



**PREVIEW**

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The next two chapters are devoted primarily to the study of Fourier techniques. Collectively, these techniques form a branch of applied mathematics called *Fourier analysis*. The field of study is named for Jean Baptiste Fourier (1768–1830), who showed that any periodic function can be represented as the sum of sinusoids with integrally related frequencies. This observation leads to the study of *Fourier series*, which will be presented first in this chapter. The purpose of the Fourier series is to express a given function as a linear combination of sine and cosine *basis functions*. In many cases, the series is simpler to analyze than the original function. Most importantly for some applications, the components of the series allow physical interpretation of the function in terms of its *frequency spectrum*.

The *Fourier transform* provides an extension of Fourier series to the analysis of nonperiodic functions. As with the series, the point of the transform is to represent a function in a manner that is easier to analyze and understand. Properties and applications of this important transform will be considered in the chapter.

The present chapter concentrates on techniques and transforms that apply to a continuous function  $f(t)$ . These include Fourier series and Fourier transforms. In Chapter 11, the discrete Fourier transform for

functions  $f(t_i)$  defined at discrete points and other transforms are considered.

It may be helpful to review the sections on orthogonal functions in Chapter 2 and Chapter 7 before studying the details of Fourier series in this chapter.

## 8.1 FOURIER SERIES

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In 1807, Fourier astounded many of his contemporary mathematicians and scientists by asserting that an arbitrary function could be expressed as a linear combination of sines and cosines. These linear combinations of the trigonometric sine and cosine functions, now called a *Fourier trigonometric series*, are applied to the analysis of *periodic* phenomena including vibrations and wave motion.

**FOURIER SERIES FORMULA** The Fourier series approximates a function  $f(t)$  by using a *trigonometric polynomial* of degree  $N$  as follows:

$$f(t) \approx \frac{a_0}{2} + \sum_{n=1}^N [a_n \cos(nt) + b_n \sin(nt)] = s_N(t), \quad (8.1)$$

where  $s_N(t)$  denotes the  $n$ th partial sum. Assuming that  $f(t)$  is continuous on the interval  $-\pi \leq t \leq \pi$ , the coefficients  $a_n$  and  $b_n$  can be computed by the formulas

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt \quad (8.2)$$

for the constant term and

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt, \quad (8.3)$$

for  $n = 1, 2, \dots, N$ . If  $f(t)$  is continuous on the interval and the derivative of  $f(t)$  exists, the series converges to  $f(t)$  at the point  $t$  when  $N \rightarrow \infty$ . Although the convergence properties of the series in Equation 8.1 will be discussed later, we will use the equality sign in Fourier series expansions, as is commonly done when the sum contains an infinite number of terms.

The series  $s_N(t)$  is the *Fourier approximation* to the function  $f(t)$  on the interval  $[-\pi, \pi]$ . From the periodicity of the trigonometric terms, it follows that

$$s_N(t + 2k\pi) = s_N(t) \quad (8.4)$$

for all  $t$  and all integers  $k$ .

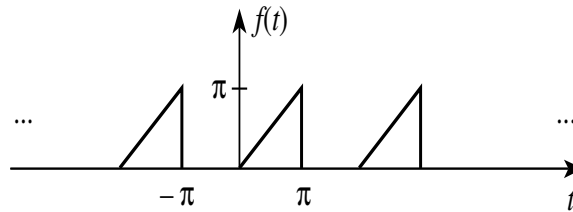
Notice that the constant term  $a_0/2$  in the series of Equation 8.1 is the average value of  $f(t)$  on the interval  $-\pi \leq t \leq \pi$  since  $a_0$  calculated by Equation 8.2 is twice the average value of  $f(t)$  over the interval. The integrals in Equation 8.3 are twice the average value of  $f(t) \cos(nt)$  and  $f(t) \sin(nt)$ , respectively. When the series is written using  $a_0/2$  as the constant term, Equation 8.3 can be used for all the coefficients  $a_n$  by letting  $n$  vary from 0 to  $N$ .

□ **EXAMPLE 8.1** *Fourier Series Example*

Consider the periodic function

$$f(t) = \begin{cases} 0, & -\pi < t < 0, \\ t, & 0 < t < \pi, \end{cases}$$

with period  $2\pi$ , as shown in Figure 8.1.



**FIGURE 8.1** Periodic function  $f(t)$  for Example 8.1

Using Equation 8.2, the Fourier coefficient  $a_0$  is

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt = \frac{1}{\pi} \int_0^{\pi} t dt = \frac{\pi}{2},$$

which yields a constant term of  $a_0/2 = \pi/4$ . The coefficients of the cosine terms are computed from Equation 8.3 as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos(nt) dt \\ &= \frac{1}{\pi} \int_{-\pi}^0 0 \cos(nt) dt + \frac{1}{\pi} \int_0^{\pi} t \cos(nt) dt. \end{aligned}$$

Integrating by parts yields

$$a_n = \frac{1}{\pi} \left[ \frac{t}{n} \sin(nt) + \frac{1}{n^2} \cos(nt) \right]_0^{\pi} = \frac{1}{\pi n^2} [\cos(n\pi) - 1] \quad (8.5)$$

for  $n = 1, 2, \dots$

Similarly, the sine terms are computed as

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin(nt) dt \\ &= \frac{1}{\pi} \int_{-\pi}^0 0 \sin(nt) dt + \frac{1}{\pi} \int_0^{\pi} t \sin(nt) dt, \end{aligned}$$

which yields

$$b_n = \frac{1}{\pi} \left[ -\frac{t}{n} \cos(nt) + \frac{1}{n^2} \sin(nt) \right]_0^\pi = -\frac{1}{n} [\cos(n\pi)] \quad (8.6)$$

for  $n = 1, 2, \dots$

The series approximation can be rewritten using the identity relationship  $\cos(n\pi) = (-1)^n$  and noticing that  $a_n = 0$  when  $n$  is an even integer. The result is

$$f(t) = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)t}{(2n-1)^2} - \sum_{n=1}^{\infty} (-1)^n \frac{\sin(nt)}{n}, \quad (8.7)$$

where the  $(2n-1)$  is introduced in the first sum to assure that only odd terms are included in that summation.

Writing out a few terms yields the series for  $f(t)$  as

$$\begin{aligned} f(t) = \frac{\pi}{4} & - \frac{2}{\pi} \cos(t) - \frac{2}{9\pi} \cos(3t) - \dots \\ & + \sin(t) - \frac{1}{2} \sin(2t) + \frac{1}{3} \sin(3t) + \dots, \end{aligned} \quad (8.8)$$

where the equality holds at points at which  $f(t)$  is continuous. At points of discontinuity  $t = n\pi$ , the series converges to  $\pi/2$ , as explained later.  $\square$

$\square$  **EXAMPLE 8.2** *MATLAB Fourier Series Example*

The accompanying MATLAB script and plots show the Fourier approximation to the function of Example 8.1 for 5 and 20 terms of the series.

**MATLAB Script** \_\_\_\_\_

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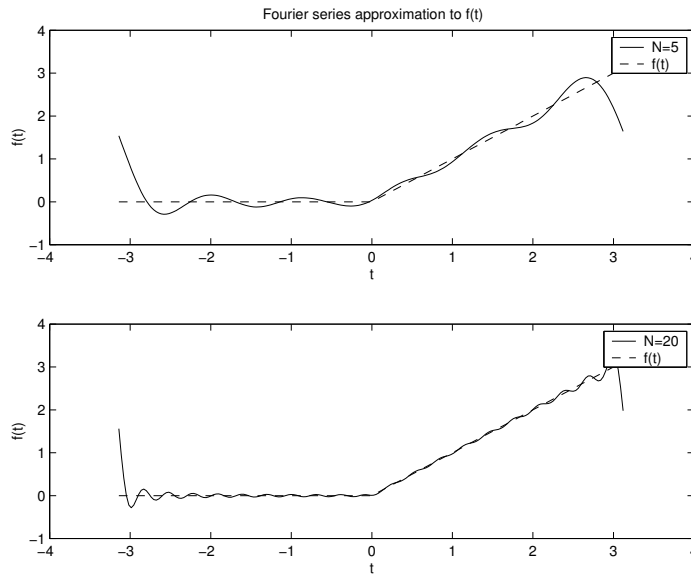
Example 8.2
% EX8_2.M Plot the Fourier series of the function f(t)
% f(t)=0 -pi < t < 0
% f(t)=t 0 < t < pi
% Plot f(t) for 5 and 20 terms in the series
clear
t = [-pi:.031:pi];           % Time points for plotting
sized=size(t);
fn = pi/4*(ones(sized));     % Fourier approximation at each t
yplt=zeros(sized);          % for plot of f(t)
% 5 terms
for n=1:5
    fn=fn+ (1/pi)*(-2*cos((2*n-1)*t)/(2*n-1)^2)-((-1)^n*sin(n*t)/n);
end
%
for k=1:length(t)           % Create f(t)
    if t(k) < 0
        yplt(k)=0;
    else
        yplt(k)=t(k);
    end
end
end

```

