between the vectors, usually measured in radians, taken from the interval $0 \leq \theta \leq \pi$.

The dot product of the 3D vectors $\mathbf{x} = [x_1, x_2, x_3]$ and $\mathbf{y} = [y_1, y_2, y_3]$ can also be shown to be

$$\mathbf{x} \cdot \mathbf{y} \;\; = \;\; x_1 y_1 + x_2 y_2 + x_3 y_3 \tag{2.25}$$

in terms of their components. Of course, for 2D vectors the terms in x_3 and y_3 are not included. Notice that the length of the vector $\mathbf{x}$ can be

computed as $\sqrt{\mathbf{x} \cdot \mathbf{x}}$, as shown by letting $\mathbf{x} = \mathbf{y}$ in Equation 2.25, which defines the dot product in terms of the components of the vectors.

The two definitions just given for the dot product are equivalent. For example, if $\mathbf{x}$ and $\mathbf{y}$ are two nonzero vectors in three dimensions that originate from a common point, the cosine of the angle between them is

$$\cos \theta = \frac{(\mathbf{x} \cdot \mathbf{y})}{(\|\mathbf{x}\| \, \|\mathbf{y}\|)} \tag{2.26}$$

$$= \frac{x_1 y_1 + x_2 y_2 + x_3 y_3}{\sqrt{x_1^2 + x_2^2 + x_3^2} \sqrt{y_1^2 + y_2^2 + y_3^2}}. \tag{2.27}$$

This relationship can be proven, as indicated by Problem 2.29.

☐ **EXAMPLE 2.5** **Use of the Dot Product**

Let $\mathbf{x} = [2, 4, 0]$ and $\mathbf{y} = [6, -4, 2]$. Thus, the vectors have length

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{4 + 16 + 0} = 2\sqrt{5},$$
$$\|\mathbf{y}\| = \sqrt{\mathbf{y} \cdot \mathbf{y}} = \sqrt{36 + 16 + 4} = 2\sqrt{14},$$

and the dot product is

$$\mathbf{x} \cdot \mathbf{y} = (2)(6) + (4)(-4) + (0)(2) = -4.$$

By combining Equation 2.24 and Equation 2.23, the angle between the vectors is computed as

$$\theta = \cos^{-1}\left[\frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|}\right]$$

$$= \cos^{-1}\left[\frac{-4}{(2\sqrt{5})(2\sqrt{14})}\right] = \cos^{-1}\left[\frac{-1}{\sqrt{70}}\right] \approx 96.9°.$$

If we define $\mathbf{d} = \mathbf{y} - \mathbf{x}$, then the distance between the endpoints of $\mathbf{y}$ and $\mathbf{x}$ is the length of $\mathbf{d}$. Thus, the distance is

$$\|\mathbf{d}\| = \|\mathbf{y} - \mathbf{x}\|$$
$$= \sqrt{(6-2)^2 + (-4-4)^2 + (2-0)^2} = \sqrt{84}$$
$$= 2\sqrt{21}.$$

☐

Orthogonal Vectors If two vectors have a nonzero length and

$$\mathbf{x} \cdot \mathbf{y} = 0,$$

the vectors are said to be perpendicular, or *orthogonal*, to each other. In 2D and 3D, the vectors thus meet at right angles, since $\theta = 90°$ when $\cos \theta = 0$. The fact that two vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal can be represented by the notation $\mathbf{x} \perp \mathbf{y}$.

Standard Unit Vectors   The 3D *unit vectors*, defined as

$$
\begin{aligned}
\mathbf{i} &= [1, 0, 0], \\
\mathbf{j} &= [0, 1, 0], \\
\mathbf{k} &= [0, 0, 1],
\end{aligned}
\tag{2.28}
$$

are called the *standard*, or *natural*, unit vectors in $\mathbf{R}^3$. The triplet $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is also called a set of *basis vectors*, since any 3D vector can be represented as a sum of scalar multiples of these three unit vectors.

The standard unit vectors all have length 1 and are mutually orthogonal, as is easily shown by forming the dot product between any pair of these vectors. Such a set of vectors is said to form an *orthonormal* set.

□ EXAMPLE 2.6   **Use of Unit Vectors**

The unit vectors are frequently used to form an alternative notation for a vector in ordinary 3D space. Thus, the vector $\mathbf{F} = [f_x, f_y, f_z]$ can be written as

$$
\mathbf{F} = f_x \mathbf{i} + f_y \mathbf{j} + f_z \mathbf{k},
\tag{2.29}
$$

where f_x, f_y, and f_z represent the lengths of the vector along the x-, y-, and z-axes, respectively.

However, the unit vectors and the dot product have far greater application than notational convenience or geometrical interpretation. For example, assume that the force on an object is defined by the vector $\mathbf{F}$. Then, the component of the force in the x-direction is given by $\mathbf{F} \cdot \mathbf{i}$. The proof is straightforward, since

$$
\mathbf{F} \cdot \mathbf{i} = f_x (\mathbf{i} \cdot \mathbf{i}) + f_y (\mathbf{j} \cdot \mathbf{i}) + f_z (\mathbf{k} \cdot \mathbf{i}) = f_x
\tag{2.30}
$$

after taking the dot product of the standard unit vectors.

<div align="right">□</div>

Application of the Dot Product   The dot product is frequently used to find the component of one vector in the direction of another. For example, an important use of the dot product is to determine the work done by a constant force $\mathbf{F}$ acting on a body moving a distance and in a direction defined by the vector $\mathbf{s}$. The magnitude of $\mathbf{s}$ defines the distance moved, but only the component of $\mathbf{F}$ in the direction of $\mathbf{s}$ does work. The calculation $\mathbf{F} \cdot \mathbf{s}$ multiplies the component of $\mathbf{F}$ in the direction of motion times the distance traveled and hence defines the work done. In vector calculus, the dot product plays a role in computing certain line and surface integrals and in the definition of the divergence of a function $(\nabla \cdot \mathbf{F})$ to be studied in later chapters.

ROW AND COLUMN VECTORS

So far in this chapter, we have represented a vector as a row of numbers, called a *row vector*. The vector $\mathbf{x}$ can also be represented as a *column vector* in the form

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.
$$

These representations of vectors will be useful in Chapter 3 when matrix operations are considered. For two vectors $\mathbf{x}$ and $\mathbf{y}$ that have the same number of components, the dot product can be computed by writing one vector as a row vector and the other as a column vector and performing matrix multiplication to yield

$$\mathbf{x} \cdot \mathbf{y} = [\, x_1, x_2, x_3 \,] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = x_1 y_1 + x_2 y_2 + x_3 y_3. \tag{2.31}$$

In some contexts, the terms *scalar product* and *inner product* are also used to describe the product of Equation 2.31.

PROJECTIONS   A very important application of the dot product of two vectors is in determining the *projection* of one nonzero vector along another. Figure 2.5 shows the geometry of the problem.
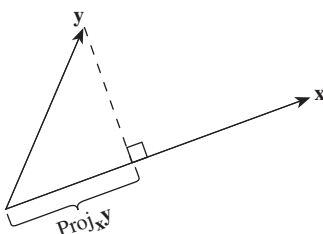
FIGURE 2.5 *Projections*

The figure shows the projection of $\mathbf{y}$ on $\mathbf{x}$ determined by dropping a line from the end of $\mathbf{y}$ that is perpendicular to $\mathbf{x}$ in the plane of $\mathbf{x}$ and $\mathbf{y}$. Thus, the length of the projection measured from the origin of the vectors to the intersection of the line with $\mathbf{x}$ is $||\mathbf{y}|| \cos \theta$. We now wish to define the projection in terms of the dot product and also define the vector that specifies the projection. This is accomplished by first observing that

$$||\mathbf{y}|| \cos \theta = \frac{(||\mathbf{x}||)||\mathbf{y}|| \cos \theta}{||\mathbf{x}||}$$

and noting that $\mathbf{x}/||\mathbf{x}||$ is a unit vector in the direction of $\mathbf{x}$ and hence defines the direction of the projection. Using the notation $\mathrm{proj}_\mathbf{x} \mathbf{y}$ to indicate the *projection* of $\mathbf{y}$ on $\mathbf{x}$, we can write

$$\mathrm{proj}_\mathbf{x} \mathbf{y} = \frac{(\mathbf{x} \cdot \mathbf{y})}{||\mathbf{x}||} \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{(\mathbf{x} \cdot \mathbf{y})}{||\mathbf{x}||^2} \mathbf{x}. \tag{2.32}$$

For the vectors in Figure 2.5, $\theta < \pi/2$ so $\mathbf{x} \cdot \mathbf{y} > 0$. If $\theta > \pi/2$, $\mathbf{x} \cdot \mathbf{y} < 0$.

□ EXAMPLE 2.7 *Projections*

a. The projection of $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$ on the vector $\mathbf{v} = \mathbf{i} + \mathbf{j}$ is given by

$$\text{proj}_{\mathbf{v}}\,\mathbf{u} = \frac{5\mathbf{v}}{(\sqrt{2})^2} = \frac{5}{2}\mathbf{i} + \frac{5}{2}\mathbf{j}.$$

b. Let $\mathbf{x}$ be a nonzero vector. Then, for any other vector $\mathbf{y}$, the vector

$$\mathbf{w} = \mathbf{y} - \frac{(\mathbf{y} \cdot \mathbf{x})}{||\mathbf{x}||^2}\mathbf{x} = \mathbf{y} - \text{proj}_{\mathbf{x}}\mathbf{y}$$

is orthogonal to $\mathbf{x}$. To show this, form the dot product of $\mathbf{w}$ and $\mathbf{x}$ and expand as

$$\mathbf{w} \cdot \mathbf{x} \;=\; \left[\mathbf{y} - \frac{(\mathbf{y} \cdot \mathbf{x})}{||\mathbf{x}||^2}\mathbf{x}\right] \cdot \mathbf{x} = \mathbf{y} \cdot \mathbf{x} - \frac{(\mathbf{y} \cdot \mathbf{x})}{||\mathbf{x}||^2}\mathbf{x} \cdot \mathbf{x}$$

$$=\; \mathbf{y} \cdot \mathbf{x} - \frac{(\mathbf{y} \cdot \mathbf{x})||\mathbf{x}||^2}{||\mathbf{x}||^2} = \mathbf{y} \cdot \mathbf{x} - \mathbf{y} \cdot \mathbf{x} = 0.$$

Using the vectors of Part a, the component of $\mathbf{u}$ perpendicular to $\mathbf{v}$ is

$$\mathbf{u}_{\perp} = (-\mathbf{i} + \mathbf{j})/2.$$

<div align="right">■</div>

W H A T   I F ?   You should draw the vectors $\mathbf{u}$ and $\mathbf{v}$ in Example 2.7 to confirm that the projection is correct. Also, show that $\mathbf{u}$ can be written as the sum of the components parallel and perpendicular to $\mathbf{v}$.

CROSS PRODUCT

The *cross product* is an operation on vectors in $\mathbf{R}^3$ that has widespread use in physics. Whereas the scalar (dot) product of two vectors yields a scalar value, the cross product of two vectors is another vector. Thus, some authors use the term *vector product* for this operation.

The cross product is indicated as $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ and satisfies the following relationship:

$$\mathbf{c} = ||\,\mathbf{a}\,||\,||\,\mathbf{b}\,||\,\sin\theta\,\mathbf{n}, \tag{2.33}$$

where $\mathbf{n}$ is a unit vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. Since the resulting vector is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, we can write

$$\mathbf{c} \perp \mathbf{a},\ \mathbf{c} \perp \mathbf{b}. \tag{2.34}$$

The cross product is defined only in three dimensions, and it is natural to apply it to vectors whose components are the value of the vector along the x-, y-, and z-axes. For the vectors $\mathbf{a} = [a_x, a_y, a_z]$ and $\mathbf{b} = [b_x, b_y, b_z]$, the expression for $\mathbf{a} \times \mathbf{b}$ can be written in terms of the vector components as

$$\mathbf{a} \times \mathbf{b} = (a_y b_z - a_z b_y)\mathbf{i} + (a_z b_x - a_x b_z)\mathbf{j} + (a_x b_y - a_y b_x)\mathbf{k}. \tag{2.35}$$

A technique to remember the expansion of Equation 2.35 is to write the array

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{2.36}$$

and expand this as a determinant in the form

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix}. \tag{2.37}$$

This is not a numerical determinant in the ordinary sense, as treated in Chapter 3, but it is a useful mnemonic to expand the cross product.

Figure 2.6 shows the relationship of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in the cross product $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ for the positive direction of the cross product vector $\mathbf{c}$. This orientation of the vectors defines a right-handed system of axes. Remember that a right-handed system is defined such that if the fingers of your right hand curl from $\mathbf{a}$ to $\mathbf{b}$, your thumb points in the direction of the cross product vector.
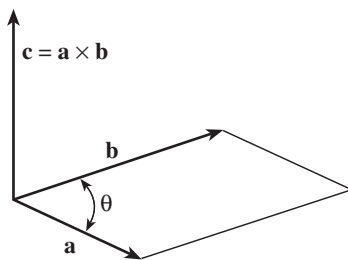

FIGURE 2.6 *Cross product of two vectors*

□ **EXAMPLE 2.8**   ***Cross Product***

A result of electromagnetic field theory is that the force $\mathbf{F}$ acting on a charge q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is given by

$$\mathbf{F} = q\,\mathbf{v} \times \mathbf{B},$$

where q is measured in coulombs and $\mathbf{B}$ has units of teslas in the mks (meter-kilogram-second) system.[4]

Let the physical values be

$$q = 1.6 \times 10^{-19} \text{ coulombs},$$
$$\mathbf{B} = 0.5\,\mathbf{i} - 0.8\,\mathbf{k} \text{ teslas},$$
$$\mathbf{v} = 10\mathbf{j} \text{ meters/second},$$

where the physical units apply to the magnitude of the vectors.

[4]The mks system is also called the SI (International System of Units) measurement system.

Then, the force acting on the charge is

$$q\,\mathbf{v} \times \mathbf{B} \;=\; q \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 10 & 0 \\ 0.5 & 0 & -0.8 \end{vmatrix}$$

$$=\; q\,[-8.0\mathbf{i} - 5.0\mathbf{k}]\,.$$

The result is $\mathbf{F} = (-12.8\mathbf{i} - 8.0\mathbf{k}) \times 10^{-19}$ newtons. Although the particle is initially moving in the y-direction, the field gives the particle components of velocity in the x- and z-directions as well.

$\square$

Cross products are used extensively in mechanics to represent such quantities as angular momentum. In electromagnetic field theory, the cross product defines the direction of the force acting on a moving charged particle in a magnetic field. The cross product will also be used again in Chapter 13, where the expression $\nabla \times \mathbf{F}$ will be introduced as the *curl* of a vector function.

Examples of Applications Table 2.5 lists a few of the applications of vectors commonly used in physics.

TABLE 2.5 *A few vector applications*

Meaning	Vector Form
Newton's second law	$\mathbf{F} = m\mathbf{a}$
Force on a charge	$\mathbf{F} = q\mathbf{E}$
Linear momentum	$\mathbf{p} = m\mathbf{v}$
Work	$W = \mathbf{F} \cdot \mathbf{s}$
Force on a charge	$\mathbf{F} = q\mathbf{v} \times \mathbf{B}$
Torque	$\mathbf{T} = \mathbf{r} \times \mathbf{F}$
Angular acceleration	$\mathbf{L} = \mathbf{r} \times \mathbf{p}$

2.5 VECTORS IN HIGHER DIMENSIONS

In elementary physics, the vectors of interest represent physical phenomena, such as the force or the electric field at a point in 2D or 3D space. However, the generalization of the vector concept to vectors having more than two or three components is useful in many areas of applied mathematics.

The notation $\mathbf{R}^n$ will denote the set of all vectors with n real components. In the previous section, $\mathbf{R}^2$ denoted the set containing all the 2D vectors with components of the form $[x, y]$. Similarly, $\mathbf{R}^3$ contains all the 3D vectors with three real components. The emphasis on the real components is made here because vectors may also have complex values for components. Unless otherwise noted in the text, the vectors discussed have real components.

When $\mathbf{x}$ is one of the vectors in the set $\mathbf{R}^n$, we say that $\mathbf{x}$ is in $\mathbf{R}^n$ or occasionally use the notation $\mathbf{x} \in \mathbf{R}^n$. Thus, the statement $\mathbf{a} \in \mathbf{R}^5$ means that $\mathbf{a}$ is a five-dimensional vector with five real components.

An n-dimensional vector $\mathbf{x}$ in $\mathbf{R}^n$ is defined as the n-tuple of real numbers written

$$\mathbf{x} = [x_1, x_2, \ldots, x_n], \tag{2.38}$$

where each x_i is a real number. For convenience, such a vector is often referred to as an *n-vector* or simply as a *vector* if n is understood.

Basic operations are defined for vectors in $\mathbf{R}^n$ that are straightforward generalizations of the familiar operations of addition, multiplication by a scalar, and defining the length of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$.

The *addition* of two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$ is defined as

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= [x_1, x_2, \ldots, x_n] + [y_1, y_2, \ldots, y_n] \\ &= [x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]. \end{aligned} \tag{2.39}$$

The *scalar multiple* of a vector $\alpha\mathbf{x}$ is formed by multiplication of each component of a vector in $\mathbf{R}^n$ by a real number α, with the result

$$\alpha\mathbf{x} = [\alpha x_1, \alpha x_2, \ldots, \alpha x_n]. \tag{2.40}$$

The term *norm* is sometimes used for the length of vectors in $\mathbf{R}^n$ when $n > 3$. The generalization of the vector length given in Equation 2.23 for a vector with n components is

$$\|\mathbf{x}\| = \left(\sum_{i=1}^{n} x_i{}^2 \right)^{1/2} = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{2.41}$$

The norm is a scalar value that measures the size of the vector in n-space. However, there are different types of norms useful in some applications.

Therefore, the norm in Equation 2.41 is properly referred to as the 2-norm to distinguish it from other norms. The 2 defines the power to which the components are raised. In physics, $\mathbf{R}^n$ with this norm is called the Euclidean n-space.

Vectors of Higher Dimensions

Many physical quantities can be represented by an n-vector. For example, suppose an analog signal sampled periodically yields n samples, denoted

$$[s_1, s_2, \ldots, s_n].$$

Let the values be represented by the vector $\mathbf{S}$, which could serve as a shorthand notation when referring to the samples.

However, since $\mathbf{S}$ is a vector, useful computations can be performed on $\mathbf{S}$. If the signal is input to a multiplier circuit that multiplies the input by α, then the sampled output is $\alpha\,\mathbf{S}$ if the input and output signals are sampled simultaneously. The norm of $\mathbf{S}$ also has physical meaning if the input signal is an electrical signal. In this case, the root-mean-square (RMS) value is the norm divided by $\sqrt{n}$.

Vector language and vectors of greater than three dimensions are used in solving problems in many areas of physics. Equations of a mechanical system are often described as having N *degrees of freedom* using N independent vectors to express the solutions. In statistical mechanics, an n-dimensional space is used. Relativity uses a four-dimensional space, and quantum mechanics even extends the space of n dimensions to one with an infinite number of components.

□

2.6 MATLAB VECTORS

One of MATLAB's fundamental data types is the n-dimensional vector. The vector `x` can be entered in the form

```
>> x=[x1 x2 ...xn]
```

where the `xi` are the components written in a row with spaces separating the values. This MATLAB form represents the vector $\mathbf{x} = [x_1, x_2, \ldots, x_n]$. The fundamental operations on vectors are defined in Table 2.6, in which the first column defines the operation, the second column shows the MATLAB format, and the third column presents the mathematical result.

TABLE 2.6 *MATLAB Vector Operations*

Operation	MATLAB Form	Mathematical Form
Vector	`[x1 x2 ...xn]`	$[x_1, x_2, \ldots, x_n]$
Addition	`x+y`	$[x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n]$
Subtraction	`x-y`	$[x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n]$
Scalar multiplication	`k*x`	$[kx_1, kx_2, \ldots, kx_n]$
Transpose	`x'`	Column vector
Dot product	`x*y'` or **dot(x,y)**	$x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$
Cross product	**cross(x,y)**	$\mathbf{x} \times \mathbf{y}$
Norm	**norm(x)**	$\|\mathbf{x}\|$
Special:		
$x_i \times y_i$	`x.*y`	$[x_1 y_1, x_2 y_2, \ldots, x_n y_n]$
x_i / y_i	`x./y`	$[x_1/y_1, x_2/y_2, \ldots, x_n/y_n]$
$x_i{}^m$	`x.^m`	$[x_1{}^m, x_2{}^m, \ldots, x_n{}^m]$
$f(x_i)$	`f(x)`	$[f(x_1), f(x_2), \ldots, f(x_n)]$
Augment	`[x y]`	$[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$
Element x_i	`x(i)`	x_i
Row vector		
Integers m, n	`x=m:n`	$x = [m, m+1, \ldots, n]$
Real a, dx, b	`x=a:dx:b`	$x = [a, a + dx, a + 2dx, \ldots, b]$

To compute the dot product of two vectors, the *transpose* of one of the vectors must be used in the product. The transpose of a vector `xt` is entered as

$>>$ `xt=[x1 x2 ...xn]'`

or as

$>>$ `xt=[x1;x2;...;xn]`

with semicolons separating the components. The result is the column vector

$$xt = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

The MATLAB command

$>>$ `xt=x'`

converts the row vector `x` into the column vector `xt`. The command **dot** computes the dot product of two vectors without the need to form the transpose of one of the vectors.