```

clf                                % Clear any figures
subplot(2,1,1),plot(t,fn,t,yplt,'--');
xlabel('t')
ylabel('f(t)')
title('Fourier series approximation to f(t)')
legend(['N=',num2str(n)],'f(t)') % Annotate the graph
% Add 15 more terms
for n=6:20
    fn=fn+ (1/pi)*(-2*cos((2*n-1)*t)/(2*n-1)^2)-((-1)^n*sin(n*t)/n);
end
subplot(2,1,2),plot(t,fn,t,yplt,'--');
xlabel('t')
ylabel('f(t)')
legend(['N=',num2str(n)],'f(t)')

```



**FIGURE 8.2** *Fourier series approximation for Example 8.1*

□

Five terms shown in the first plot of Figure 8.2 do not give a very good approximation, particularly near the endpoints of the function. The second plot showing a 20-term series is much closer to the actual function. Both approximations are converging to  $\pi/2$  at the endpoints.

**ORTHOGONAL FOURIER TERMS** From the discussion of orthogonal functions in Chapter 2, Chapter 6, and Chapter 7, the integrals for the Fourier coefficients in Equation 8.3 can be viewed as the normalized inner product of the function  $f(t)$  and the sine and cosine trigonometric functions.

Recall that in Chapter 2, the *inner product* of the trigonometric functions on the interval  $[-\pi, \pi]$  was defined as

$$\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)g(t) dt, \quad (8.9)$$

where the factor  $1/\pi$  was introduced to normalize the inner product for the Fourier trigonometric functions.

Then, the trigonometric terms in the Fourier series consist of functions that form an orthonormal set, since for integers  $k$  and  $m$

$$\begin{aligned} \langle \cos(kt), \cos(mt) \rangle &= \begin{cases} 1, & k = m \neq 0, \\ 0, & k \neq m, \end{cases} \\ \langle \sin(kt), \sin(mt) \rangle &= \begin{cases} 1, & k = m \neq 0, \\ 0, & k \neq m, \end{cases} \\ \langle \cos(kt), \sin(mt) \rangle &= 0, \quad \text{for all } k, m. \end{aligned} \quad (8.10)$$

Thus, the Fourier coefficients in the expansion of a function  $f(t)$  from Equation 8.3 can be written as

$$\begin{aligned} a_k &= \langle f(t), \cos(kt) \rangle, & k = 0, 1, \dots, \\ b_k &= \langle f(t), \sin(kt) \rangle, & k = 1, 2, \dots \end{aligned} \quad (8.11)$$

Notice that the constant term  $a_0$  is computed as

$$a_0 = \langle f(t), 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) dt, \quad (8.12)$$

which is the inner product of  $f(t)$  and the  $\cos(kt)$  term in Equation 8.11 for  $k = 0$ .

#### EVEN AND ODD FUNCTIONS

The Fourier approximation of even and odd functions can be computed with significantly less effort than that needed for functions without such symmetry. The properties that define an even or odd function are as follows:

1. An *even* function has a graph that is symmetric with respect to the vertical axis ( $t = 0$ ) and satisfies the equation

$$f(-t) = f(t). \quad (8.13)$$

2. An *odd* function is symmetric with respect to the origin and satisfies the equation

$$f(-t) = -f(t). \quad (8.14)$$

For the *even* function  $f_e(t)$ , a range of integration that is symmetrical about the vertical axis, where  $t = 0$ , yields the result

$$\int_{-\pi}^{\pi} f_e(t) dt = 2 \int_0^{\pi} f_e(t) dt.$$

The integral over a symmetrical range about the vertical axis for an *odd* function  $f_o(t)$  is zero; that is,

$$\int_{-\pi}^{\pi} f_o(t) dt = 0.$$

Based on these results, if  $f(t)$  is an *even function*,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nt)], \quad (8.15)$$

where

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(t) dt, \quad (8.16)$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(t) \cos(nt) dt. \quad (8.17)$$

If  $f(t)$  is an *odd function*,

$$f(t) = \sum_{n=1}^{\infty} [b_n \sin(nt)], \quad (8.18)$$

where

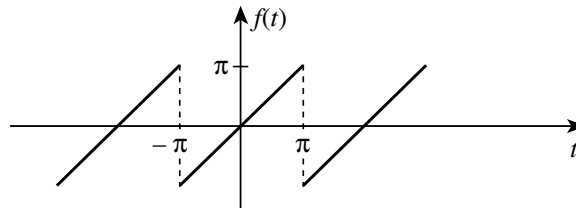
$$b_n = \frac{2}{\pi} \int_0^{\pi} f(t) \sin(nt) dt. \quad (8.19)$$

□ **EXAMPLE 8.3** *Fourier Series Odd Function Example*

Consider the Fourier series for the odd periodic function

$$f(t) = t, \quad -\pi < t < \pi,$$

shown in Figure 8.3. Each term  $a_i = 0$ , since there is no constant term or terms in  $\cos(nt)$ .



**FIGURE 8.3** *Odd periodic function for Example 8.3*



Applying Equation 8.19, the coefficients of the sine terms are

$$b_n = \frac{2}{\pi} \int_0^\pi t \sin(nt) dt, \quad n = 1, 2, \dots,$$

which can be integrated by parts to yield

$$b_n = \frac{2}{\pi} \left[ \frac{-t}{n} \cos(nt) + \frac{1}{n^2} \sin(nt) \right]_0^\pi = -\frac{2}{n} \cos(n\pi).$$

Using the fact that  $\cos(n\pi) = (-1)^n$ , the resulting series can be written as

$$f(t) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt).$$

The result is a series of sine terms as expected for the approximation of an odd function. Note that  $f(0) = 0$ , as required, and  $f(n\pi) = 0$  for any integer  $n$ . At the discontinuities, the series converges to the midpoint.  $\square$

$[-T/2, T/2]$   
INTERVAL

On the interval  $[-T/2, T/2]$ , the limits of integration for the Fourier series can be changed from  $[-\pi, \pi]$  by assigning to the integration variable  $t$  the value  $2\pi t/T$ . The period of the function is thus  $T$ .

Assuming that  $f(t)$  is continuous on the interval  $-T/2 \leq t \leq T/2$ , the coefficients  $a_n$  and  $b_n$  can be computed by the formulas

$$\begin{aligned} a_0 &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) dt, \\ a_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos\left(\frac{2n\pi t}{T}\right) dt, \\ b_n &= \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin\left(\frac{2n\pi t}{T}\right) dt, \end{aligned} \quad (8.20)$$

where  $n = 1, 2, \dots$  is any positive integer.

The Fourier series on the interval  $[-T/2, T/2]$  is thus

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right]. \quad (8.21)$$

**Frequency Components** Assuming the variable  $t$  represents time, the function  $f(t)$  repeats every  $T$  seconds. The *frequency* associated with the fundamental sinusoid in the series of Equation 8.21 is  $f_0 = 1/T$ , measured in cycles per second, or hertz. The parameter

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

is the frequency in radians per second.

Since  $2n\pi/T = 2n\pi f_0 = n\omega_0$ , the series in Equation 8.21 can be written

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)], \end{aligned} \quad (8.22)$$

which emphasizes the components in terms of their frequencies.

The first term in cosine or sine is called the *fundamental* component, and the other terms are the *harmonics* with frequencies that are integer multiples of the fundamental component's frequency. Thus, the frequencies of the Fourier series terms are

$$f_0, 2f_0, 3f_0, \dots,$$

although some of the components may be zero for a particular Fourier series. However,  $f(t)$  is a continuous function of time, and this aspect of the Fourier series is emphasized when the series is used to approximate  $f(t)$ . In other applications, the frequencies of the components are of primary interest.

Sometimes a function of a spatial variable  $x$  is of interest. If the function has period  $\lambda$  meters, the function repeats as

$$f(x + \lambda) = f(x).$$

Then, the variable  $t$  in Equation 8.22 is replaced by  $x$ , and the frequency components are defined by replacing  $f_0$  with  $1/\lambda$ . The spatial equivalent of  $\omega_0$  is

$$k = \frac{2\pi}{\lambda},$$

measured in inverse units of length. Such a formulation of Fourier series is used frequently in problems involving optics. In optical applications, the values  $nk$  are called *spatial frequencies*. Thus,  $\lambda$  represents the wavelength of the light wave being analyzed.

□ **EXAMPLE 8.4** *Fourier series square wave example*

A square wave of amplitude  $A$  and period  $T$  shown in Figure 8.4 can be defined as

$$f(t) = \begin{cases} A, & 0 < t < \frac{T}{2}, \\ -A, & -\frac{T}{2} < t < 0, \end{cases}$$

with  $f(t) = f(t + T)$ , since the function is periodic.

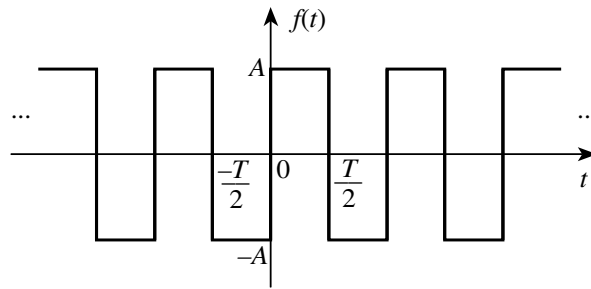


FIGURE 8.4 Square wave of Example 8.4

The first observation is that  $f(t)$  is odd, which yields the result that  $a_0 = 0$  and  $a_i = 0$  for every coefficient of the cosine terms. Letting  $\omega_0 = 2\pi/T$ , the coefficients  $b_n$  are

$$b_n = 2 \left( \frac{2}{T} \right) \int_0^{T/2} A \sin(n\omega_0 t) dt.$$

The result is

$$f(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

where  $(2n-1)$  is introduced to assure that only odd terms are included in the summation. The sine waves that make up the Fourier series for the odd square wave are

$$f(t) = \frac{4A}{\pi} \left[ \sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \dots \right],$$

so the series consists not only of sine terms, as expected, but also odd harmonics appear. This is due to the rotational symmetry of the function since the wave shapes on alternate half-cycles are identical in shape but reversed in sign. Such waveforms are produced in certain types of rotating electrical machinery.  $\square$

## SYMMETRIES

Symmetry in a function  $f(t)$  should be exploited to reduce the computational effort of finding the Fourier coefficients. Generally, the symmetry exists either about a vertical line or a horizontal line. Several types of symmetry are presented in Table 8.1. Even and odd symmetry were discussed previously. Rotational symmetry exists about the zero axis, and the waveshape of alternate half-cycles is identical but reversed in sign. This symmetry is also called *half-wave symmetry* since the integrals for the Fourier coefficients need be taken over only half a period.

There is no constant term when rotational symmetry exists. If the function is also odd, only odd-harmonic sine terms will appear in the Fourier series, as was the case for the odd square wave of Example 8.4. An even function with rotational symmetry will have a Fourier series consisting of odd-harmonic cosine terms.

**TABLE 8.1** *Symmetry of the function  $f(t)$*

<i>Symmetry</i>	<i>Fourier Series</i>
Even function $f(-t) = f(t)$	Cosine terms only
Odd function $f(-t) = -f(t)$	Sine terms only
Rotational $f(t) = -f(-t + T/2)$	Odd harmonics only

COMPLEX  
SERIES

The Fourier series of Equation 8.1 contains a series of sines and cosines and thus involves real functions. It is often convenient to write the series for a function  $f(t)$  with period  $T$  as a sum of exponential functions in the form

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t}, \quad (8.23)$$

where  $\omega_0 = 2\pi/T$  as before and the coefficients  $\alpha_n$  are the complex Fourier coefficients.

By substituting the identities

$$\begin{aligned} \cos(n\omega_0 t) &= \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}, \\ \sin(n\omega_0 t) &= \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}, \end{aligned} \quad (8.24)$$

in the trigonometric form of the series in Equation 8.21, the relationship between the trigonometric and exponential coefficients is found to be

$$\begin{aligned} \alpha_0 &= \frac{a_0}{2}, \\ \alpha_n &= \frac{a_n - ib_n}{2} \quad \text{for } n > 0, \\ \alpha_{-n} &= \frac{a_n + ib_n}{2}. \end{aligned} \quad (8.25)$$

Notice that  $\alpha_{-n}$  is the complex conjugate of the term  $\alpha_n$ .<sup>1</sup> Thus, the series in Equation 8.23 becomes

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} [\alpha_n e^{in\omega_0 t} + \alpha_{-n} e^{-in\omega_0 t}]. \quad (8.26)$$

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<sup>1</sup>These results hold when  $f(t)$  is a real-valued function. For complex-valued functions, the real and imaginary parts can be treated separately as real functions.

**Orthogonality** To find the coefficients in Equation 8.23, each side is multiplied by  $e^{-im\omega_0 t}$  and integrated over the period to yield

$$\int_{-T/2}^{T/2} f(t)e^{-im\omega_0 t} dt = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-T/2}^{T/2} e^{i(n-m)\omega_0 t} dt. \quad (8.27)$$