There are a number of special operations listed in the table that are not standard vector operations. MATLAB operators multiply (.*)

or divide (./) two vectors element by element when the period operator follows the vector. Also, each element of a vector may be raised to a power. Most of the MATLAB functions, such as **exp** or **sqrt**, can be used to compute the function of each element of a vector.

The n components of a vector **x** can be augmented with the elements of a vector **y** having n components with the command

$>>$ z=[x y]

after which **z** becomes a row vector with $2n$ elements. The ith element of a vector **x** is designated as x(i). Thus, the third element of **x** is x(3). Finally, a vector with fixed increments between the elements can be generated. For example, the command

$>>$ x=[1:0.5:3]

creates the mathematical vector $\mathbf{x} = [1, 1.5, 2.0, 2.5, 3.0]$.

☐ **EXAMPLE 2.10**   *MATLAB Operations*

The accompanying MATLAB sessions illustrate various operations from previously presented discussions and examples. In the first script, the command **abs** is used to compute the absolute value of a number. Next, the function clbindec is called to convert a binary number to decimal using Equation 2.2. The binary digits are entered as elements in a vector. After the conversion, the result is displayed showing 111110_2 converted to 62. The function is also displayed for convenience. It is stored on disk as file CLBINDEC.M.

MATLAB Script

```
Example 2.10
>>x=-5
x =
    -5
>>y=abs(x)
y =
     5
>>xbin=[1 1 1 1 1 0]; % Convert binary to decimal
>>Ndec=clbindec(xbin)
Ndec =
    62
%
function ndec=clbindec(xbin)
% CALL:  ndec=clbindec(xbin), Convert positive binary number xbin
%  to decimal value ndec. No error check if xbin is not binary
m=length(xbin);
ndec=0;
for I=1:m
 ndec=ndec+xbin(I)*2^(m-I);
end
```

The second session shows the vector operations of Example 2.5 performed with MATLAB commands. The commands compute the norms of the vectors, the dot product, and the angle between the vectors in radians and degrees.

MATLAB Script

```
Example 2.10
%  Norm, Dot Product and Angle
>>x1=[2 4 0]
x1 =
     2     4     0
>>y1=[6 -4 2]
y1 =
     6    -4     2
>>% Compute norms
>>nx1=norm(x1)
nx1 =
    4.4721
>>ny1=norm(y1)
ny1 =
    7.4833
>>% Dot product
>>xdoty=x1*y1'
xdoty =
    -4
>>% Angle between vectors
>>theta=acos(xdoty/(nx1*ny1))
theta =
    1.6906
>>thetadeg=theta*180/pi
thetadeg =
   96.8646
>>quit
```

□

Symbolic MATLAB   The *Symbolic Math Toolbox* has commands to manipulate vectors defined in symbolic form. For example, the command **symop** performs symbolic operations on vectors and matrices. To compute the norm of a vector and the dot product of vectors, the symbolic command **transpose** is useful.

2.7 PROPERTIES OF VECTORS

Consider arbitrary vectors $\mathbf{x}$ and $\mathbf{y} \in \mathbf{R}^n$ and a scalar α. A basic property we assume to be true for any vectors in $\mathbf{R}^n$ is that the sum of the vectors is also a vector in $\mathbf{R}^n$. The scalar multiple of any vector in $\mathbf{R}^n$ is also a vector in $\mathbf{R}^n$. These statements are called *closure* properties and are stated mathematically as

$$\mathbf{x} + \mathbf{y} \in \mathbf{R}^n \quad \text{and} \quad \alpha\mathbf{x} \in \mathbf{R}^n$$

for vectors in $\mathbf{R}^n$.

Other properties of vectors in $\mathbf{R}^n$ define various addition and multiplication operations. For vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ and scalars α and β, these vectors satisfy the following relationships for addition and multiplication by a scalar.

The properties of *vector addition* are as follows:

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.   Commutative law

2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.   Associative law

3. $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$, where $\mathbf{0} = [0, 0, ..., 0]$.   Zero element

4. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.

In Property 4, the negative of $\mathbf{x}$ is defined as $(-1)\mathbf{x} = -\mathbf{x}$ or in terms of the components,

$$-\mathbf{x} = \left[-x_1, -x_2, ..., -x_n\right].$$

Let α and β be scalars. Then, the scalar multiplication properties of vectors are

5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$.

6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.

7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$.

8. $\mathbf{x} = 1 \times \mathbf{x}$

☐ **EXAMPLE 2.11**   *Commutative Law*

There are several ways to define the basic properties of vectors with real elements. Our approach is to postulate these properties without proof, as has just been done. For example, the commutativity property of vectors is a proposition that can be proven but was assumed. Thus, we state that addition of two vectors is *commutative* because

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}. \tag{2.42}$$

Another approach to presenting the basic properties of vectors is to prove the relationships using the axioms that were presented earlier for real numbers. The properties of vectors listed in this section can all be proven from the definition of vector addition and multiplication of a vector by a scalar using the properties of real numbers. For example, given that $\alpha + \beta = \beta + \alpha$ for any real numbers α and β, proving the commutative law for vectors is straightforward. Since the vectors have real components, the definition of $\mathbf{x} + \mathbf{y}$ previously given by Equation 2.39 yields the jth component of the sum as

$$x_j + y_j \quad \text{for } j = 1, 2, \ldots, n. \tag{2.43}$$

Then, by the commutative law for real numbers, each component can also be written as

$$y_j + x_j \quad \text{for } j = 1, 2, \ldots, n. \tag{2.44}$$

This latter sum is $\mathbf{y} + \mathbf{x}$, so vector addition is commutative, as stated in Equation 2.42.

$\blacksquare$

In the relationships that hold for vectors, the associative, commutative, and distributive laws of multiplication that apply to real numbers are not listed. Instead, the dot product of two vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as the sum of the products of the respective components.

The dot product of two n-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i. \tag{2.45}$$

From the definition, it is clear that the dot product is commutative, since $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$. A useful equation relates the norm of a vector in $\mathbf{R}^n$ and the dot product. Letting $\mathbf{x} = \mathbf{y}$ in Equation 2.45, the square of the norm is computed as

$$\| \mathbf{x} \|^2 = \mathbf{x} \cdot \mathbf{x}, \tag{2.46}$$

using the norm as previously defined in Equation 2.41.

Unfortunately, visualization of the geometric properties of vectors as in $\mathbf{R}^2$ and $\mathbf{R}^3$ is difficult when the dimension is greater than three. However, it is useful to define the concept of orthogonal vectors in $\mathbf{R}^n$.

ORTHOGONAL VECTORS AND INNER PRODUCT

In terms of the dot product, the nonzero vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be *orthogonal* if $\mathbf{x} \cdot \mathbf{y} = 0$. Vectors in $\mathbf{R}^n$ that are mutually orthogonal and of unit length (norm is 1) are called *orthonormal*. Thus, a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is orthonormal if

$$\mathbf{x}_i \cdot \mathbf{x}_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

The vector $\mathbf{0}$ is considered to be orthogonal to every vector in $\mathbf{R}^n$, since $\mathbf{x} \cdot \mathbf{0} = 0$ for every vector $\mathbf{x} \in \mathbf{R}^n$. In the next section, orthonormal sets of vectors will be used to represent vectors in $\mathbf{R}^n$.

When dealing with vectors in $\mathbf{R}^n$ and in some other applications, the dot product is often called the *inner product*. The operation is defined by the notation

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \tag{2.47}$$

for two vectors in $\mathbf{R}^n$.

The notation for a set of orthonormal vectors can be simplified by using the inner product and introducing the *Kronecker delta* with the definition

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Then, a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is an orthonormal set if and only if

$$\langle \mathbf{x}_i, \mathbf{x}_j \rangle = \delta_{ij} \qquad i, j = 1, 2, \ldots, n.$$

2.8 COMPLEX VECTORS

The set of all n-dimensional vectors with complex components will be designated as $\mathbf{C}^n$. The vectors in the set have the form

$$\mathbf{z} = [z_1, z_2, \ldots, z_n],$$

where each component is the complex number $z_i = x_i + iy_i$, $i = 1, 2, \ldots n$. The vector $\bar{\mathbf{z}}$ has the ith component $\bar{z}_i = x_i - iy_i$. The scalars in $\mathbf{C}^n$ are taken from $\mathbf{C}$. With these understandings, addition and scalar multiplication is "componentwise," just as for real vectors.

The dimension n cannot be understood in the same geometric way as for real vectors, because each component is really 2-dimensional. Thus, the individual vectors in $\mathbf{C}^2$ could be visualized much like the individual vectors in $\mathbf{R}^4$.

It is possible to define inner product and length for complex vectors. As introduced earlier in the chapter, the length of the complex number $z = x + iy$ considered as a vector in xy-space is

$$||x + iy|| = (z\bar{z})^{1/2} = \sqrt{(x + iy)(x - iy)} = \sqrt{x^2 + y^2}.$$

Since x and y are real numbers, the length of the vector is a positive number. Thus, if z is not zero, $z\bar{z}$ is always real and positive.

With complex vectors, we wish to define an inner product such that the length of a vector $\mathbf{z}$ is a positive number. Consider the vectors

$$\mathbf{z} = [z_1, z_2, \ldots, z_n] \quad \text{and} \quad \mathbf{w} = [w_1, w_2, \ldots, w_n]$$

with complex components. The complex conjugate of such vectors is formed by conjugating each component of the vectors. The *inner product* of complex vectors $\mathbf{z}$ and $\mathbf{w}$ can be defined as

$$\langle \mathbf{z}, \mathbf{w} \rangle = \bar{z}_1 w_1 + \cdots + \bar{z}_n w_n, \qquad \text{(2.48)}$$

in which the conjugate of the first vector is used. For complex vectors, the inner product is thus formed by first conjugating $\mathbf{z}$ and then forming the sum of the products of the corresponding elements of $\bar{\mathbf{z}}$ and $\mathbf{w}$. Then, the norm or length of the vector $\mathbf{z}$ can be defined as

$$||\mathbf{z}|| = \langle \mathbf{z}, \mathbf{z} \rangle^{1/2} = \langle \mathbf{z}, \mathbf{z} \rangle^{1/2}.$$

If $\mathbf{z} \neq \mathbf{0}$, it is simple to prove that $\langle \mathbf{z}, \bar{\mathbf{z}} \rangle > 0$.

For complex vectors with real components, the dot product in $\mathbf{R}^n$ and the inner product are identical since $z = \bar{z}$ if z is a real number and so it follows that $\mathbf{z} = \bar{\mathbf{z}}$ for real vectors.

WHAT IF?   Suppose the definition of Equation 2.48 is given as $\langle \mathbf{z}, \bar{\mathbf{w}} \rangle$. Try both definitions with the vectors $z_1 = [i, 0]$ and $z_2 = [1, 0]$ and show that the norms are different. Which definition does the MATLAB command **dot** use for complex vectors?

2.9 VECTOR SPACES

Thus far we have moved from the easily visualized vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ to the more "abstract" vectors in $\mathbf{R}^n$. However, except for the cross product, the various vector properties and operations are equally valid independent of the dimension of the vectors. One further generalization dealing with vectors is of great importance in both theoretical studies and certain practical problems.

Our final generalization for vectors in $\mathbf{R}^n$ introduces the concept of a *vector space*. The vector space is a set of vectors together with the rules for vector addition and multiplication by a scalar. Vectors in $\mathbf{R}^n$ are said to form a vector space. These vectors satisfy the closure condition and the eight properties of vectors listed previously.

☐ EXAMPLE 2.12 **Vector Space**

The introduction of n-dimensional vectors and vector spaces represents an important generalization from the 2D and 3D space of the physical world. Advantages of the extension will be shown in a number of ways in this text. The main principle is that once a fact about vector spaces in general is known to be true, we can apply that fact to every vector space, rather than having to prove the fact for each new vector space.

For example, the number zero (0) is itself a vector space that is sometimes called the *trivial* vector space. Thus, zero satisfies all the properties of a vector space such as $0 + 0 = 0$, $\alpha \times 0 = 0$, and $0 = 1 \times 0$. Note that the number 1 does not form a vector space since it violates the closure property; that is, $1 + 1 = 2$ but 2 is not in the space. Extending this argument slightly shows that the vector $\mathbf{0}$ in $\mathbf{R}^n$ also forms a vector space. We will study much more interesting vector spaces in the sections to follow.

The set of points in $\mathbf{R}^2$ that lie on a line passing through the origin constitutes a vector space. The points $[x, y]$ are defined by the equation $y = mx$ where m is a scalar constant. Letting $\mathbf{x} = [x_1, y_1]$ and $\mathbf{y} = [x_2, y_2]$ with $y_1 = mx_1$ and $y_2 = mx_2$, it is easy to show that the closure property and the other eight properties for vector spaces are satisfied. However, the set of points on a line not passing through the origin ($y = mx + b$, $b \neq 0$) does not form a vector space, as you are asked to prove in Problem 2.12.

☐

SUBSPACES

In some problems, a subset of the vectors in $\mathbf{R}^n$ is important, rather than the entire set. If the vectors in the subset satisfy the closure property of addition and scalar multiplication, the vectors also form a vector space called a subspace of $\mathbf{R}^n$. A set of vectors S that is a subset of the vectors in $\mathbf{R}^n$ forms a vector space if α is a scalar in $\mathbf{R}$ and for every vector $\mathbf{x}$ and $\mathbf{y}$ in S,

1. $\mathbf{x} + \mathbf{y} \in S$.   Closure under addition

2. $\alpha\,\mathbf{x} \in S$, α a scalar. Closure under scalar multiplication

3. $\mathbf{0} \in S$.

4. $-\mathbf{x} \in S$.

A vector space with these properties is said to be a *subspace* of $\mathbf{R}^n$. The vector spaces $\mathbf{R}, \mathbf{R}^2, \ldots$, including $\mathbf{R}^n$ itself are subspaces of $\mathbf{R}^n$. Actually, Properties 3 and 4 follow from Property 2 and the fact that for a vector in $\mathbf{R}^n$, $0 \times \mathbf{x} = \mathbf{0}$. Similarly, Property 4 can be derived from Property 2 and the fact that $-\mathbf{x} = (-1)\mathbf{x}$ for vectors in $\mathbf{R}^n$.

There are two important properties of a subspace that are worth mentioning explicitly:

1. Every subspace of a vector space contains $\mathbf{0}$.

2. Any subspace of a vector space is a vector space, with the operations inherited from the original space.

If the vectors in the subset of $\mathbf{R}^n$ form a subspace, it is not necessary to verify that they satisfy the eight properties of vectors in a vector space because the properties are satisfied in the larger space and will be satisfied in every subspace. Conversely, if the subset does not form a subspace, some of the properties associated with a vector space will not apply.

LINEAR INDEPENDENCE AND BASES

Representing one mathematical quantity in terms of a combination of (presumably) simpler quantities is useful in many applications. As shown in an earlier section of this chapter, a 3D vector may be expressed as a sum of scalar multiples of three orthogonal unit vectors. In this section, this concept is extended to vectors in $\mathbf{R}^n$.

Extending the previous discussion of unit vectors in $\mathbf{R}^3$, it is natural to write a vector $\mathbf{x}$ in $\mathbf{R}^n$ in the form

$$\mathbf{x} \;=\; [x_1, x_2, \ldots, x_n] \;=\; x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n, \qquad \textbf{(2.49)}$$

where

$$\begin{aligned}
\mathbf{e}_1 &= [1, 0, \ldots, 0],\\
\mathbf{e}_2 &= [0, 1, \ldots, 0],\\
&\;\;\vdots\\
\mathbf{e}_n &= [0, 0, \ldots, 1].
\end{aligned} \qquad \textbf{(2.50)}$$

This set of unit vectors forms an orthonormal set, and any vector in $\mathbf{R}^n$ can be written as a sum of scalar multiples of these vectors. The importance of this fundamental set of vectors for $\mathbf{R}^n$ will be explored in more detail shortly.

☐ **EXAMPLE 2.13**   *Vector in terms of Unit Vectors*

In $\mathbf{R}^3$, the standard unit vectors would be $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. In most applications, they are written $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, as used in Example 2.6. Thus, the vector

$$\mathbf{x} = [1, 1.5, 3, 7, 8]$$

can be written in terms of the unit vectors in $\mathbf{R}^5$ as follows:

$$\mathbf{x} = 1\mathbf{e}_1 + 1.5\mathbf{e}_2 + 3\mathbf{e}_3 + 7\mathbf{e}_4 + 8\mathbf{e}_5. \tag{2.51}$$

A vector component in the direction of a unit vector is determined by taking the inner product of the vector and the unit vector. For the vector $\mathbf{x}$, the component in the direction of $\mathbf{e}_1$ is thus

$$\langle \mathbf{x}, \mathbf{e}_1 \rangle = 1\langle \mathbf{e}_1, \mathbf{e}_1 \rangle + 1.5\langle \mathbf{e}_1, \mathbf{e}_2 \rangle + \cdots + 8\langle \mathbf{e}_1, \mathbf{e}_5 \rangle = 1,$$

since the unit vectors form an orthonormal set. In terms of projections as defined in Equation 2.32, the component of $\mathbf{x}$ in the direction of $\mathbf{e}_1$ is

$$\mathrm{proj}_{\mathbf{e}_1} \mathbf{x} = \frac{\langle \mathbf{e}_1, \mathbf{x} \rangle}{||\mathbf{e}_1||^2} \mathbf{e}_1 = 1\mathbf{e}_1.$$

☐

Linear Independence of Vectors   Many problems involving vector spaces are simplified if we find a set of vectors that can be used to generate any vector in the space. Such a set is called a *basis* for the space. The properties of a basis set will be explored after the concept of linear independence is defined.

A vector $\mathbf{y}$ is called a *linear combination* of a set of vectors designated $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ if $\mathbf{y}$ can be written as a sum of scalar multiples of these vectors in the form

$$\mathbf{y} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_n \mathbf{x}_n, \tag{2.52}$$

where each α_j is a scalar. The set of vectors

$$\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \tag{2.53}$$

is described as *linearly independent* if none of the vectors can be written as a linear combination of the others. An equivalent statement expresses the fact that a linear combination of independent vectors cannot be summed to yield the zero vector unless each coefficient is zero.

A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ in $\mathbf{R}^n$ is said to be linearly independent if the equation

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_n \mathbf{x}_n = \mathbf{0} \tag{2.54}$$

is true only when the scalars $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. Otherwise, the vectors are linearly *dependent*, and at least one of them is a linear combination of the others. Thus, if the relation in Equation 2.54 does hold with not all the α's equal to 0, the vectors are *linearly dependent*.

For two vectors, the definition means that the vectors are linearly dependent if and only if one is a scalar multiple of the other. When more than two vectors are involved, the test for independence results in a set of equations for the coefficients.

Linearly Independent Vectors
To test the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for linear independence, we form the equation

$$a_1 \, \mathbf{i} + a_2 \, \mathbf{j} + a_2 \, \mathbf{k} = \mathbf{0},$$

which is equivalent to

$$[a_1, a_2, a_3] = [0, 0, 0].$$

By equating components, the vector equation leads to the conclusion

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0.$$

Hence, the three unit vectors in $\mathbf{R}^3$ are linearly independent. In fact, it is easy to show that any set of mutually orthogonal nonzero vectors from $\mathbf{R}^n$ is linearly independent.

The vectors $[3, 2, 3], [1, 1, 0], [0, 1, -3]$ can be shown to be linearly dependent by solving the system of equations resulting from setting

$$a_1 \, [3, 2, 3] + a_2 \, [1, 1, 0] + a_3 \, [0, 1, -3] = [0, 0, 0]. \tag{2.55}$$

The system

$$\begin{array}{rcrcrcl}
3a_1 & + & a_2 & & & = & 0 \\
2a_1 & + & a_2 & + & a_3 & = & 0 \\
3a_1 & & & - & 3a_3 & = & 0
\end{array} \tag{2.56}$$

has a solution $[a_1, a_2, a_3] = [1, -3, 1]$, among others. Thus, the first vector is a linear combination of the other two since

$$[3, 2, 3] = 3 \, [1, 1, 0] - 1 \, [0, 1, -3]. \tag{2.57}$$

Although matrices and determinants are not treated until Chapter 3, a useful result from matrix theory can be applied to determine if n vectors in $\mathbf{R}^n$ are independent. First, form the matrix with the vectors as columns. Then, if the determinant of the matrix is nonzero, the vectors are independent. If the determinant is zero, the vectors are dependent. The determinant formed by the unit vectors is

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1, \tag{2.58}$$

so the vectors are independent. The determinant for the other set is

$$\begin{vmatrix} 3 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & -3 \end{vmatrix} = 0, \tag{2.59}$$

so the vectors are dependent.