Since the terms with different exponents are orthogonal, all terms but that for which  $m = n$  are zero for the integral on the right-hand side. Thus,

$$\int_{-T/2}^{T/2} f(t)e^{-im\omega_0 t} dt = \int_{-T/2}^{T/2} e^{-in\omega_0 t} e^{in\omega_0 t} dt = \alpha_n T,$$

so that dividing both sides  $T$  yields the coefficients

$$\alpha_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t)e^{-in\omega_0 t} dt. \quad (8.28)$$

□ EXAMPLE 8.5 **Complex Series Square Wave Example**

Consider the odd square wave of Example 8.4 and the complex Fourier coefficients

$$\alpha_n = \frac{1}{T} \int_{-T/2}^0 (-A)e^{-in\omega_0 t} dt + \frac{1}{T} \int_0^{T/2} (A)e^{-in\omega_0 t} dt, \quad (8.29)$$

which leads to the series

$$f(t) = \frac{2A}{i\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}, \quad (8.30)$$

as defined in Equation 8.23.

This form contains complex coefficients, but the series can be written in terms of sine waves by combining the corresponding terms for positive and negative arguments. To determine the coefficients, the amount of difficulty is about the same for the trigonometric series and the complex series. However, the complex series perhaps has an advantage when the magnitude of the coefficients are of interest.

Each coefficient has the form

$$\alpha_n = \frac{2A}{in\pi} = \frac{2A}{n\pi} e^{-i\pi/2}, \quad n = \pm 1, \pm 3, \dots,$$

and the coefficients for even values,  $n = 0, \pm 2, \dots$ , are zero. Notice that the coefficients decrease as the index  $n$  increases. The use of these coefficients to compute the *frequency spectrum* of  $f(t)$  is considered later.

The trigonometric series is derived from the complex series by expanding the complex series of Equation 8.30 as

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t} \\ &= \dots - \frac{2A}{3\pi i} e^{-i3\omega_0 t} - \frac{2A}{\pi i} e^{-i\omega_0 t} + \frac{2A}{\pi i} e^{i\omega_0 t} + \frac{2A}{3\pi i} e^{i3\omega_0 t} + \dots \end{aligned}$$

and recognizing the sum of negative and positive terms for each  $n$  as  $2 \sin(n\omega_0 t)$ . The trigonometric series becomes

$$f(t) = \frac{4A}{\pi} \left( \sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \dots \right) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

which is the result of Example 8.4. □

#### FOURIER SPECTRUM FOR PERIODIC FUNCTIONS

In general, the Fourier series of a periodic function with period  $T$  seconds contains the fundamental sinusoid and numerous harmonics some of which may be zero. The plot of the magnitude of the frequency components is called the *amplitude*, or *frequency spectrum*. The frequency components are spaced  $f_0 = 1/T$  hertz apart. On a graph, the spectrum is a series of points (or lines) and is called a *discrete spectrum*. A discrete Fourier spectrum is characteristic of all periodic functions.

***Spectrum of Trigonometric Series*** An alternative method of expressing the trigonometric Fourier series of Equation 8.22 is to write the series with terms of the form

$$c_n \cos(2\pi n f_0 t + \theta_n), \quad (8.31)$$

which we will call a *shifted cosine series* due to the phase shift  $\theta_n$  in each term. To derive the relationship between the coefficients  $c_n$  and the coefficients of the complete cosine and sine series, we set the  $n$ th term in the cosine expansion equal to the  $n$ th component of the original series as follows:

$$c_n \cos(nt + \theta_n) = a_n \cos(nt) + b_n \sin(nt), \quad (8.32)$$

where  $a_n$ ,  $b_n$ , and  $\theta_n$  are known.

Using the identity

$$c_n \cos(nt + \theta_n) = c_n \cos(nt) \cos(\theta_n) - c_n \sin(nt) \sin(\theta_n), \quad (8.33)$$

and expanding the left-hand side of Equation 8.32 leads to the relationships,

$$\begin{aligned} c_n \cos(\theta_n) &= a_n, \\ c_n \sin(\theta_n) &= -b_n, \end{aligned}$$

to be solved for  $c_n$  and  $\theta_n$ . Squaring these equations and adding them together yields the solution for  $c_n$ , and taking the ratio determines  $\theta_n$ . The result is

$$c_n = \sqrt{a_n^2 + b_n^2} \quad \text{and} \quad \theta_n = \tan^{-1} \left( -\frac{b_n}{a_n} \right).$$

Notice that the sign of the argument of the tangent is negative.

In terms of the fundamental frequency  $f_0$ , the shifted cosine series equivalent to Equation 8.22 can be written

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [c_n \cos(2\pi n f_0 t + \theta_n)], \quad (8.34)$$

in which the numerical value of  $c_n$  is the *amplitude* and the angle  $\theta_n$  is the *phase* of the  $n$ th harmonic. In many problems, physical units are associated with the coefficients. For example, if  $f(t)$  represents a voltage wave that varies with time, the components yield the values of the voltages at each frequency that must be combined to reproduce  $f(t)$ . The phase indicates how the individual components must be shifted from the  $t = 0$  axis before they are combined.

When plotted versus frequency, the magnitudes of the constant ( $a_0/2$ ) and the set of harmonic amplitudes,  $c_n, n = 1, 2, \dots$ , are referred to as the *amplitude spectrum*, and the plot of phase shifts ( $\theta_n$ ) is the *phase spectrum*. The constant  $a_0/2$  corresponding to zero frequency is often called the dc (direct current) component in problems involving electrical circuits.

**Spectrum of Complex Series** The complex series presented as Equation 8.23,

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t} = \sum_{n=-\infty}^{\infty} \alpha_n e^{i2\pi n f_0 t},$$

has coefficients  $\alpha_n$  that are in general complex. Thus, the actual real sinusoidal terms are composed of terms that have both positive and “negative” frequency components. This use of the negative frequencies should not be disturbing and is given no physical interpretation. After all, the basis functions as complex exponentials have no physical meaning either. The spectrum is usually plotted as  $\alpha_n$  and  $\theta_n$  versus  $n$  or frequency  $n f_0$  for both negative and positive values of  $n$ .

Since the complex series coefficients are complex, they can be written

$$\begin{aligned} \alpha_n &= |\alpha_n| e^{i\theta_n}, \\ \alpha_n^* &= |\alpha_n| e^{-i\theta_n}. \end{aligned}$$

The sum of two terms for the  $n$ th harmonic yields the term  $f_n$  as

$$\begin{aligned} f_n &= |\alpha_n| e^{i\theta_n} e^{in\omega_0 t} + |\alpha_n| e^{-i\theta_n} e^{-in\omega_0 t} \\ &= |\alpha_n| \left[ e^{i(n\omega_0 t + \theta_n)} + e^{-i(n\omega_0 t + \theta_n)} \right] \\ &= 2|\alpha_n| \cos(2\pi n f_0 t + \theta_n). \end{aligned}$$

Comparing this result with the coefficients for the shifted cosine series of Equation 8.34 shows that  $c_n = 2|\alpha_n|$  if the complex series is to be converted to the shifted cosine series. Thus, the one-sided amplitude

spectrum ( $n = 1, 2, \dots$ ) for the complex series consists of the terms  $2|\alpha_n|$ . The phase is given by the terms  $\theta_n$  just as for the trigonometric series. If both positive and negative frequency components are shown, the terms  $|\alpha_n|$  are plotted. The zero frequency term is the same for any representation.

**Real Functions** If  $f(t)$  is a real and even function of  $t$ , the coefficients  $\alpha_n$  are real and even functions of  $n$ . If  $f(t)$  is a real and odd function of  $t$ , then the coefficients  $\alpha_n$  are imaginary and odd functions of  $n$ . In either case, plotting  $|\alpha_n|$  results in a real and even discrete frequency spectrum for the amplitudes.

**Power in the signal** It is an important result of alternating current theory that the power associated with a periodic wave of voltage or current  $f(t)$  is proportional to the mean-square value of  $f(t)$ . The mean-square formula is

$$\overline{f^2(t)} = \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 dt, \quad (8.35)$$

which is seen to be the average of the square of  $f(t)$ . For a pure sinusoid, the average value of its square is one-half the peak value. Thus, for the  $n$ th harmonic  $c_n \cos(n\omega_0 t)$ , the result is  $c_n^2/2$ .

Applying the mean-square formula to any of the Fourier series representations leads to the power spectrum for the function. The power is computed by squaring the appropriate series and dividing the integral over the period by the period itself to compute the mean-square value of each component. All the cross terms average to zero since the trigonometric functions are orthogonal.

The average power in the time signal must equal the power computed by the Fourier series. A rigorous statement of this fact is called Parseval's theorem. This important result was presented in Chapter 7.

*Comment:* If the function  $f(t)$  represents a voltage signal (volts) or a current (amperes), the power is not strictly given by Equation 8.35 when the signal is applied to a circuit. For example, the average power dissipated in a resistor of  $R$  ohms would be  $\overline{P(t)} = \overline{f(t)^2}/R$  watts when  $f(t)$  represents the voltage across the resistor. Generally, when no confusion would result, the power in a periodic signal is considered to be given by Equation 8.35.



FOURIER SERIES SUMMARY For a periodic signal  $f(t)$  with period  $T$ , the various forms of the Fourier series and the power associated with the signal are shown in Table 8.2.

**TABLE 8.2** *Fourier Series Representation*

<i>Series</i>	<i>Power</i>
<i>Sine and cosine series:</i>	
$\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2n\pi f_0 t) + b_n \sin(2n\pi f_0 t)]$	$\left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$
where	
$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(2n\pi f_0 t) dt$	
$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(2n\pi f_0 t) dt$	
<i>Shifted cosine:</i>	
$\frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(2\pi n f_0 t + \theta_n)$	$\left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} c_n^2$
where	
$c_n = \sqrt{a_n^2 + b_n^2}, \quad \theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$	
<i>Complex series:</i>	
$\sum_{n=-\infty}^{\infty} \alpha_n e^{i2\pi n f_0 t}$	$\sum_{n=-\infty}^{\infty}  \alpha_n ^2$
where	
$\alpha_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i2\pi n f_0 t} dt$	

Comparing the power relations in the table, the coefficients of the shifted cosine series are related to those for the sine and cosine series as

$$c_n^2 = a_n^2 + b_n^2,$$

with  $c_0 = a_0/2$ . The complex series coefficients are related to the coefficients of the trigonometric series as

$$\alpha_n^2 = \frac{1}{2}(a_n^2 + b_n^2)$$

with  $\alpha_0 = a_0/2$ .

□ EXAMPLE 8.6 *MATLAB Fourier Spectrum Example*

The spectrum for the square wave of Figure 8.4 can be derived from the complex series of Equation 8.30 in Example 8.5,

$$f(t) = \frac{2A}{i\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}.$$

The coefficients of the series consists of the terms

$$\alpha_n = \frac{2A}{n\pi} e^{-i\pi/2}, \quad n = \pm 1, \pm 3,$$

since  $i^{-1} = e^{-i\pi/2}$ . Thus, the amplitude is  $(2A/n\pi)$  with phase  $+90^\circ$  for each negative frequency component and  $-90^\circ$  for each positive frequency components. Figure 8.5 shows the one-sided amplitude spectrum as computed by the M-function CLSPEC1.M. This MATLAB script creates the spectrum. To specify the power in the signal, the series

$$P = |\alpha_0|^2 + 2 \sum_{n=1}^{\infty} |\alpha_n|^2$$

would be computed. The function `clptdscf` actually plots the line spectrum.

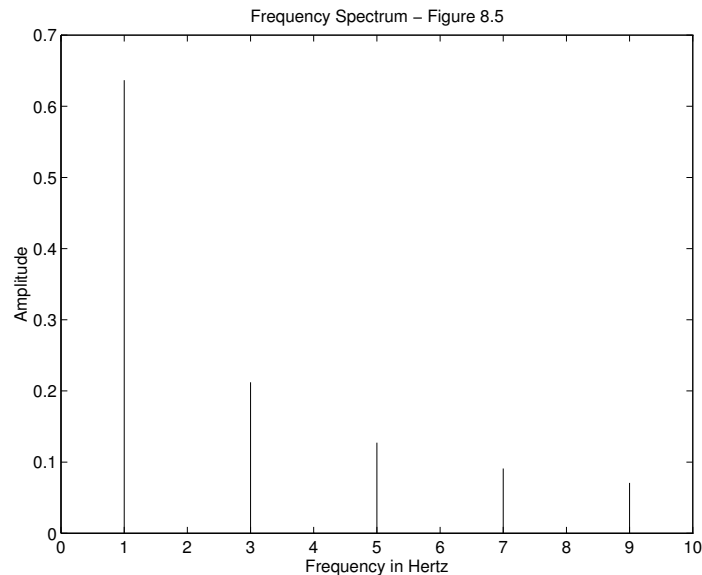


FIGURE 8.5 *Spectrum of square wave of Example 8.4*

The MATLAB command `stem` also plots discrete data sequences. We created `clptdscf` because it gives more control over the output format.

## MATLAB Script

---

Example 8.6