■

Basis Vectors   A *basis* for a vector space consists of a set of vectors that can be used to uniquely generate every vector in the space. Such a set of basis vectors is said to *span* the space. Important results from linear algebra allow us to determine the number of vectors that span the space and the characteristics of the vectors that can serve as the basis.

A set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is said to form a *basis* for the vector space of $\mathbf{R}^n$ if the following two conditions hold:

1. The set of vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ is linearly independent.

2. Every vector in $\mathbf{R}^n$ can be written as a linear combination of the independent vectors.

Considering the vector space $\mathbf{R}^n$, it can be shown that every set of n linearly independent vectors in $\mathbf{R}^n$ is a basis for the space. Furthermore, although the basis set is not unique, every basis for $\mathbf{R}^n$ has exactly n vectors. Thus, if $\mathbf{x}$ is a vector in $\mathbf{R}^n$, then $\mathbf{x}$ can be written as

$$\mathbf{x} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \cdots + \alpha_n \mathbf{x}_n, \tag{2.60}$$

where each α_j is a scalar if the vectors $\mathbf{x}_j$, for $j = 1, 2, \ldots, n$, are linearly independent and span the vector space.

The *dimension* of the vector space is the number of vectors in the basis, and this number is sometimes referred to as the number of *degrees of freedom* of the space. In a vector space of dimension n, such as $\mathbf{R}^n$, not more than n vectors can be linearly independent. Also, any set of basis vectors that span the space can contain no fewer than n vectors. Any subspace of $\mathbf{R}^n$ containing m vectors $(m \leq n)$ has dimension m and requires m basis vectors. Thus, every basis of $\mathbf{R}^2$ must contain two independent vectors, and any three vectors in $\mathbf{R}^2$ must be linearly dependent.

☐ **EXAMPLE 2.15**   *Orthonormal Basis for* $\mathbf{R}^n$

The orthonormal set of unit vectors

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \tag{2.61}$$

defined in Equation 2.50 forms a basis for $\mathbf{R}^n$. This set is often called the *standard basis*, or the *natural basis*, for the space. However, this set is not unique. Any set of n linearly independent vectors from $\mathbf{R}^n$ could serve as a basis. For example, the set of vectors formed by multiplying each vector in the natural basis by a nonzero constant would also serve as a basis for $\mathbf{R}^n$.

In $\mathbf{R}^1$, any nonzero vector can serve as a basis. This is evident since any two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ that lie along a line (collinear) can be written as $\mathbf{b} = k\mathbf{a}$, where k is a nonzero number. To write this in the form to test for linear dependence, let $k = -\alpha/\beta$ so that the relationship between the vectors becomes

$$\alpha \mathbf{a} + \beta \mathbf{b} = \mathbf{0},$$

where neither α or β is zero. Considering the definition of linear dependence, we conclude that two collinear vectors are always linearly dependent.

In two dimensions, two noncollinear vectors $\mathbf{a}$ and $\mathbf{b}$ are linearly independent. Thus, every vector in a plane can be represented as

$$\mathbf{c} = k_1\mathbf{a} + k_2\mathbf{b},$$

with the proper choice of the constants k_1 and k_2. The 2D vector space can be defined by the equation

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}.$$

By a similar argument, we can show that any vector in space can be represented as

$$\mathbf{d} = k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c},$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three noncoplanar vectors in $\mathbf{R}^3$.

Generalizing these observations, we conclude that every n-dimensional vector can be represented as the linear combination of n linearly independent vectors. It is also clear that every set of more that n vectors must be linearly dependent.

$\square$

A fundamental result of linear algebra is that n linearly independent vectors in $\mathbf{R}^n$ form a *basis* for $\mathbf{R}^n$. Thus, any vector in $\mathbf{R}^n$ can be expressed uniquely as a linear combination of the basis vectors. If $\mathbf{u}_1, \dots, \mathbf{u}_n$ are linearly independent and $\mathbf{v}$ is any vector in $\mathbf{R}^n$, then

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i\mathbf{u}_i = \alpha_1\mathbf{u}_1 + \cdots + \alpha_n\mathbf{u}_n, \tag{2.62}$$

where the α_i are uniquely determined. To solve for the coefficients, each component can be written as

$$v_k = \alpha_1 u_{1k} + \cdots + \alpha_n u_{nk} = \sum_{i=1}^{n} \alpha_i u_{ik}, \tag{2.63}$$

where the notation u_{ik} means the kth component $(k = 1, \dots, n)$ of the vector $\mathbf{u}_i, i = 1, \dots, n$. Thus, the ith vector in the linearly independent set has components

$$\mathbf{u}_i = (u_{i1}, u_{i2}, \dots, u_{in}).$$

Equation 2.63 represents an $n \times n$ system of linear equations that can be solved for the coefficients in Equation 2.62, since the $\alpha_i, i = 1, \dots, n$ are unique.

Now suppose that a set of vectors $\mathbf{u}_i, i = 1, \dots, n$ in $\mathbf{R}^n$ are orthogonal. It can be shown that the vectors are linearly independent. The expansion of Equation 2.62 for any vector in $\mathbf{R}^n$ is again

$$\mathbf{v} = \sum_{i=1}^{n} \alpha_i\mathbf{u}_i, \tag{2.64}$$

but in this case the coefficients are easy to find. Simply take the inner (dot) product of both sides of Equation 2.64 with each $\mathbf{u}_i$ and use the orthogonality property. Thus,

$$\langle \mathbf{v}, \mathbf{u}_i \rangle = \sum_{i=1}^{n} \alpha_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle = \alpha_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle. \qquad (2.65)$$

Solving for α_i, the result for the coefficients is

$$\alpha_i = \frac{\langle \mathbf{v}, \mathbf{u}_i \rangle}{\langle \mathbf{u}_i, \mathbf{u}_i \rangle} \qquad (2.66)$$

for $i = 1, \ldots, n$. Notice that if the vectors $\mathbf{u}_i$ are orthonormal, each coefficient α_i is simply the inner product of the vector $\mathbf{v}$ and $\mathbf{u}_i$ since $\langle \mathbf{u}_i, \mathbf{u}_i \rangle = 1$.

ORTHOGONAL SETS OF VECTORS

The importance of orthonormal sets is that they are equivalent to linearly independent sets in many aspects, but they also have an inner product associated with them. It is often more convenient to work with orthonormal sets because of the computational simplifications that arise. Two theorems relate linearly independent and orthonormal sets.

■ **THEOREM 2.1**   ***Orthogonal sets***

An orthonormal set of vectors is linearly independent.

To show this result, let $\mathbf{u}_1, \ldots, \mathbf{u}_n$ be an orthonormal set and form the vector equation

$$c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_n \mathbf{u}_n = \mathbf{0}, \qquad (2.67)$$

where the c_i's $(i = 1, 2, \ldots, n)$ are constants. The set of vectors will be linearly dependent if the only constants that satisfy Equation 2.67 are

$$c_1 = c_2 = \cdots = c_n = 0.$$

Taking the inner product of both sides of Equation 2.67 with $\mathbf{u}_1$ yields

$$\langle c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_n \mathbf{u}_n, \mathbf{u}_1 \rangle = \langle \mathbf{0}, \mathbf{u}_1 \rangle,$$

which can be written

$$c_1 \langle \mathbf{u}_1, \mathbf{u}_1 \rangle + c_2 \langle \mathbf{u}_2, \mathbf{u}_1 \rangle + \cdots + c_n \langle \mathbf{u}_n, \mathbf{u}_1 \rangle = 0.$$

Since $\langle \mathbf{u}_i, \mathbf{u}_1 \rangle = \delta_{i1}$, the conclusion is that $c_1 = 0$. Now taking the inner product of Equation 2.67 successively with $\mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ shows that $c_2 = 0, \ldots, c_n = 0$. Summarizing, the results, we find that all of the constants are zero so the vectors are linearly independent.

Construction of an Orthonormal Set Once an orthonormal basis is found for a vector space, many computations are simplified since the vectors are orthogonal. For example, finding the coefficients in the expansion of an arbitrary vector in terms of the basis vectors is considerably simplified using the dot product, as previously shown.

Given a basis for a vector space, is there a way to construct an orthonormal basis? Fortunately, the answer is yes, as will be shown in the proof of the important theorem that states the existence of such an orthonormal set.

■ **THEOREM 2.2**   ***Orthonormal vectors***

For every linearly independent set of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, *there exists an orthonormal set of vectors*

$$\mathbf{u}_1, \ldots, \mathbf{u}_n$$

such that each $\mathbf{u}_j, j = 1, 2, \ldots, n,$ *is a linear combination of* $\mathbf{x}_1, \ldots, \mathbf{x}_j.$

To prove this theorem, an orthonormal set of vectors will be constructed using the Gram-Schmidt process.

Gram-Schmidt Process   The *Gram-Schmidt process* is used to construct an orthonormal set from an independent set. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be a linearly independent set in a vector space. An orthonormal set $\mathbf{u}_1, \ldots, \mathbf{u}_n$ can be constructed by the following procedure:

1. Pick $\mathbf{x}_1$ and form a unit vector

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{||\mathbf{x}_1||}$$

so that $||\mathbf{u}_1|| = 1$.

2. Pick another element, say, $\mathbf{x}_2$, and form its projection on $\mathbf{u}_1$ by forming the vector $(\mathbf{x}_2 \cdot \mathbf{u}_1)\mathbf{u}_1$ and then letting

$$\mathbf{y}_2 = \mathbf{x}_2 - (\mathbf{x}_2 \cdot \mathbf{u}_1)\mathbf{u}_1.$$

Then, create the unit vector

$$\mathbf{u}_2 = \frac{\mathbf{y}_2}{||\mathbf{y}_2||}.$$

Notice that from the definition of $\mathbf{y}_2$ that it cannot be zero because that would imply that $\mathbf{x}_2$ and $\mathbf{u}_1$ are linearly dependent.

3. Continue in this way, successively computing $\mathbf{u}_1, \ldots, \mathbf{u}_j$, and form

$$\mathbf{y}_{j+1} = \mathbf{x}_{j+1} - (\mathbf{x}_{j+1} \cdot \mathbf{u}_1)\mathbf{u}_1 - \cdots - (\mathbf{x}_{j+1} \cdot \mathbf{u}_j)\mathbf{u}_j,$$

with

$$\mathbf{u}_{j+1} = \frac{\mathbf{y}_{j+1}}{||\mathbf{y}_{j+1}||}.$$

☐ **EXAMPLE 2.16**   ***Gram-Schmidt Process***

We first show that the vectors $\mathbf{x}_1 = (1, -1, 2)$ and $\mathbf{x}_2 = (1, 0, -1)$ are linearly independent. Then, we use the Gram-Schmidt process to produce an orthonormal set to be used as the basis for any vector in a plane in $\mathbf{R}^3$.

The vectors are linearly independent, since

$$c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 = 0$$

implies that $c_1 = c_2 = 0$, as you can easily prove by expanding the equation by components of the vectors. Applying the Gram-Schmidt process,

$$\mathbf{u}_1 = \frac{\mathbf{x}_1}{||\mathbf{x}_1||} = \frac{(1, -1, 2)}{\sqrt{6}}$$

and

$$\begin{aligned}
\mathbf{y}_2 &= \mathbf{x}_2 - (\mathbf{x}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 \\
&= (1, 0, -1) + (\frac{1}{6}, -\frac{1}{6}, \frac{1}{3}) = (\frac{7}{6}, -\frac{1}{6}, -\frac{2}{3}).
\end{aligned}$$

Then

$$\mathbf{u}_2 = \frac{\mathbf{y}_2}{||\mathbf{y}_2||} = \frac{(\frac{7}{6}, -\frac{1}{6}, -\frac{2}{3})}{\sqrt{66/36}} = \frac{(7, -1, -4)}{\sqrt{66}}.$$

Since two *noncollinear* vectors can be used to define a plane, the points in the plane defined by the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ can be represented as

$$\alpha_1 \mathbf{x}_1 + \beta_1 \mathbf{x}_2,$$

where the coefficients are real numbers. Another representation is

$$\alpha_2 \mathbf{u}_1 + \beta_2 \mathbf{u}_2.$$

The representation in terms of $\mathbf{u}_1$ and $\mathbf{u}_2$ has the advantage that the vectors are perpendicular and form an orthonormal set.

☐

ABSTRACT VECTOR SPACES

In the earlier sections of this chapter, it was assumed that the vector space under discussion contained vectors with n real components. Although the concept of $\mathbf{R}^n$ as a vector space has great use in many applications, it is possible to define more general vector spaces. For example, a vector space can be defined in which the elements are complex numbers. Another vector space could be defined that consists of all continuous real-valued functions of a real variable. An example subspace of this vector space would be the set of all polynomials.

This section discusses the extension to vector spaces consisting of functions. Later chapters apply the technique to matrices, Fourier analysis, and the solution of differential equations. The approach frequently leads not only to simplified mathematical analysis and computation but also to useful physical insights in many problems of interest in physics and engineering.

Suppose V contains a collection of elements that may be vectors in $\mathbf{R}^n$, matrices, functions, or other elements. Further, assume that there are two operations given in the definition of the space, called addition and scalar multiplication. V is called an *abstract vector space* if the elements $\mathbf{a}, \mathbf{b}$, and $\mathbf{c} \in V$ and α and β are scalars with the closure properties

$$\mathbf{a} + \mathbf{b} \in V \quad \text{and} \quad \alpha\mathbf{a} \in V$$

and the additional properties:

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.   Commutative law for addition

2. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.   Associative law for addition

3. There is zero element such that $\mathbf{a} + 0 = \mathbf{a}$.

4. There is an inverse element such that $\mathbf{a} + (-\mathbf{a}) = 0$.

5. $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$.

6. $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$.

7. $(\alpha\beta)\mathbf{a} = \alpha(\beta\mathbf{a})$.

8. There is an identity element such that $1 \times \mathbf{a} = \mathbf{a}$.

Comparing the properties of the abstract vector space with those of $\mathbf{R}^n$ previously defined shows that the properties are the same. The term *abstract* is used only to indicate that the vector space under discussion may consist of objects other than the vectors in $\mathbf{R}^n$. When no confusion could arise, the abstract vector spaces discussed are simply termed *vector spaces*. Also, when vector spaces of functions are considered, the boldface notation will be discontinued and functions will be represented in their ordinary mathematical form.

□ **EXAMPLE 2.17**   *Polynomials*

The set of all polynomials

$$P(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \qquad (2.68)$$

forms a vector space. In this case, vector addition is polynomial addition and scalar multiplication is multiplication of $P(x)$ by a constant.

Polynomials such as $1, x, x^2, \ldots, x^n$ are linearly independent, since one of these functions cannot be written as a linear combination of the others that is valid for all x. However, the vector space does not have a finite number of polynomials that span the space. In this case, the space is said to be an *infinite-dimensional* vector space. A number of vector spaces containing functions will have this property. This does not mean that the vector space does not have a basis but that the basis contains an infinite number of functions.

□

2.10 VECTOR SPACES OF FUNCTIONS

We now come to perhaps the most important generalizations of vector theory in the chapter as they pertain to techniques in advanced mathematics. Reaching this point has involved extending the theory of vectors in $\mathbf{R}^2$ and $\mathbf{R}^3$ to those in $\mathbf{R}^n$ and then considering abstract vector spaces. The key points to be considered here include

1. Generalizing the dot product for vectors to the inner product for functions;

2. Extending the idea of length of a vector to define the norm of a function;

3. Introducing the concept of expressing a function in terms of a linear combination of orthogonal functions based on the expansion of a vector in terms of the basis vectors for the vector space.

The discussion will be rather short, but these subjects will be revisited in the chapters that treat differential equations, approximation of functions, and Fourier analysis, as well as elsewhere in the text.

INNER
PRODUCT AND
NORM

When functions instead of vectors in $\mathbf{R}^n$ are the elements of a vector space, it is necessary to carefully specify the operations on the elements that are allowed. For example, it is possible to define a norm that measures the "length" of a function. Since a continuous function $f(x)$ is like a vector with a continuous range of components, adding the squares of the values obviously leads to an infinite result. However, a finite result is possible when the summation is replaced by integration. If the dot product for vectors is generalized to the *inner* product for functions, the norm can then be defined accordingly.

As for vectors, the basic operations for functions are addition and multiplication by a scalar. If $f(x)$ and $g(x)$ are continuous real-valued functions, the addition and scalar multiplication are defined as

1. $(f + g)(x) = f(x) + g(x)$,

2. $(\alpha f)(x) = \alpha f(x)$,

where α is a scalar and x is real.

The *inner product* of $f(x)$ and $g(x)$ is defined on the interval $[a, b]$ as the integral of the product

$$\langle f, g \rangle = \int_a^b f(x)\, g(x)\, dx. \tag{2.69}$$

Based on the definition of the inner product, the norm is written as

$$\|f\| = \langle f, f \rangle^{1/2} = \left[\int [f(x)]^2 \, dx \right]^{1/2}. \qquad \textbf{(2.70)}$$

If the inner product of two nonzero functions is zero, the functions are said to be *orthogonal*.

□ EXAMPLE 2.18   *MATLAB Norm*

The norm of $\sin x$ on the interval $[0, 2\pi]$ is computed as

$$
\begin{aligned}
\| \sin x \| &= \langle \sin x, \sin x \rangle^{1/2} \\
&= \left[\int_0^{2\pi} [\sin x]^2 \, dx \right]^{1/2} = \left[\frac{1}{2} \int_0^{2\pi} [1 - \cos(2x)] \, dx \right]^{1/2} \\
&= \left[\frac{1}{2} \left(x - \frac{\sin(2x)}{2} \right) \bigg|_0^{2\pi} \right]^{1/2} = \sqrt{\pi} \approx 1.7725.
\end{aligned}
$$

MATLAB Integration. The accompanying MATLAB script shows the computation of the norm of the sine both numerically and symbolically. The MATLAB command **quad** is used to compute the integral of a function. The format of the command is

>> quad('function',a,b)

where `function` is the name of a function file defining the function and `a,b` defines the interval of integration. The second script shows the function to return the values of $\sin^2(x)$. The norm is then the square root of the result of the integration performed by **quad**. MATLAB integration routines are treated in more detail in Chapter 14.

The symbolic command **int** will attempt to integrate a function defined symbolically. The limits $[a, b]$ can be defined as symbolic or numeric expressions. In the script, **int** is used to compute the symbolic indefinite and definite integral of the sine. The command **digits** determines the accuracy of the symbolic computations when they are converted to numerical values. Command **sym** returns the symbolic square root of the symbolic value `normsqn`. Then, the command **double** converts the symbolic representation of the square root of π to a numerical value.

MATLAB Script ————————————————————————————

```
Example 2.18
% EX2_15.M Compute the norm of sin(x) on the interval [0,2pi]
%  Compare symbolic and numerical result
%    (This script requires the Symbolic Math Toolbox)
% Numerical value; call function sinsq to compute (sin(x))^2
normsin1=sqrt(quad('sinsq',0,2*pi));
digits(5)                      % Define numerical format
normsin1=sym(normsin1,'d')     % Numerical value
%
% Symbolic
```

```
normsq=int('sin(x)^(2)');          % Perform symbolic integration
normsq=simple(normsq)              % Simplify the result
normsqn=int('sin(x)^2',0,2*pi)     % Definite integral
norm2=sym(normsqn^(1/2))           % Symbolic square root
normsin=double(norm2)              % Convert to a number
%
% Edited results from M-file EX2_15.M
%
>>ex2_15
normsin1 =  1.7725                 % Numerical results
normsq   = -1/4*sin(2*x)+1/2*x     % Symbolic integration
normsqn  = pi                      % Symbolic definite integral
norm2    = pi^(1/2)                % Symbolic Square root
normsin  = 1.7725                  % Numerical value of symbolic result
%
```

The function `sinsq` returns the squared value of the sine of the argument `x` passed to the function when it is called by the M-file.

MATLAB Script ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

```
Example 2.18
function yout=sinsq(x)
% CALL: yout=sinsq(x) returns the square of sine(x)
yout=sin(x).^2;
```

□

ORTHOGONAL
FUNCTIONS

Consider a set of functions $\phi_n(x), n = 1, 2, \ldots$, each of which is continuous on the interval $[a, b]$. The set is *orthogonal* if the inner products of different nonzero functions in the set are zero. Thus, the set of functions is orthogonal if

$$\langle \phi_m(x), \phi_n(x) \rangle = \int_a^b [\phi_m(x)\phi_n(x)]dx = 0, \qquad m \neq n, \qquad \textbf{(2.71)}$$

and no $\phi_n(x)$ is identically zero except perhaps at a finite number of points. Furthermore, the system is *orthonormal* if the functions satisfy Equation 2.71 and

$$\langle \phi_n(x), \phi_n(x) \rangle = \int_a^b [\phi_n^2(x)]dx = 1, \qquad n = 1, 2, \ldots. \qquad \textbf{(2.72)}$$

Suppose a function $f(x)$ is continuous on the interval $[a, b]$. Then, following an approach similar to that of expanding a vector in terms of orthonormal basis vectors, we postulate that $f(x)$ can be expressed as

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \qquad \textbf{(2.73)}$$

where the coefficient c_m is determined as

$$c_m = \int_a^b f(x)\phi_m(x)\,dx. \qquad (2.74)$$

Of course, much is left to be said about the expansion of Equation 2.73. Questions about the accuracy of the approximation if a finite number of terms is used and other considerations to guarantee that the series actually represents $f(x)$ will be explored in other chapters. Here we will simply present a most famous set of orthogonal functions that have enormous utility in mathematics and science.