```
% CLSPEC1.M Plot positive frequency spectrum of square wave
% The components are  $2/(n \pi)$ ; n odd.
% Plot 10 components of the discrete spectrum [f F]
% by calling function clptdscf
%
clear
xunit='Hertz';           % Units of frequency
f=[0:1:10];             % Frequency scale
Fn=zeros(1,11);        % Row vector of 11 elements
% Frequency spectrum
for n=1:5 % Compute 5 positive components
    Fn(2*n-1)=2/((2*n-1)*pi);
end
Fn=[0 Fn];             % Add the zero value
%
clptdscf(f,Fn,xunit)   % Call for plot
```

---

The function `clptdscf` plots a discrete function ( $f, F$ ) in  $f$  units specified by the input `xunit` and the units will be displayed. The title of the graph must be input from the keyboard when the function executes.

## MATLAB Script

---

Example 8.6

```
function clptdscf(f,F,xunit)
% CALL: clptdscf(f,F,xunit) Plot a discrete spectrum [f F]
% Input to function is
% f - frequencies
% F - spectral values
% xunit - units of frequency (Hz or rad/sec)
% Input title of graph from keyboard
nl=length(f);           % Number of f points
fmin=min(f);           % and range
fmax=max(f);
Fmax=max(F);
% Plotting range, lengthen axes by 10%
Fmaxp=Fmax+.1*Fmax;
fminp=fmin-.1*fmax;
fmaxp=fmax+.1*fmax;
%
title1=input('Title ', 's' );
clf                     % Clear any figures
axis([fminp fmaxp 0 Fmaxp]) % Manual scaling
for I=1:nl,
    fplots=[f(I) f(I)];
    Fplot=[0 F(I)];
    plot(fplots,Fplot) % Plot one line at a time
hold on
end
```