Trigonometric Functions   The set of functions

$$\cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx$$

defined for $-\pi \leq x \leq \pi$ span the vector space of trigonometric sums of the form

$$T(x) = \sum_{k=1}^{n} [a_k \cos kx + b_k \sin kx]. \qquad (2.75)$$

The set of trigonometric basis functions can be shown to be orthonormal with respect to the inner product,

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} [f(x)\,g(x)]\,dx. \qquad (2.76)$$

Example 2.19 discusses these functions further.

A coefficient such as a_k is found by taking the inner product of each side of Equation 2.75 with $\cos kx$ and similarly for b_k using $\sin kx$ in the inner product. Thus,

$$
\begin{aligned}
a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} [\,T(x)\cos kx\,]\,dx, \\
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} [\,T(x)\sin kx\,]\,dx \qquad k = 1, 2, \ldots.
\end{aligned}
\qquad (2.77)
$$

The numbers a_k and b_k are called the kth *Fourier coefficients*.

Fourier Techniques   The applications of Fourier techniques are discussed in detail in Chapter 8 and Chapter 11. The reader who is familiar with Fourier series to approximate functions should note that the trigonometric series in Equation 2.75 is not the complete series because it does not include a constant term. The issue here is the orthogonality of the trigonometric functions, not the approximation of arbitrary functions by a Fourier trigonometric series as treated in Chapter 8.

☐ **EXAMPLE 2.19**   *Orthogonal Functions*

The functions $\sin x$ and $\cos x$ are orthogonal over the interval $[-\pi, \pi]$, since

$$\langle \sin x, \cos x \rangle \;=\; \int_{-\pi}^{\pi} \sin x \cos x \, dx = 0 \tag{2.78}$$

over this interval. This is easily seen if you recognize that the integrand is odd over a full period of the sinusoids.[5] Otherwise, make the substitution

$$\frac{1}{2}\sin 2x = \sin x \cos x$$

and integrate. For general integrals of this form, integrate by parts.

By integrating, we find that

$$\langle \cos nx, \cos mx \rangle \;=\; \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases} \tag{2.79}$$

These values for the inner product show that the cosine terms divided by $\sqrt{\pi}$ form an orthonormal set so that

$$\left\langle \frac{\cos nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}} \right\rangle , = 1.$$

The result for $\langle \sin nx, \sin nx \rangle$ is the same. Thus, the factor $1/\pi$ is used to normalize the integral in Equation 2.76.

Polynomials can also be orthogonal over restricted intervals. For example, over the interval $-1 \leq x \leq 1$, the set of polynomials

$$P_0(x) = 1, \quad P_1(x) = x, \quad \text{and} \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

are orthogonal, as you are asked to show in Problem 2.17. These are known as *Legendre polynomials*.

☐

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[5]Integrals of even and odd functions are discussed in Chapter 8.

2.11 REINFORCEMENT EXERCISES AND EXPLORATION PROBLEMS

■

REINFORCEMENT EXERCISES
In these problems, do the computations by hand unless otherwise indicated, and then check the solution with MATLAB for problems that have numerical or symbolic results.

P2.1. **Binary system** Show that the binary system with elements (0,1) satisfies the properties of real numbers if the operations are defined as

+	0	1		×	0	1
0	0	1		0	0	0
1	1	0		1	0	1

P2.2. **Associativity** Using a simple example show that vector subtraction is not associative. Vector subtraction is defined as

$$\mathbf{a} - \mathbf{b} \; = \; \mathbf{a} + (-\mathbf{b}).$$

P2.3. **Sum of series** Sum the series

$$N = \sum_{i=0}^{\infty} d_i \, 2^{-i} = d_0 + d_1 \times 2^{-1} + d_2 \times 2^{-2} + \cdots$$

when each $d_i = 1$, and find the error when the series is truncated after m terms.

P2.4. **Complex numbers** For the complex number $z = 3 - 4i$, compute the following:
(a) z^2 (b) $z\bar{z}$ (c) $z/\bar{z}$ (d) $|z^2|$

P2.5. **Polar form of complex number** Write the complex number $z = \sqrt{3} + i$ in polar form and find its magnitude and angle in the complex plane.

P2.6. **Perpendicular vectors** Determine the value α so that vectors

$$\mathbf{x} \; = 2\,\mathbf{i} + \alpha\,\mathbf{j} + \mathbf{k} \text{ and } \mathbf{y} \; = 4\,\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k}$$

are perpendicular. Compute $\mathbf{x} \cdot \mathbf{y}$ to verify the result.

P2.7. **Unit vector** Prove that if $\mathbf{x}$ is any nonzero vector, $\mathbf{u} \; = \; \mathbf{x}/\sqrt{\mathbf{x} \cdot \mathbf{x}}$ is a unit vector.

P2.8. **Angle between vectors** Find the angles in the triangle with the vertices:

$$[2, -1, 0], \quad [5, -4, 3], \quad \text{and} \quad [1, -3, 2].$$

P2.9. **Cross product** Show that the magnitude of the cross product $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

P2.10. **Basis vectors** Given the following vectors, compute the sums and write them in terms of the standard basis vectors, and compute the dot product:

    a. $\mathbf{x} = [0, 1, 4, -3]$, $\mathbf{y} = [2, 8, 6, -4]$;

    b. $\mathbf{x} = [3, -5, 0, 8]$, $\mathbf{y} = [6, 1, -7, -2]$.

P2.11. **Vectors and subspaces** Show which of the following subsets of $\mathbf{R}^3$ are subspaces and which are not:

a. The set of vectors in $\mathbf{R}^3$ with first component 1;

b. All vectors with $x_3 = 0$;

c. The set of vectors with nonzero first component.

P2.12. Subspace Using the properties of vector spaces, prove the following:

a. The set of vectors in $\mathbf{R}^2$ that lie on a line passing through the origin ($y = mx$) form a vector space.

b. The set of vectors in $\mathbf{R}^2$ that lie on a line not passing through the origin ($y = mx + b$, $b \neq 0$) does not form a vector space.

Why should it be obvious that the set in (b) is not a subspace?

P2.13. Linear independent vectors Determine if the following vectors are linearly independent:

a. $[1, 2, -1, 2], [-2, -5, 3, 0], [1, 0, 1, 10]$;

b. $[3, 0, 0, 2], [1, 0, 0, 4]$.

P2.14. Independence of polynomials Are the following polynomials independent?

$$x^2 - 1, \ x^2 + x - 2, \ x^2 + 3x + 2$$

P2.15. Complex vectors If we insist that the norm be a positive number, show that the ordinary definition of dot product for real vectors does not hold in the vector space containing complex numbers of the form $z = x + iy$, where x and y are real numbers and i is imaginary. What is the proper definition?

P2.16. Inner product Compute the inner product of the following functions on the interval $[-\pi, \pi]$ with n and m distinct positive integers:

a. $\langle \sin mx, \sin nx \rangle$;

b. $\langle \cos mx, \cos nx \rangle$;

c. $\langle \cos mx, \sin nx \rangle$;

d. $\langle \cos nx, \cos nx \rangle$;

e. $\langle \sin nx, \cos nx \rangle$.

Which functions are orthogonal?

P2.17. Orthogonal Legendre functions Show that the Legendre polynomials in Example 2.19 are orthogonal over the interval in $\mathbf{R}^1$ such that $-1 \leq x \leq 1$.

P2.18. MATLAB roundoff error Add the MATLAB value 10^{-6} in a loop N times, and compare the result with the value $N \times 10^{-6}$. If there is a difference, explain it. Try values such as $N = 100$ and $N = 1000$.

P2.19. MATLAB conversion Write a MATLAB function to convert an N-digit octal (base 8) number to decimal. Test the function by converting 0.502_8.

P2.20. MATLAB vector operations For the vectors

$$\mathbf{x} = [0, -1, 2, 3] \quad \text{and} \quad \mathbf{y} = [5, 1, 2, -3],$$

compute the following:

a. $\mathbf{x} + \mathbf{y}$;

b. $3\,\mathbf{x}$;

c. $\mathbf{x} \cdot \mathbf{y}$;

d. $\|\,\mathbf{x}\,\|$ and $\|\,\mathbf{y}\,\|$.

P2.21. MATLAB cross product   Write a MATLAB function to compute the cross product of two 3D vectors. Test the function with the vectors in Example 2.8. Compare your version with the MATLAB command **cross**.

P2.22. MATLAB orthogonal vectors   Are the vectors

$$\mathbf{x} \;=\; [\sqrt{2}/2, -1] \qquad \text{and} \qquad \mathbf{y} \;=\; [1, \sin(45^\circ)]$$

orthogonal? Do the problem analytically and using MATLAB.

P2.23. MATLAB vector norm   Write a MATLAB function to compute the norm and the corresponding unit vector for a vector in $\mathbf{R}^n$. Test the result with the vector $\mathbf{x} \;=\; [2, 3, 6]$.

EXPLORATION PROBLEMS

P2.24. MATLAB roundoff   Write a MATLAB program using the command **floor** to input a real number r and an integer k and display r rounded to k decimal places. Test the program with numbers such as π.

Hint: Investigate the MATLAB **round** and **chop** commands.

P2.25. DeMoivre's theorem   Given the complex numbers

$$\begin{aligned} z_1 &= r_1(\cos\theta + i\sin\theta), \\ z_2 &= r_2(\cos\phi + i\sin\phi) \end{aligned}$$

and the result that

$$z_1 z_2 = r_1 r_2 \left[\cos(\theta + \phi) + i\sin(\theta + \phi)\right],$$

prove DeMoivre's theorem,

$$z^n = r^n \left(\cos n\theta + i\sin n\theta\right),$$

if $z = r(\cos\theta + i\sin\theta)$ and n is an integer.

Hint: Don't forget the cases $n \le 0$.

P2.26. Complex functions   Using the results of the previous problem, show that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

P2.27. Complex plane   For the equation

$$\left|\frac{z}{z - 1}\right| = 2,$$

what is the locus of points in the complex (xy) plane?