```

title(title1)
ylabel('Amplitude')
xlabel(['Frequency in ', xunit])

```

□

□ EXAMPLE 8.7 *Fourier spectrum*

Consider the even, periodic pulse train in Figure 8.6. This is an important test signal in electronics and the signal is also of interest because of the characteristics of its Fourier components. The period is  $T$ , the amplitude is  $A$ , and the pulse has duration  $\tau$  in each period. The function is an even function with average value over a period of  $A\tau/T$ .

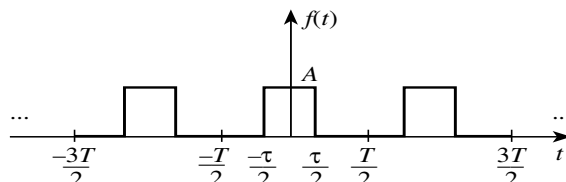


FIGURE 8.6 Periodic train of rectangular pulses

Letting  $\omega_0 = 2\pi/T$ , the coefficients of the complex series are

$$\alpha_n = \frac{1}{T} \int_{-T/2}^{T/2} A e^{-in\omega_0 t} dt = \frac{1}{T} \int_{-\tau/2}^{\tau/2} A e^{-in\omega_0 t} dt.$$

Integrating and substituting  $-2 \sin(n\omega_0\tau/2)$  for the resulting exponentials and then multiplying and dividing by the term  $\omega_0\tau/2$  yields

$$\alpha_n = \frac{A\tau}{T} \frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}.$$

The coefficient  $\alpha_0$  is determined as  $A\tau/T$  by l'Hôpital's rule. Notice that the coefficient

$$\alpha_0 = \frac{A\tau}{T} = \frac{\text{area of pulse}}{\text{period}}.$$

Defining the function

$$\text{sinc } x \equiv \frac{\sin x}{x} \tag{8.36}$$

with  $x = n\omega_0\tau/2$  leads to the series

$$f(t) = \frac{A\tau}{T} + 2\frac{A\tau}{T} \sum_{n=1}^{\infty} \text{sinc}(n\omega_0\tau/2) \cos(n\omega_0 t)$$

since the coefficients of the cosine series are twice the values of those in the complex series.

*Comment:* The sinc function is frequently defined as

$$\text{sinc } t = \frac{\sin \pi t}{\pi t}.$$

This is the convention used in the MATLAB *Signal Processing Toolbox* for the function `sinc`. Letting  $t = x/\pi$  converts the MATLAB version to the form in Equation 8.36.

The sinc function determines the magnitude of the frequency components when it is evaluated at the points  $n\omega_0\tau/2$ . This function will be studied later as it plays an important role in Fourier analysis using Fourier transforms.

As a specific example, let  $A = 1$ ,  $\omega_0 = 1$  radian/second and  $\tau = \pi/2$  seconds for the pulse train of Figure 8.6. Thus, the period  $T = 2\pi$  seconds and  $A\tau/T = 1/4$ . The accompanying MATLAB script computes and plots the spectrum and the Fourier series representation of pulse train for 20 components in the series.

### MATLAB Script

---

Example 8.7

```
% CLSPEC2.M Plot the positive amplitude spectrum and the
% Fourier series representation for the pulse train with
% A=1, period T=2*pi, and pulse width tau=pi/2.
n=1:10; % Number of components
Wn=zeros(size(n));
Wn=2*(1/pi)*(sin(n*pi/4)./n); % Frequency spectrum n=1,2,...
Wn=[1/4,Wn]; % Add dc term
n=[0,n];
%
t=[-3*pi:.02:3*pi]; % Range of t
f=zeros(size(t));
for k=1:1:20; % f(t) with 20 terms
f=f+(2/(k*pi))*sin(k*pi/4)*cos(k*t); % in series
end
f=1/4+f; % Add dc value A*tau/T
% Put in a zero line and plot frequency
fzero=zeros(size(n));
clf % Clear any figures
subplot(2,1,1),plot(n,Wn,'*',n,fzero,'-');
xlabel('w radians per second')
ylabel('(2/pi)*sinc(n*pi/4)')
title('Fourier Series of Pulse Train')
% Plot f(t)
subplot(2,1,2),plot(t,f)
xlabel('t time in seconds')
ylabel('f(t)')
```

---

Figure 8.7 shows the Fourier series spectrum for positive frequencies and the approximation to the pulse train using 20 terms. If a two-sided spectrum were plotted, the terms  $\alpha_1, \dots, \alpha_n, \dots$ , would be divided by 2.

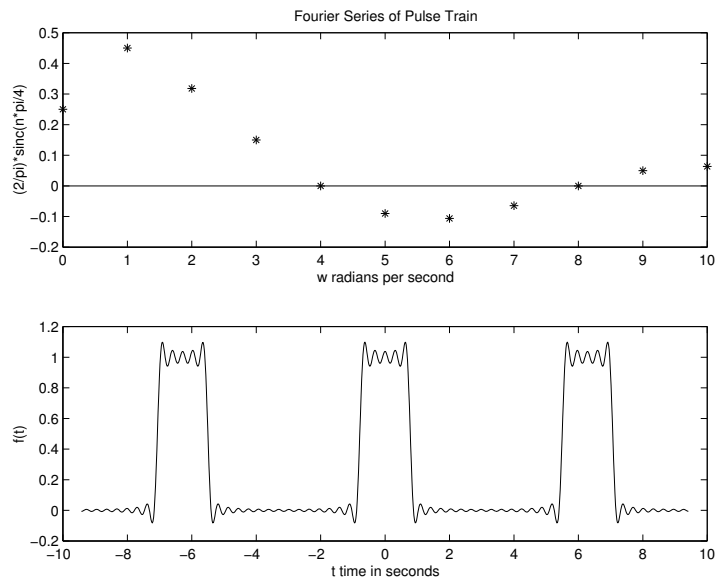


FIGURE 8.7 Spectrum and approximation for periodic pulse train of Example 8.7

You should analyze the effect on the spectrum of varying  $T$  and the effect on the reconstructed  $f(t)$  of varying the number of terms in the series.

□

## 8.2 PROPERTIES OF FOURIER SERIES

This section treats properties of Fourier series that are useful for practical and theoretical purposes. Topics include the time shift property, conditions for convergence, and the Gibbs phenomenon. Theorems define the relationship between the Fourier series and its derivative and integral.

### TIME SHIFT

For a periodic function  $f(t)$  with period  $T$ , the complex Fourier series is

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t}, \quad (8.37)$$

with  $\omega_0 = 2\pi/T$ . If the function is shifted to the right along the  $t$ -axis by an amount  $\tau > 0$ , the shifted function can be represented as

$$g(t) = f(t - \tau). \quad (8.38)$$

Thus, a shifted version of  $f(t)$  is obtained by substituting the variable  $(t - \tau)$  for  $t$ . Let the coefficients of the Fourier series expansion of  $g(t)$  be  $\hat{\alpha}_n$ , so that

$$g(t) = \sum_{n=-\infty}^{\infty} \hat{\alpha}_n e^{in\omega_0 t}.$$

With the substitution of  $t - \tau$ , Equation 8.37 becomes

$$\begin{aligned} g(t) = f(t - \tau) &= \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0(t-\tau)} \\ &= \sum_{n=-\infty}^{\infty} \alpha_n e^{-in\omega_0\tau} e^{in\omega_0 t}. \end{aligned} \quad (8.39)$$

Thus, the Fourier coefficients for the shifted function are obtained as

$$\hat{\alpha}_n = \alpha_n e^{-in\omega_0\tau},$$

indicating the coefficients of the shifted function have the same magnitude as those for the original function, but the phase of each component is shifted by  $-n\omega_0\tau$  radians if  $\tau$  is in seconds. Also, if  $\tau < 0$ , the function is considered to be shifted to the left from the origin  $t = 0$ . In effect, in either case of a shift left or right,  $\tau$  becomes the new origin for  $f(t)$  since  $g(\tau) = f(0)$  in Equation 8.38. Figure 8.14 of Problem 8.5 shows an example of a shifted pulse.

#### CONVERGENCE OF FOURIER SERIES

The general conditions for existence and convergence of the Fourier series,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)], \quad (8.40)$$

are fairly complicated, and several references listed in the Annotated Bibliography at the end of the chapter present a rigorous discussion of Fourier series. For example, a strong condition for convergence is that if  $f(t)$  is a periodic function with continuous derivatives through the second order for all  $t$ , the Fourier series of  $f(t)$  converges uniformly to  $f(t)$  for all  $t$ . However, many functions of interest are not so smooth, and we even wish to consider functions with discontinuities.

For engineering applications, the sufficient conditions for convergence of the Fourier series considering only the function and its first derivative are usually appropriate. Assume that  $f(t)$  is piecewise continuous and at  $t_0$ ,  $f'(t_0)$  exists. Then, the series converges to  $f(t_0)$  when  $t = t_0$ . At each point of discontinuity, assume the function takes the average value

$$f(t) = \frac{1}{2}[f(t+) + f(t-)], \quad (8.41)$$

where  $f(t+)$  means the limit of the function from the right and  $f(t-)$  is the limit from the left. Then, the Fourier series for  $f(t)$  converges to  $f(t)$  at every value of  $t$ .

In summary, the Fourier series for the functions  $f(t)$  considered in this text will converge to the function at points in an interval where  $f(t)$  is continuous and  $f'(t)$  exists. At points of discontinuity where there is a finite *jump* in the function, the series converges to the average value of the right-hand and left-hand limits as defined in Equation 8.41. This behavior is obvious in the Fourier series approximation of Figure 8.2.

**Gibbs Phenomenon** Convergence of the Fourier series as previously discussed assumed that infinitely many terms in the Fourier series were used to approximate a function. In practice, the Fourier series of Equation 8.40 is summed with a finite number of terms as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^N [a_n \cos(nt) + b_n \sin(nt)]. \quad (8.42)$$

When the Fourier series converges, we expect the series to approximate the function in the sense that the square error is minimized by the Fourier coefficients, as discussed in Chapter 7. This error tends to zero as the number of terms  $n \rightarrow \infty$  if  $f(t)$  is continuous in the interval of interest.

An interesting and apparently strange behavior, called the *Gibbs phenomenon* of the Fourier series, occurs near a point of discontinuity. When a finite number of terms of the series is summed, the series shows an oscillatory error, particularly near the discontinuity. This is evident in Figure 8.2 and Figure 8.7. Taking more terms in the series shows that the error does not greatly decrease very close to the points of discontinuity. However, as more terms are taken, the oscillations move closer to the point of discontinuity. In theory, the Gibbs phenomenon would occur at the point of discontinuity, even if an infinite number of terms is taken in the series.<sup>2</sup>

One other feature of a Fourier series that approximates a function with discontinuities is that the  $n$ th coefficient is divided by  $n$ . For continuous functions, the coefficients decrease as  $1/n^2$ , as stated in Problem 8.8. In general, the smoother the function, the more rapidly the coefficients decrease with increasing  $n$ .

**Integration and Differentiation of Fourier Series** Consider the integral of  $f(t)$  in terms of its Fourier series

$$\int_{t_1}^{t_2} f(t) dt = \frac{a_0}{2} \int_{t_1}^{t_2} dt + \int_{t_1}^{t_2} \sum_{n=1}^{\infty} [a_n \cos(nt) + b_n \sin(nt)] dt. \quad (8.43)$$

---

<sup>2</sup>The Gibbs phenomenon is not just a mathematical curiosity. Any real system attenuates the high-frequency components of an input signal higher in frequency than a certain value. For example, a rapidly switched signal approximating a square wave viewed on an oscilloscope that cannot pass all the frequency components of the signal as determined by Fourier analysis will appear to exhibit the Gibbs phenomenon in the displayed signal.



If  $f(t)$  is piecewise continuous on the interval  $[t_1, t_2]$ , the Fourier series of  $f(t)$  can be integrated term by term and the resulting series will converge to the integral of  $f(t)$  in the interval. Notice that integration will have the effect of dividing the coefficients in the original series by  $n$  and hence increases the rate of convergence. In terms of frequency, integration tends to smooth the function by reducing the magnitudes of higher harmonics.

Differentiation of the Fourier series, in effect, multiplies the original series by  $n$  and thus increases the magnitude of the coefficients. Using the complex series, the derivative is

$$\begin{aligned}\frac{df(t)}{dt} &= \frac{d}{dt} \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} (in\omega_0) \alpha_n e^{in\omega_0 t}.\end{aligned}\quad (8.44)$$

At higher frequencies as  $n$  increases, each term in the series is multiplied by a large number and the higher frequency components increase rapidly. However, if  $f(t)$  is continuous and  $f'(t)$  is piecewise continuous, then the Fourier series for  $f'(t)$  can be obtained by differentiating the series for  $f(t)$ .

#### APPLICATIONS OF FOURIER SERIES TO DIFFERENTIAL EQUATIONS

Consider the  $n$ th-order differential equation with constant coefficients subject to a harmonic series of sinusoids input as the forcing function. The differential equation is thus

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = \sum_{n=1}^N A_n e^{in\omega_0 t}.\quad (8.45)$$

Following the discussion in Chapter 5, if the frequencies  $n\omega_0$  are not those of the characteristic equation, the assumed solution is

$$y(t) = \sum_{n=1}^N \alpha_n e^{in\omega_0 t},\quad (8.46)$$

using the method of undetermined coefficients. Substituting the  $k$ th term in the differential equation leads to the relationship

$$\alpha_k [(i\omega_k)^n + a_{n-1}(i\omega_k)^{n-1} + \cdots + a_1(i\omega_k) + a_0] = A_k.$$

Letting  $H^{-1}(ik\omega_0)$  designate the term in brackets, the solution for the  $k$ th coefficient of the solution is

$$\alpha_k = H(ik\omega_0) A_k,$$

and the complete particular solution is

$$y(t) = \sum_{n=1}^N H(in\omega_0) A_n e^{in\omega_0 t}.\quad (8.47)$$

□ EXAMPLE 8.8 *Fourier Series DE Example*

Consider the simple circuit of Figure 8.8 consisting of a resistor  $R$  and capacitor  $C$ . The input voltage is designated  $f(t)$  and the output voltage across the capacitor is  $y(t)$ .

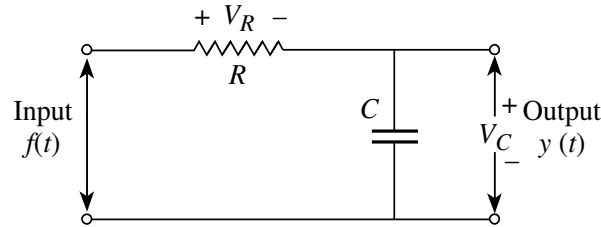


FIGURE 8.8 *RC circuit*

To derive the differential equation for the circuit, apply Kirchhoff's voltage law, with the result

$$f(t) = V_R(t) + V_C(t).$$

Using Ohm's law,  $V_R = Ri(t)$ , where  $i(t)$  is the current through the circuit. Since the current is proportional to the change in voltage across the capacitor,

$$i(t) = C \frac{dV_C(t)}{dt},$$

and Kirchhoff's law can be written

$$f(t) = RC \frac{dV_C(t)}{dt} + V_C(t).$$

Letting  $y(t) = V_C(t)$  leads to the resulting equation

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}f(t).$$

If the input voltage can be written as

$$f(t) = \sum_{n=1}^N A_n e^{in\omega_0 t},$$

the solution according to Equation 8.47 is

$$y(t) = \sum_{n=1}^N H(in\omega_0) \times A_n e^{in\omega_0 t},$$

and the function  $H(in\omega_0)$  is

$$H(in\omega_0) = \frac{1}{1 + in\omega_0 RC}.$$

In this context, the function  $H(in\omega_0)$  for a given frequency  $\omega = n\omega_0$  is called the *frequency response function* for the circuit being modeled by the differential equation. The magnitude  $|H(in\omega_0)|$  defines the amplification (or attenuation)

factor at frequency  $n\omega_0$ . The angle of  $H(in\omega_0)$ , designated  $\arg[H(in\omega_0)]$ , gives the phase shift of the input. For this circuit, the results are

$$\begin{aligned} |H(in\omega_0)| &= \frac{1}{\sqrt{1 + (n\omega_0 RC)^2}}, \\ \arg[H(in\omega_0)] &= -\tan^{-1}(n\omega_0 RC). \end{aligned} \quad (8.48)$$

Problem 8.9 at the end of the chapter further explores this approach to determining the response of a system to a Fourier series input.

The  $RC$  circuit can be used to model (approximately) the input circuits of many instruments. Given the values of  $R$  and  $C$ , a frequency analysis of the circuit shows that at the radian frequency  $\omega = 1/RC$ , the input frequency component is attenuated by the factor  $1/\sqrt{2}$ . The value  $f = \omega/2\pi$  hertz defines the *bandwidth* of the instrument. Although an input circuit does not completely cut off frequencies above a specific value, frequency components higher in frequency than the bandwidth are considered to be eliminated from the signal for the purposes of a simple analysis. □

## 8.3 FOURIER TRANSFORMS

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Fourier series, as previously presented, were used to represent arbitrary periodic functions as a linear combination of a set of *complete, orthogonal* functions, namely, the set of harmonic sines and cosines or exponentials. These functions are ideal for periodic functions. The frequencies present in the series are integer multiples of  $2\pi/T$ , where  $T$  is the period of the function being expanded. However, the application of Fourier series goes much further, since the harmonic functions are *eigenfunctions* of derivative and integral operators. Thus, the Fourier series can be used to solve differential equations and to describe the response of time-invariant linear systems.

We now turn to functions that are not periodic. A Fourier analysis of such functions requires all frequencies, so the sum of functions of discrete frequencies must be replaced by an integral. The result is the *Fourier transform*. As with Fourier series, the transform has a wide range of applications. A few are listed in Table 8.3. Even if the reader is not familiar with every area of application, it is clear from the examples in the table that Fourier transforms play a very important role in engineering and physics.

**TABLE 8.3** Applications of Fourier transforms

Area	Application
Linear systems	The Fourier transform of the output of a linear system is the product of the system transfer function and the Fourier transform of the input signal.
Optics	A Fourier transform relationship exists between the light distribution on the object and image focal planes of a lens.
Random process	The power density spectrum of a random process is the Fourier transform of the auto correlation function of the process.
Quantum mechanics	The momentum and position of a particle are related by the Fourier transform.
Partial differential equations	Fourier series and Fourier transforms are used to solve various equations.

**DEFINITION OF THE FOURIER TRANSFORM** Let the function  $f(t)$  be *piecewise continuous* for  $-\infty < t < \infty$ , and let  $\int_{-\infty}^{\infty} |f(t)| dt$  exist in the sense that the result is finite. This latter condition is called *absolute convergence* of the integral. Then, the Fourier transform of  $f(t)$  exists and is defined as

$$\mathcal{F}[f(t)] = F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt. \quad (8.49)$$

The transform  $F(i\omega)$  represents the *frequency spectrum* of  $f(t)$ , and it may be complex even though  $f(t)$  is real. The magnitude  $|F(i\omega)|$  is called the amplitude spectrum of  $F(i\omega)$ . Using the notation  $F(i\omega)$  emphasizes the fact that the Fourier transform is a function of a complex variable.

□ **EXAMPLE 8.9** *Fourier Transform Example*

Consider the piecewise continuous function  $f(t)$  defined as

$$f(t) = \begin{cases} 0, & t < 0, \\ |A|e^{-\alpha t}, & t \geq 0, \end{cases}$$

with  $\alpha > 0$ .

The function has a Fourier transform since

$$\int_{-\infty}^{\infty} |f(t)| dt = \int_0^{\infty} |A|e^{-\alpha t} dt = \frac{|A|}{\alpha}$$

is finite for  $\alpha > 0$ .

The Fourier transform is thus

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} |A|e^{-\alpha t} e^{-i\omega t} dt$$

$$\begin{aligned}
&= \int_0^{\infty} |A|e^{-(\alpha+i\omega)t} dt \\
&= -|A|\frac{e^{-(\alpha+i\omega)t}}{\alpha+i\omega}\bigg|_0^{\infty} = \frac{|A|}{\alpha+i\omega} = F(i\omega). \quad (8.50)
\end{aligned}$$

The amplitude spectrum is determined by computing  $|F(i\omega)|$ , and the phase spectrum is  $\tan^{-1}(\omega/\alpha)$ . For this example,

$$|F(i\omega)| = \frac{|A|}{|\alpha+i\omega|} = \frac{|A|}{\sqrt{\alpha^2+\omega^2}}.$$

□

The Fourier transform represents a function of time in terms of its frequency content. The energy associated with the signal can also be determined from its Fourier transform.

□ **EXAMPLE 8.10** *MATLAB Fourier Transform Example*

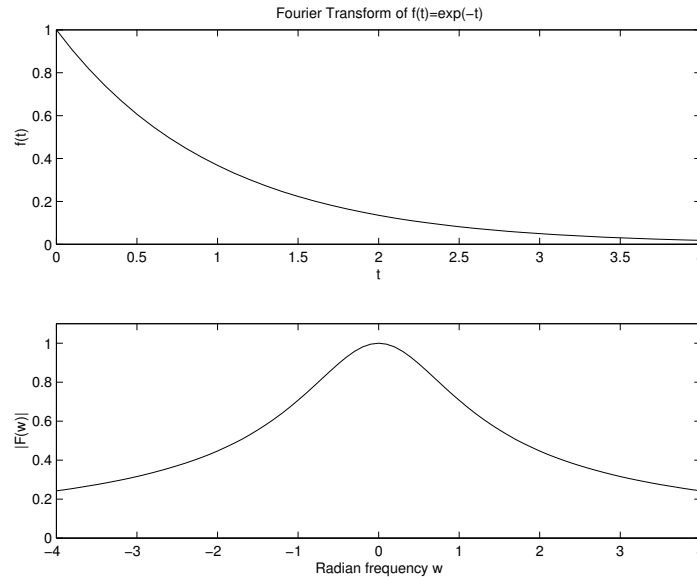
The Fourier spectrum of

$$f(t) = \begin{cases} 0, & t < 0, \\ e^{-t}, & t \geq 0. \end{cases}$$

is

$$|F(i\omega)| = \frac{1}{\sqrt{1+\omega^2}},$$

as shown in Example 8.9, with  $A = 1$  and  $\alpha = 1$ . Figure 8.9 shows the plot of  $f(t)$  and  $|F(i\omega)|$  created by the accompanying MATLAB script. The spectrum is always symmetric around  $\omega = 0$  when  $f(t)$  is real.



**FIGURE 8.9** *Fourier transform for Example 8.10*

**MATLAB Script**

Example 8.10

---