P2.28. MATLAB complex numbers   Find the following:

a. the solution of $z^3 + 1 = 0$ (or the three cube roots of -1);
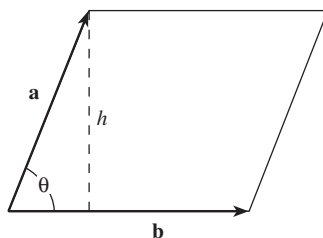
b. $(3 + 4i)^{1/3}$.

Then, compute the results using MATLAB.

P2.29. Dot product   Prove the following theorem concerning the dot product.
If θ is the angle between two vectors $\mathbf{x}$ *and* $\mathbf{y}$ *in* $\mathbf{R}^3$, *then*

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}|| \, ||\mathbf{y}||}.$$

Hint: Apply the law of cosines to the triangle with sides $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{x} - \mathbf{y}$.

P2.30. Area of a parallelogram   Consider the parallelogram defined by two vectors $\mathbf{a}$ and $\mathbf{b}$ at an angle θ, as shown in Figure 2.7.

FIGURE 2.7 *Parallelogram*

Show that the area of the parallelogram is defined by

$$A = ||\mathbf{a}|| \, ||\mathbf{b}|| \, \sin\theta,$$

which is the cross product of the two vectors. Test the result with the vectors $\mathbf{a} = [3, 1, 4]$ and $\mathbf{b} = [-2, 5, 3]$.

Hint: The area is $17\sqrt{3}$.

P2.31. Distance in polar coordinates   Determine the distance between the points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$ in the polar plane.

Hint: Write the points in rectangular coordinates and determine the distance.

Test the results with the points $P_1(1, 0)$ and $P_2(\sqrt{2}, \pi/4)$ with the angle measured in radians.

P2.32. Polynomials Consider the polynomial of degree n

$$P(x) = a_0 + a_1\,x + a_2\,x^2 + \cdots + a_n\,x^n.$$

Define the polynomial as the inner product of two vectors. Then, compare the results with the value computed by the MATLAB command **polyval** for the polynomial by testing both results with the polynomial

$$P(x) = x^5 + 5x^3 + 3x^2 + 4x + 2$$

evaluated at the points $1, 2, 3$ and -1.

P2.33. Projections Given two vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbf{R}^n$, write the vector $\mathbf{x}$ as the sum of a vector $\mathbf{x}_{\|}$ parallel to $\mathbf{y}$ and a vector $\mathbf{x}_{\perp}$ perpendicular to $\mathbf{y}$. Verify the results.

Hint: Draw two typical vectors in $\mathbf{R}^2$ to define the equations.

Test the expressions for $\mathbf{x}_{\|}$ and $\mathbf{x}_{\perp}$ for the vectors

$$\mathbf{x} = [4, -5, 3] \qquad \text{and} \qquad \mathbf{y} = [2, 1, -2].$$

P2.34. Orthogonal vectors Consider a linear combination of the vectors

$$
\begin{aligned}
\mathbf{u}_1 &= (1, 1, 1, 1) \\
\mathbf{u}_2 &= (-1, -1, 1, 1) \\
\mathbf{u}_3 &= (-1, 1, -1, 1) \\
\mathbf{u}_4 &= (-1, 1, 1, -1)
\end{aligned}
$$

to represent the vector $\mathbf{v} = (2, 3, -1, 4)$. Show the vectors are linearly independent and find the expansion.

P2.35. MATLAB Gram-Schmidt process Write a MATLAB program to implement the Gram-Schmidt procedure. Test the program with the vectors from Example 2.16.

P2.36. Orthonormal set of vectors Use the Gram-Schmidt process to produce an orthonormal set from the complex vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 1 - i \\ i \end{bmatrix}.$$

P2.37. Test your computer Determine how fast your computer multiplies and what range of numbers is permitted by doing the following:

    a. Create a program to estimate the number of floating-point multiplications per second (flops) for your computer.

    b. Find the largest and smallest MATLAB numbers allowed by your computer.

P2.38. Summing-up questions

    a. Give an example of a set of orthogonal vectors that are not linearly independent.

    b. Give an example of a set of linearly independent vectors that are not mutually orthogonal.

    c. Suppose that $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ are four vectors in $\mathbf{R}^3$. Can these vectors be linearly independent? Can these vectors be a basis for $\mathbf{R}^3$?

    d. Find all the vectors orthogonal to the vectors

$$[1, 1, 1] \quad \text{and} \quad [1, -1, 0].$$

2.12 ANNOTATED BIBLIOGRAPHY

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2. Grossman, Stanley I., *Elementary Linear Algebra*, Saunders College Publishing, Fort Worth, TX, 1991. *A good introduction to linear algebra with a number of MATLAB examples.*

3. Henrici, Peter. *Elements of Numerical Analysis*, John Wiley and Sons, New York, 1964. *A classic treatment of numerical methods, including errors that arise during numerical computation.*

4. Hill, David R., *Experiments in Computational Matrix Algebra*, Random House, New York, 1988. *The text shows the use of MATLAB to solve many problems in linear algebra and vector analysis.*

5. Mathews, John H. and K. D. Fink, *Numerical Methods Using MATLAB*, Prentice Hall, Upper Saddle River Cliffs, NJ, 1999. *The text provides a good discussion of machine arithmetic and errors.*

6. Strang, Gilbert, *Linear Algebra and Its Applications*, Saunders College Publishing, Fort Worth, TX, 1988. *A readable and interesting treatment of linear algebra.*

2.13 ANSWERS

2.1   Binary system In the binary system, letting x, y, and z be 0 or 1, the associative, commutative, and distributive laws are satisfied for both addition and multiplication. Identity elements obviously exist since $x + 0 = x$ and $1 \times x = x$. The equation $1 + 1 = 0$ shows that the system has an additive inverse.

2.3   Sum of series The series

$$S = \sum_{i=0}^{\infty} d_i \, 2^{-i} \qquad (2.80)$$

is a *geometric series* in the form $S = \sum_{k=1}^{\infty} a \, r^{k-1}$, with $a = 1$ and $r = 1/2$. To sum the series, form the sum $S_N = a + ar + ar^2 + \cdots + ar^{N-1}$ and the difference $S_N - rS_N = a - ar^N$, and solve for S_N. The sum of the geometric series for N terms is thus $S_N = \frac{a(1-r^N)}{1-r}$. For the specific case of Equation 2.80,

$$S_N = \frac{1 - \left(\frac{1}{2}\right)^N}{1 - \frac{1}{2}},$$

which converges to 2 as N goes to infinity. The error for m terms is thus $S - S_m = 2^{-m+1}$.

2.5 **Polar form of complex number** The complex number $z = \sqrt{3} + i$ has the magnitude and angle

$$
\begin{aligned}
r &= \sqrt{3+1} = 2 \\
\theta &= \tan^{-1}(1/\sqrt{3}) = \frac{\pi}{6} + 2n\pi.
\end{aligned}
$$

2.7 **Unit vector** Assuming that $\mathbf{x}$ is a nonzero vector,
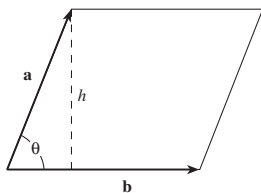
$$
\mathbf{u} = \mathbf{x}/\sqrt{\mathbf{x} \cdot \mathbf{x}} = \frac{\mathbf{x}}{(||\mathbf{x}||^2)^{1/2}}.
$$

Thus, forming $\mathbf{u} \cdot \mathbf{u}$, the result is

$$
\frac{\mathbf{x}}{||\mathbf{x}||} \cdot \frac{\mathbf{x}}{||\mathbf{x}||} = \frac{1}{||\mathbf{x}||^2}(\mathbf{x} \cdot \mathbf{x}) = 1
$$

so that $\mathbf{u}$ is a unit vector.

2.9 **Cross product** Consider the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ that meet at an angle of θ degrees, as shown in Figure 2.8.

FIGURE 2.8 *Parallelogram formed by vectors* $\mathbf{a}$ *and* $\mathbf{b}$

The area of the parallelogram is the base time the height, or $h||\mathbf{b}||$. From the figure, the area is $h||\mathbf{b}|| = ||\mathbf{a}|| \sin \theta ||\mathbf{b}|| = ||\mathbf{a} \times \mathbf{b}||$.

2.11 **Vectors and subspaces**

    a. Considering the sum of two vectors in the space

$$
[1, x_2, x_3] + [1, y_2, y_3] = [2, x_2 + y_2, x_3 + y_3],
$$

      it is clear that the sum of the vectors is not in the vector space since the sum has first component 2. Thus, the vectors in $\mathbf{R}^3$ with first component 1 do not form a subspace.

    b. All vectors with $x_3 = 0$ form a subspace.

    c. The set of vectors with nonzero first component form a subspace.

2.13 **Linear independent vectors**

    a. Since the vectors have the relationship $-5\mathbf{x}_1 - 2\mathbf{x}_2 + \mathbf{x}_3 = \mathbf{0}$, the vectors are dependent.

    b. The equations $\alpha[3, 0, 0, 2] + \beta[1, 0, 0, 4] = [0, 0, 0, 0]$ have the only solution $\alpha = \beta = 0$, so the vectors are independent.

2.15 **Complex vectors** Consider two complex vectors with components

$$
\mathbf{z}_1 = x_1 + iy_1, \quad \mathbf{z}_2 = x_2 + iy_2.
$$

The ordinary dot product produces $\mathbf{z}_1 \cdot \mathbf{z}_2 = x_1 x_2 + (iy_1)(iy_2) = x_1 x_2 - y_1 y_2$, which could be a negative value. Forming the dot product with the conjugate vector yields the correct value

$$
\mathbf{z}_1 \cdot \bar{\mathbf{z}}_2 = x_1 x_2 + (iy_1)(-iy_2) = x_1 x_2 + (y_1)(y_2).
$$

Chapter 2 ■ NUMBERS AND VECTORS

2.17   Orthogonal Legendre functions   Integrating $P_0 P_1$ yields

$$\langle P_0, P_1 \rangle = \int_{-1}^{1} x \, dx = 0.$$

Similarly, $\langle P_0, P_2 \rangle = \langle P_1, P_2 \rangle = 0$ on the interval $[-1, 1]$.

2.18   MATLAB roundoff error   This problem is discussed in the December 1993 issue of the MathWorks *MATLAB Digest*, Volume 1, number 5. Since 10^{-6} is not exactly representable in the binary floating-point format, there is a possibility of round-off error at each step of a calculation. The difference between the product $N \times 10^{-6}$ and N sums of 10^{-6} ranges from about -1.7×10^{-19} for $N = 100$ to 7.935×10^{-14} for $N = 100,000$.