```

% EX8_10.M Plot f(t)=exp(-t), t=[0,4] and
% the magnitude of the Fourier transform
%
t=[0:.1:4];           % Range in time
foft=exp(-t);        %
% Plot magnitude of F(w)
w=[-4:.1:4];
Fw=1./(sqrt(1+w.^2)); % Fourier transform
%
clf                   % Clear any figures
subplot(2,1,1), plot(t,foft)
xlabel('t')
ylabel('f(t)')
title('Fourier Transform of f(t)=exp(-t)')
subplot(2,1,2), plot(w,Fw)
axis([-4 4 0 1.1])
xlabel('Radian frequency w')
ylabel('|F(w)|')

```

---

Try the MATLAB symbolic command **fourier** or **int** to perform the Fourier integration of  $f(t)$ .

□

**Energy in a Signal** The energy  $E$  in a signal is given by the relationship  $E = P \times T$  where  $P$  is the power and  $T$  is the time interval. For a periodic signal  $f(t)$  with period  $T$ , the power is given by

$$P = \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2 dt = \sum_{n=-\infty}^{\infty} |\alpha_n|^2,$$

as discussed previously in connection with Fourier series. The Fourier series also provides a direct measure of the power by virtue of Parseval's theorem. In this case, the  $\alpha_n$  are the coefficients for the complex series.

For the Fourier transform as previously defined to exist, the signal must have a finite energy over the interval of interest. Thus, if the signal is defined on the interval  $[-\infty, \infty]$ , the power would be zero since  $P = E/T$  and  $T$  goes to infinity. Instead of power, the energy in the signal is calculated for signals with Fourier transforms. The relationship between the signal energy in the time domain and the frequency domain is again given by *Parseval's theorem* in the form

$$E = \int_{-\infty}^{\infty} [f(t)]^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(i\omega)|^2 d\omega.$$

Integrating  $|F(i\omega)|^2$  between  $\omega_1$  and  $\omega_2$  gives the energy contributed by the frequency components between  $\omega_1$  and  $\omega_2$  to the total energy of the signal.

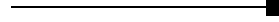
**Properties of the Fourier transform** Various important properties of the Fourier transform will be stated as theorems.

■ THEOREM 8.1

**Linearity**

Assume that  $\mathcal{F}[f(t)] = F(i\omega)$  and  $\mathcal{F}[g(t)] = G(i\omega)$ . The Fourier transform is linear since

$$\mathcal{F}[\alpha f(t) + \beta g(t)] = \alpha F(i\omega) + \beta G(i\omega).$$



■ THEOREM 8.2

**Time Shifting**

Shifting  $f(t)$  in time simply changes the phase of the Fourier transform, so that

$$\mathcal{F}[f(t - t_0)] = e^{-i\omega t_0} F(i\omega).$$



Notice that the Fourier spectrum of the time-shifted function is not changed from that of the unshifted function.

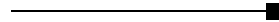
■ THEOREM 8.3

**Time and Frequency Scaling**

Scaling the time axis by a nonzero axis yields

$$\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

for the Fourier transform, where  $a$  is a real, nonzero constant.



If  $a < 1$ , the time function is spread out on the time axis, and the spectrum of  $F(i\omega)$  is compressed by a similar amount.

□ EXAMPLE 8.11

**Fourier Pulse Example**

The even rectangular pulse of height  $A$  and width  $\tau$  is defined as

$$P(t) = \begin{cases} A, & -\frac{\tau}{2} \leq t \leq \frac{\tau}{2}, \\ 0, & |t| > \frac{\tau}{2}. \end{cases}$$

The Fourier transform is

$$\begin{aligned} \mathcal{F}[P(t)] &= \int_{-\infty}^{\infty} P(t)e^{-i\omega t} dt = A \int_{-\tau/2}^{\tau/2} e^{-i\omega t} dt \\ &= -A \frac{e^{-i\omega t}}{i\omega} \Big|_{-\tau/2}^{\tau/2} = -A \frac{e^{-i\omega\tau/2} - e^{i\omega\tau/2}}{i\omega}. \end{aligned}$$

This result expressed in terms of the sine function is  $(2A/\omega) \sin(\omega\tau/2)$ . Multiplying the numerator and denominator by  $\tau/2$  yields the Fourier transform as

$$P(i\omega) = A\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2} = A\tau \operatorname{sinc}(\omega\tau/2). \quad (8.51)$$

This is the product of the area under the pulse times  $\sin(\omega\tau/2)/(\omega\tau/2)$ . □

□ EXAMPLE 8.12 **MATLAB Pulse Example**

Figure 8.10 shows the Fourier transform for two pulses as described in Example 8.11. The positive frequencies of the transform are shown for different pulse widths. The accompanying MATLAB script was used to plot the transforms for the two pulses. Each pulse has amplitude  $A = 1$ . One pulse has a pulse width of 16 seconds and the other 4 seconds.

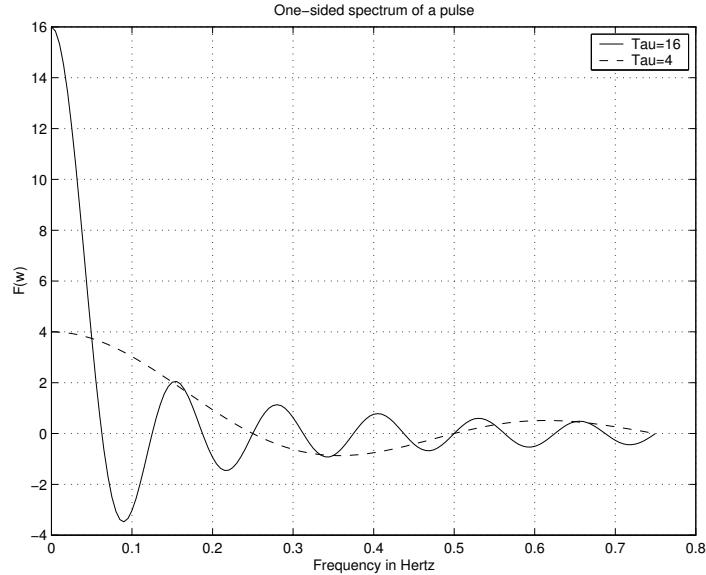


FIGURE 8.10 Fourier transform of two pulses with different pulse widths

A number of conclusions about the Fourier transform can be drawn from comparing the two transforms in the figure. Using Equation 8.51 with  $A = 1$ , the transform of a pulse is

$$P(i\omega) = \tau \frac{\sin(\omega\tau/2)}{\omega\tau/2},$$

and the argument of the sine is  $\omega\tau/2 = \pi f\tau$ . First, considering the definition of the Fourier transform, it is obvious that at  $f = 0$ ,

$$P(0) = \int_{-\infty}^{\infty} f(t) dt,$$

which is the area under the pulse.

Also, at the points

$$\frac{\omega\tau}{2} = \pi f\tau = n\pi, \quad \text{or} \quad f = n \left( \frac{1}{\tau} \right),$$

the transform has zeros. For the 16-second pulse, the first zero of the transform occurs at  $f = 1/16$  or 0.0625 hertz. The transform of the 4 second pulse crosses



zero first at  $f = 0.25$  hertz. Another important feature is that as the time pulse becomes shorter, the Fourier transform spreads out in frequency and decreases in amplitude. Although it can be proven rigorously for certain functions, we simply state that if  $W_t$  is the width of the time function and  $W_f$  is the width of the frequency spectrum in hertz, then

$$W_t W_f \geq K,$$

where  $K$  is a constant that would depend on the function being considered. The width of a function that is not a pulse is usually taken to be the distance between points where the function has a value of at least  $1/\sqrt{2}$ , or 0.707, of its peak value. More important than the exact details, which must be worked out for an individual problem, is the fact that if the function in time or frequency becomes more narrow, the function in the other domain becomes broader. In the present example, as the pulse is shortened by a factor of 4, we expect the range of significant frequencies in the transform to be increased by approximately 4. This is shown in the figure.

#### MATLAB Script

Example 8.12

```
% EX8_12.M Plot the Fourier transform (w>0) of a pulse for various
% widths. Pulse width is tau = 16 and 4 seconds.
f=[0:.005:.75];           % Frequency
f=f + eps;                 % Avoid a divide by zero
F1=zeros(size(f));
F2=zeros(size(f));
F3=zeros(size(f));
tau=16                     % Pulse width in seconds
F1=(1/pi)*(sin(pi*f*tau))./f;
tau=tau/4;                 % Pulse width = 4 seconds
F2=(1/pi)*(sin(pi*f*tau))./f;
plot(f,F1,'-',f,F2,'--')
title('One-sided spectrum of a pulse')
xlabel('Frequency in Hertz')
ylabel('F(w)')
grid
legend('Tau=16','Tau=4')
```

□

#### RELATIONSHIP TO FOURIER SERIES

Comparing the coefficients of the Fourier series of Example 8.7 for a periodic pulse train of rectangular pulses and the Fourier transform of Example 8.11 for a single pulse shows that the series coefficients are

$$\alpha_n = \frac{1}{T} \int_{-\tau/2}^{\tau/2} f(t) e^{-in\omega_0 t} dt = \frac{A\tau}{T} \frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}$$

and the transform is

$$\mathcal{F}[f(t)] = F(i\omega) = \int_{-\tau/2}^{\tau/2} f(t) e^{-i\omega t} dt = A\tau \frac{\sin(\omega\tau/2)}{\omega\tau/2}.$$

By comparing the two results, it is clear that designating the transform  $F(i\omega) = \mathcal{F}[f(t)]$ ,

$$\frac{F(n\omega_0)}{T} = \frac{A\tau \sin(n\omega_0\tau/2)}{T n\omega_0\tau/2}.$$

Thus, we conclude that the Fourier series coefficients are obtained by *sampling* the Fourier transform at the points  $n\omega_0$  and dividing by the period  $T$ . However, the Fourier series itself is a continuous function of time, but the Fourier transform is a function of  $\omega$  in the frequency domain.

## APPLICATIONS TO DIFFERENTIAL EQUATIONS

The Fourier transform is often used to solve differential equations and also to analyze linear time-invariant systems. As the following theorem states, the Fourier transform of the derivative of a function is simply a constant times the Fourier transform of the function.

### ■ THEOREM 8.4

#### *Differentiation*

Assume that  $f(t)$  is piecewise continuous and that  $f(t)$  and its derivative  $f'(t)$  have absolutely convergent integrals for all  $t$ . Then,

$$\mathcal{F}\left[\frac{df}{dt}\right] = (i\omega)F(i\omega).$$

By extension, the  $n$ th derivative  $f^{(n)}(t)$  has the Fourier transform

$$\mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(i\omega)$$

if  $f(t)$  is piecewise continuous and the derivatives of  $f(t)$  have absolutely convergent integrals.

The theorem is proven by substituting  $df/dt$  in the Fourier transform and integrating by parts using the fact that  $f(t) \rightarrow 0$  as  $t \rightarrow \pm\infty$  since we assume that the Fourier transform exists.

Applying the differentiation theorem to the differential equation,

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t) \quad (8.52)$$

by forming the Fourier transform of both sides of the equation, we find that

$$[(i\omega)^n + a_{n-1}(i\omega)^{n-1} + \cdots + a_1(i\omega) + a_0]Y(i\omega) = F(i\omega).$$

The solution for  $y(t)$  could be found by solving the transformed equation as

$$Y(i\omega) = \frac{F(i\omega)}{[(i\omega)^n + a_{n-1}(i\omega)^{n-1} + \cdots + a_1(i\omega) + a_0]}$$

and taking the inverse Fourier transform. An example will demonstrate this application to the solution of differential equations.

□ EXAMPLE 8.13 **Fourier Solution of Differential Equation**

Consider the second-order equation

$$m\ddot{y}(t) + \beta\dot{y}(t) + ky(t) = f(t), \quad (8.53)$$

for which it is desired to compute the output frequency spectrum. The equation can be rewritten as

$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = \frac{f(t)}{m} \quad (8.54)$$

using the substitutions  $\omega_n^2 = k/m$  and  $2\zeta\omega_n = \beta/m$ . This form of the equation is written in terms of the *natural frequency* of oscillation  $\omega_n$  that would be obtained if the damping factor  $\beta$  were zero. The factor  $\zeta$  is a measure of the damping and thus can range from zero in an ideal undamped system to any positive value in a physical system.

Taking the Fourier transform of Equation 8.54 yields the result

$$(-\omega^2 + i2\zeta\omega_n\omega + \omega_n^2)Y(i\omega) = \frac{F(i\omega)}{m}.$$

Dividing both sides by  $\omega_n^2$  to normalize the frequency and solving for  $Y(i\omega)$  gives the *frequency response* of the output as

$$Y(i\omega) = \frac{F(i\omega)/\omega_n^2 m}{[1 - (\omega/\omega_n)^2] + i[2\zeta\omega/\omega_n]}.$$

Using the relationship  $\omega_n^2 m = k$ , the amplitude of the output spectrum is thus

$$|Y(i\omega)| = \frac{1/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}} |F(i\omega)|. \quad (8.55)$$

If  $\phi_F(i\omega)$  is the phase of  $F(i\omega)$ , the phase of the output is

$$\arg(|Y(i\omega)|) = -\tan^{-1} \left[ \frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2} \right] + \phi_F(i\omega).$$

□

**Frequency Response (Optional)** The term *frequency response* is often used to describe the steady-state response to a sinusoidal input by a linear, time-invariant system with zero initial conditions. Such systems are described by linear differential equations with constant coefficients, as previously discussed in Chapter 5. The frequency response at one frequency is computed as the ratio of the amplitudes of the Fourier transform of the output signal to the Fourier transform of the input signal at the specified frequency. If the frequency of the input signal is varied from zero to the highest frequency of interest, the equation or plot of the output responses versus frequency is often called the *frequency response* for the system.

The importance of the frequency response stems from the fact that sinusoidal inputs or combinations of sinusoidal inputs to systems arise in many applications. In particular,

1. Any periodic signal of interest can be represented as a linear combination of sinusoidal components (Fourier series).
2. Many natural phenomena are sinusoidal such as the simple harmonic motions or signals generated in mechanical and electrical systems.
3. Sinusoidal signals are important in communications and in the generation of electrical power.

The frequency response of a system governed by a differential equation can be determined by applying the Fourier transform to the equation and taking the ratio of the output and input transforms. If the input function is  $f(t)$  with transform  $F(i\omega)$  and the response is  $y(t)$  with transform  $Y(i\omega)$ , the input/output relationship  $Y(i\omega) = H(i\omega)F(i\omega)$  is shown in Figure 8.11. The function  $H(i\omega)$  is called the *transfer function* for the system. As a function of  $\omega$ , the transfer function defines the frequency response of the system. If  $F(i\omega)$  and  $Y(i\omega)$  are known, then the transfer function can be determined as

$$H(i\omega) = \frac{Y(i\omega)}{F(i\omega)},$$

where  $H(i\omega)$  is in general a complex function of  $\omega$  that describes the amplitude and phase of the system response at each radian frequency  $\omega$  to a sinusoidal input at that frequency.

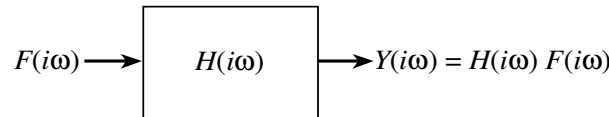


FIGURE 8.11 Frequency analysis of a linear system

□ EXAMPLE 8.14 **MATLAB Frequency Response of Differential Equation**

Writing the differential equation of Example 8.13 as

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = \frac{f(t)}{m},$$

the magnitude of the transfer function for the system is

$$\begin{aligned} |H(i\omega)| &= \frac{|Y(i\omega)|}{|F(i\omega)|} \\ &= \frac{1/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}}, \end{aligned} \quad (8.56)$$

according to our previous discussion of frequency response. Plotting the magnitude of the transfer function of Equation 8.56 and the phase angle would determine the frequency response of the system to the input of a sinusoidal

wave of radian frequency  $\omega$ . Obviously, as  $\omega$  is varied over a range of interest, the transfer function gives the frequency response to any sinusoid in that frequency range. Also, the product

$$Y(i\omega) = H(i\omega)F(i\omega)$$

determines the effect of the system on the frequency spectrum of the input function  $f(t)$ .

As a specific example, consider the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = P(t), \quad (8.57)$$

where  $P(t)$  is the pulse

$$P(t) = \begin{cases} A, & 0 \leq t \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Using our previous results, the system parameters are

$$\begin{aligned} 2\zeta\omega_n &= 3, \\ \omega_n^2 &= 2, \\ m &= 1, \end{aligned}$$

so that  $k = m\omega_n^2 = 2$ .

Using the result of Equation 8.55, the transform of the output is

$$|Y(i\omega)| = \frac{1/2}{\sqrt{[1 - (\omega/\sqrt{2})^2]^2 + [2\zeta\omega/\sqrt{2}]^2}} |P(i\omega)|,$$

where  $P(i\omega)$  is the transform of the pulse.

The Fourier transform of the pulse in this example is computed as the Fourier transform of the even pulse as given in Equation 8.51. The input pulse has amplitude  $A = 1$ , pulse width  $\tau = 1$  second, and is shifted to the right by  $t_0 = 1/2$  second. Using the shift theorem,

$$P(i\omega) = \frac{\sin(\omega/2)}{\omega/2} e^{-i\omega/2}.$$

The result for the magnitude of the spectrum of the pulse is

$$|P(i\omega)| = \left| \frac{\sin(\omega/2)}{\omega/2} \right|,$$

because the magnitude of the phase shift  $\exp(-i\omega/2) \exp(i\omega/2) = 1$ .

Combining these results, the output signal has a frequency spectrum

$$|Y(i\omega)| = \frac{1/2}{\sqrt{[1 - (\omega/\sqrt{2})^2]^2 + [2\zeta\omega/\sqrt{2}]^2}} \left| \frac{\sin(\omega/2)}{\omega/2} \right|.$$

Setting  $\zeta = 3/(2\omega_n)$ , the response can be simplified to the form

$$|Y(i\omega)| = \frac{1}{\sqrt{\omega^4 + 5\omega^2 + 4}} \left| \frac{\sin(\omega/2)}{\omega/2} \right|,$$

which is suitable for plotting.

The results from the accompanying MATLAB script are shown in Figure 8.12. It is evident that the output response of the system has a much narrower frequency spread than that of the input pulse. The system cannot respond to the higher frequencies in the input signal, and these are attenuated in the output.

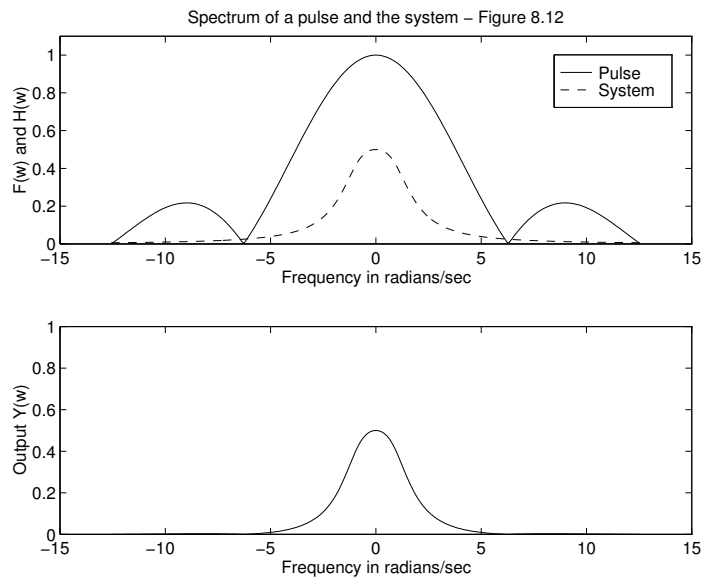


FIGURE 8.12 Frequency response of the system of Example 8.14

#### MATLAB Script

```

Example 8.14
% EX8_14.M Plot frequency response of a system with pulse input
% y'' + 3 y' + 2 y = P(t); Pulse width is tau = 1 second.
w=[-4*pi:.1:4*pi];           % Frequency range
w=w + eps;                   % Avoid a divide by zero
Y=zeros(size(w));
P=zeros(size(w));
H=zeros(size(w));
%
A=1;                          % Pulse amplitude
tau =1;                       % Pulse width in seconds
alp=w*tau/2;
P=(A*tau)*abs((sin(alp))./alp); % Transform of pulse
H=1./((sqrt(w.^4+w.^2+4)));    % Transform of system
Y=P.*H;                       % Transform of output
%
clf
subplot(2,1,1),plot(w,P,'-',w,H,'--')
axis([-15 15 0 1.1])
title('Spectrum of a pulse and the system - Figure 8.12')

```

```

xlabel('Frequency in radians/sec')
ylabel('F(w) and H(w)')
legend('Pulse', 'System')
subplot(2,1,2), plot(w,Y);
axis([-15 15 0 1])
xlabel('Frequency in radians/sec')
ylabel('Output Y(w)')

```

□

## 8.4 REINFORCEMENT EXERCISES AND EXPLORATION PROBLEMS

In these problems, do the computations by hand unless otherwise indicated and then check those that yield numerical or symbolic results with MATLAB.

### REINFORCEMENT EXERCISES

**P8.1. Fourier series of odd function** Compute the Fourier series for the odd function

$$f(x) = Ax, \quad -\lambda/2 < x < \lambda/2.$$

**P8.2. Fourier series of clock signal** Consider the computer clock signal shown in Figure 8.13, with a pulse rate of 8 million pulses per second ( $f_c = 8$  Megahertz) and amplitude of 4 volts and a pulse width of 0.05 microseconds. Find the Fourier series.

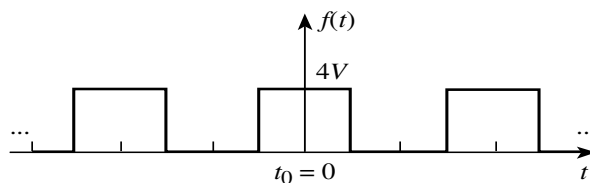


FIGURE 8.13 Clock signal

**P8.3. Fourier series** Compute the complex Fourier series for the following periodic function:

$$g(t) = \begin{cases} 1, & 0 \leq t \leq \pi, \\ -1, & -\pi \leq t < 0. \end{cases} \quad (8.58)$$

**P8.4. Fourier series** Compute the Fourier series for the function

$$y(t) = \sin^5 t.$$

*Hint:* Expand in terms of exponentials.

**P8.5. Fourier shift theorem** The shifted pulse train of Figure 8.14 consists of a pulse of width  $d$  in each period of length  $T$ . Each pulse is shifted by  $t_0$  from the center of the period. Using the Fourier time shift theorem, find the Fourier series for the pulse train.

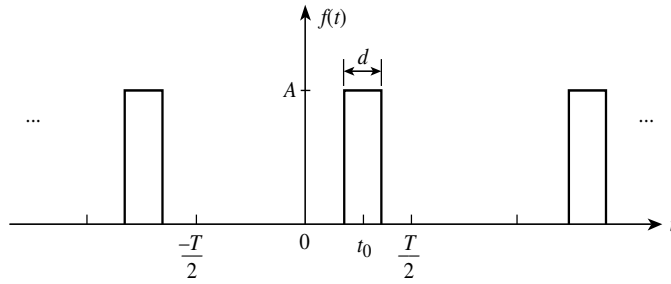


FIGURE 8.14 Shifted periodic pulse train

**P8.6. Power series sum** Using the Fourier series of the function

$$f(t) = \begin{cases} 0, & -\pi < t < 0, \\ t, & 0 < t < \pi, \end{cases}$$

determine the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

**P8.7. Power in Fourier series** The average power dissipated in a resistor of  $R$  ohms is

$$\bar{P} = \frac{\bar{V}^2}{R} \text{ watts,}$$

where  $\bar{V}$  is the root-mean-square (rms) value of voltage in volts.

- Compute the average power that an odd square wave with amplitude  $A$  volts dissipates in a 1-ohm resistor by computing the rms value.
- Compute the power by summing the components of the Fourier series.
- Suppose the repetition rate of the pulses is 10,000 times a second, so that the fundamental frequency is 10 kilohertz and the signal is filtered by a system that passes only frequencies up to 60 kilohertz. Find the power in the 1-ohm resistor at the output of the system if the amplitude of the square wave is  $A = 1$  volt.

**P8.8. Convergence of Fourier series** Let  $f(t)$  be a continuous, even function. Show that the Fourier series coefficients  $a_n$  decrease as  $1/n^2$ .

**P8.9. Fourier series solution of differential equations** Consider the mass, spring, damper system governed by the following differential equation:

$$\ddot{y} + 4\dot{y} + 40y = P(t),$$

where  $P(t)$  is a periodic function. Solve for  $y(t)$  when the input function is defined as follows:

$$P(t) = \begin{cases} 9.81, & 0 \leq t \leq \frac{T}{2}, \\ -9.81, & \frac{T}{2} < t < T. \end{cases}$$

Let  $y(0) = \dot{y}(0) = 0.0$  and  $T = 1$  second. Express the solution as a Fourier series.



**P8.10. Fourier transform and series** Consider the period pulse train with a pulse of width  $\tau$  centered in a period  $T$  with  $T > \tau$ . Compare the Fourier series coefficients of the complex series with the Fourier transform of a pulse of width  $\tau$  centered at  $t = 0$ . Derive a relationship between the Fourier transform magnitude and the magnitude of the Fourier coefficients.

**P8.11. Fourier transform of triangular pulse** Compute the Fourier transform of the triangular pulse shown in Figure 8.15.

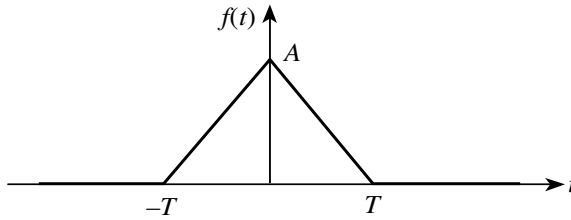


FIGURE 8.15 Triangular pulse

Express the answer as magnitude and phase functions and plot the result.

**P8.12. Fourier transform and rise time** Compute the Fourier transform of the pulse with a finite rise time shown in Figure 8.16.

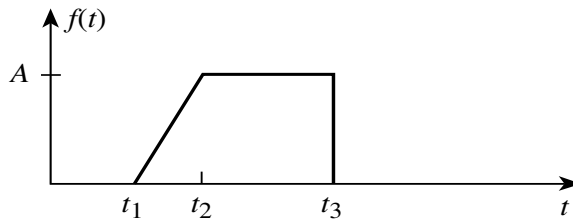


FIGURE 8.16 Pulse with finite rise time

**P8.13. Fourier transform of double gate** Compute the Fourier transform of the *double gate* function shown in Figure 8.17.

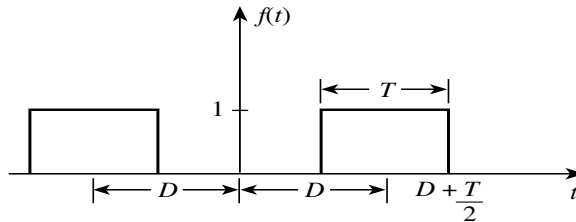


FIGURE 8.17 Double-gate function

The pulse train consists of two pulses, each of width  $T$ , separated by a distance  $2D$ .

**P8.14. Energy spectrum** Using the results of Example 8.10 for the signal  $f(t) = \exp(-t)$ ,  $t \geq 0$  and zero elsewhere, compute the following:

- The energy in the signal using  $f(t)$  and  $F(i\omega)$ ;

- b. The magnitude of the component of the signal at 1 hertz.

**P8.15. Differential equation** Solve the differential equation of Equation 8.57 and show how the input pulse is changed in time by the system.

**P8.16. MATLAB Fourier series** Plot the Fourier spectrum of the clock pulse in Problem 8.2 up to 48 Megahertz.

**P8.17. MATLAB Fourier series** Plot the series for Problem 8.3 over the interval  $(-\pi, \pi)$  for the following approximations:

- 1 term;
- 5 terms;
- 10 terms.

## EXPLORATION PROBLEMS

**P8.18. MATLAB Fourier Series** Plot the input function  $P(t)$  and the response  $y(t)$  on the same graph versus time for the equation of Problem 8.9.

**P8.19. MATLAB Fourier transform** Plot the Fourier transform for positive frequencies for the pulse of width  $\tau$  centered at the origin. On the same plot, compare the transforms for  $\tau = 4, 8, 16$  seconds.

**P8.20. Physical systems** Draw the mechanical and electrical systems that are equivalent to the system described in Example 8.14 and explain Figure 8.12 in terms of the response of the systems.

**P8.21. MATLAB Fourier Series** Determine the Fourier series of the *sawtooth* function

$$f(x) = |x| \quad \text{for } |x| \leq P/2$$

and  $f(x + P) = f(x)$ .

- Write the Fourier series for  $f(x)$  in terms of  $L = P/2$ , the half-period.
- Using MATLAB, plot the exact function over several periods for  $P = 16$  from  $[-P, P]$  and show the approximation for various number of non-constant terms in the series on one figure, say,  $n = 3, 5, 7, 9, 11$ .
- Pause 2 seconds between plots and annotate the graph with the number of non-constant terms shown.
- Use the **zoom** command and comment on the accuracy of the approximation for the final plot.

*Note: To plot the graphs on one figure, it will be necessary to experiment with MATLAB commands such as **hold**, **pause(m)** and **text**.*

**P8.22. MATLAB Bode plot** The *Signal Processing Toolbox* in the student edition of MATLAB has a function (**bode**) that plots the magnitude of the frequency response and phase for a transfer function. The MATLAB function to plot the frequency response requires that the equation be written in state-space form.

The state-space equation for Equation 8.54 is

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -\omega_n^2 x_1(t) - 2\zeta\omega_n x_2(t) + \frac{f(t)}{m}, \end{aligned}$$

as explained in Chapter 5. The MATLAB command `bode` will plot the frequency response of the system

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u}, \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u},\end{aligned}$$

where  $\mathbf{u}$  is a vector of input functions,  $\mathbf{x}$  is the state vector for the system, and  $\mathbf{y}$  is the vector of output functions. In our case,  $B = [0 \ 1]^T$ ,  $u = f(t)/m$  and  $y = x_1$  so  $C = [1 \ 0]^T$  and  $D = 0$ .

The resulting plot represents the magnitude of the frequency response in decibels  $[20\log_{10}(|H(i\omega)|)]$  versus radian frequency on a logarithmic scale. The phase is plotted in degrees. Make a Bode plot of the frequency response of the system treated in Example 8.14. The plot should use  $\omega_n = \sqrt{2}$ ,  $\zeta = 1.06$ , and  $m = 1$ .

*Comment:* The Bode plot was developed by H. W. Bode to analyze the characteristics of amplifiers using feedback. Although it is primarily used by electrical engineers to analyze circuits and control systems, the Bode plot is convenient to define the frequency response of any system since it indicates the change in amplitude and phase of the frequency components of any input that can be represented in the frequency domain.

## 8.5 ANNOTATED BIBLIOGRAPHY

1. Bracewell, Ronald N., *The Fourier Transform and Its Applications*, McGraw-Hill Book Company, New York, 1986. *This text presents a variety of applications for the Fourier series, Fourier transform, and a variety of related transforms.*
2. Kaplan, Wilfred, *Operational Methods For Linear Systems*, Addison Wesley Publishing Company, Reading, MA, 1962. *A fairly rigorous treatment of linear systems in general, and Fourier methods in particular.*
3. Strum, Robert D., and Donald E. Kirk, *Contemporary Linear Systems*, PWS Publishing Company, Boston, MA, 1994. *Another in the BookWare Companion Series, this text covers Fourier methods with numerous MATLAB examples.*

## 8.6 ANSWERS

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**8.1 Fourier series of odd functions** Since the function  $f(x) = x$  centered at zero with period  $\lambda$  is an odd function, the coefficients of the sine series are

$$\begin{aligned} b_m &= \frac{2A}{\lambda} \int_{-\lambda/2}^{\lambda/2} x \sin mkx \, dx \\ &= \frac{2A}{\lambda} \left[ \frac{\sin(mkx)}{(mk)^2} - x \frac{\cos(mkx)}{mk} \right]_{-\lambda/2}^{\lambda/2} \\ &= -\frac{2A}{mk} \cos m\pi, \end{aligned}$$

where  $k = 2\pi/\lambda$ . The series is

$$f(x) = \frac{-2A}{k} \sum_{m=1}^{\infty} \frac{(-1)^m}{m} \sin mkx = \frac{A\lambda}{\pi} \left[ \sin kx - \frac{1}{2} \sin 2kx + \cdots \right].$$

**8.3 Fourier series** Using the results of Example 8.5, the complex series is

$$f(t) = \frac{2A}{i\pi} \sum_{n=-\infty}^{\infty} \frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}$$

with  $\omega_0 = 1$  and  $A = 1$ .

**8.5 Fourier shift theorem** Notice that if the pulse train of Figure 8.14 was shifted by  $t_0$  to the left, the function would be even. Hence, the Fourier series can be written

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos[n\omega_0(t - t_0)]$$

with  $\omega_0 = \frac{2\pi}{T}$ .

The series coefficients are

$$a_n = \frac{2}{T} \int_{-d/2}^{d/2} A \cos(n\omega_0 t) \, dt, \quad n = 0, 1, \dots,$$

considering the pulse train as an even function.

The coefficients become

$$\begin{aligned} \frac{a_0}{2} &= \frac{Ad}{T} = \frac{\text{area of pulse}}{\text{period}}, \\ a_n &= \frac{2Ad}{T} \text{sinc}(n\omega_0 d/2). \end{aligned}$$

Thus, the complete series is

$$f(t) = \frac{Ad}{T} + \frac{2Ad}{T} \sum_{n=1}^{\infty} \text{sinc}(n\omega_0 d/2) \cos[n\omega_0(t - t_0)].$$

## 8.7 Power in Fourier series

- a. The average power in a signal is

$$\bar{P} = \frac{1}{T} \int_{-T/2}^{T/2} |f(t)|^2 dt,$$

so for the square wave

$$\bar{P} = \frac{1}{T} \int_{-T/2}^{T/2} A^2 dt / R = A^2 \text{ watts},$$

with  $R = 1$  ohm;

- b. The Fourier series of the square wave is

$$f(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n-1)\omega_0 t]}{(2n-1)}.$$

Thus, the power dissipated by a 1-ohm resistor by Parseval's theorem is

$$\begin{aligned} \bar{P} &= \left(\frac{a_0}{2}\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \\ &= \frac{1}{2} \sum_{n=1}^{\infty} b_n^2 \\ &= \frac{1}{2} \left(\frac{4A}{\pi}\right)^2 \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \text{ watts.} \end{aligned}$$

Using the result of Problem 8.6, the sum of the odd series is  $\pi^2/8$  so the result is  $A^2$  watts, as in Part a.

- c. Since the system passes frequencies up to 60 kHz, only the first three terms of the Fourier series of frequencies 10, 30 and 50 kHz are present in the output. The average power dissipated in a 1-ohm resistor at the output is thus

$$\bar{P}_0 = \frac{1}{2} \left(\frac{4A}{\pi}\right)^2 \left[1 + \frac{1}{9} + \frac{1}{25}\right] = 0.933 \text{ watts.}$$

Thus, comparing this power with the result of Part a with  $A = 1$ , we see that the first, third, and fifth harmonic contain 93.3% of the average power associated with a square voltage wave.

**8.9 Fourier series solution of differential equations** To find the steady-state forced motion of the system, it is necessary only to determine the change in amplitude and phase shift for each component of the input forcing function. If the input is written as a Fourier series, the response will be another Fourier series with different coefficients. The frequencies in the output series are identical to those in the input since the frequencies are unchanged by a linear system. Thus, multiplying the coefficients of the input series by the transfer function of the system evaluated at the appropriate frequency will yield the output coefficients.

The transfer function of the system is given by Equation 8.56 as

$$|H(i\omega)| = \frac{1/k}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}},$$

where

$$\zeta = \frac{\beta}{2\sqrt{km}} = \frac{4}{2\sqrt{40}} = \frac{1}{\sqrt{10}}.$$

for this particular system. This transfer function must be evaluated at  $\omega = i(2n - 1)\omega_0$  as described in Example 8.8.

The Fourier series of the input square wave is

$$f(t) = \frac{4A}{\pi} \sum_{n=1}^{\infty} \frac{\sin[(2n - 1)\omega_0 t]}{(2n - 1)}$$

according to the results of Example 8.4. In this case,  $A = 9.81$  and  $\omega_0 = 2\pi/1$ .

The output series can be written

$$y(t) = \sum_{n=1}^{\infty} |\rho_n| \frac{\sin[(2n - 1)\omega_0 t]}{(2n - 1)}.$$

Since the coefficients of the input series are  $4A/\pi(2n - 1)$ ,

$$\rho_n = \frac{\frac{1}{k} \left( \frac{4A}{\pi(2n - 1)} \right)}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta\omega/\omega_n]^2}} e^{i\phi},$$

and the phase angle is

$$\phi = -\tan^{-1} \frac{2\zeta\omega/\omega_n}{1 - (\omega/\omega_n)^2}.$$

**8.11 Fourier transform of triangular pulse** The analytic expression describing the triangular pulse is

$$P_T(t) = \begin{cases} A \left( 1 - \frac{|t|}{T} \right), & t < |T|, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the Fourier transform is

$$F(\omega) = \int_{-T}^0 A \left( 1 + \frac{t}{T} \right) e^{-i\omega t} dt + \int_0^T A \left( 1 - \frac{t}{T} \right) e^{-i\omega t} dt.$$

After integrating, a little algebra shows that

$$F(\omega) = AT \operatorname{sinc}^2 \left( \frac{\omega T}{2} \right).$$

**8.13 Fourier transform of double gate** The double gate function is simply two pulses of width  $T$ . One is shifted by  $t = D$  and the other, by  $-D$ . Thus, the Fourier transform is

$$\begin{aligned} F(\omega) &= e^{-i\omega D} AT \operatorname{sinc} \left( \frac{\omega T}{2} \right) + e^{i\omega D} AT \operatorname{sinc} \left( \frac{\omega T}{2} \right) \\ &= 2AT \cos(\omega D) \operatorname{sinc} \left( \frac{\omega T}{2} \right). \end{aligned}$$

**8.15 Differential equation** For the equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = P(t)$$

the solution is

$$y(t) = \begin{cases} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}, & 0 \leq t \leq 1, \\ e^{-t}(e-1) + \frac{1}{2}e^{-2t}(1-e^2), & t > 1. \end{cases}$$

assuming  $y(0) = 0$  and  $\dot{y}(0) = 0$ . The solution to the pulse input is thus a rising exponential until  $t = 1$  and then a decaying exponential for  $t > 0$ . In terms of frequency, the frequency spectrum of the input pulse with sharp changes in amplitude has been filtered by the system so that higher frequencies are attenuated. The resulting output is a much smoother function of time.