Fourier Series

3.1-3.3

Kamen and Heck

And

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3.1 Representation of Signals in Terms of Frequency Components

- $x(t) = \sum_{k=1,N} A_k \cos(\omega_k t + \theta_k)$, $-\infty < t < \infty$
- Fourier Series Representation
  - Amplitudes
  - Frequencies
  - Phases

Remember:

$\cos(\omega_k t+\theta_k)=\cos(\omega_k t)\cos(\theta_k)-\sin(\omega_k t)\sin(\theta_k)$

- Example 3.1 Sum of Sinusoids
  - 3 sinusoids—fixed frequency and phase
  - Different values for amplitudes (see Fig.3.1-3.4)
  - ?? Is $x(t)$ periodic? Can you show it?

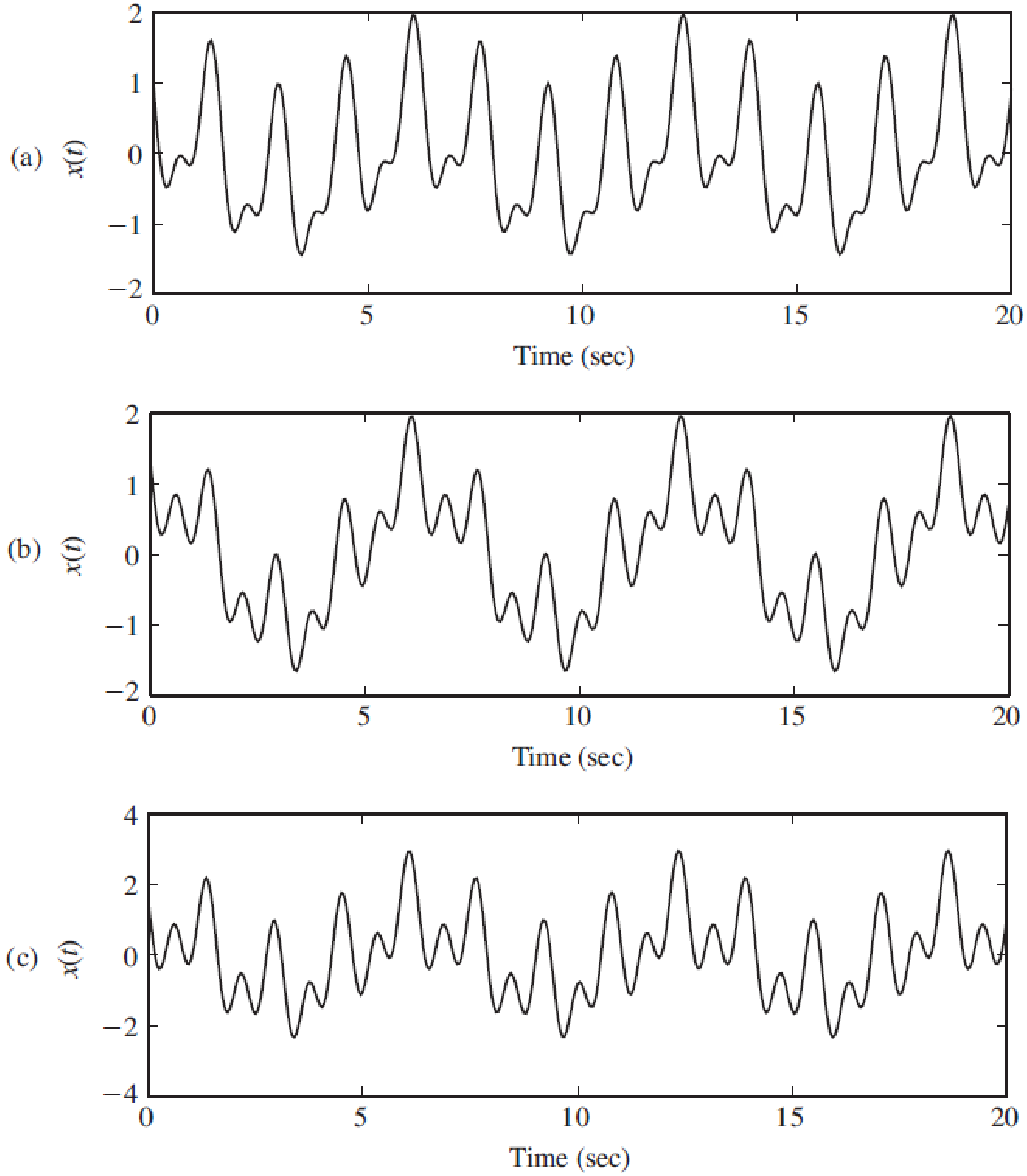
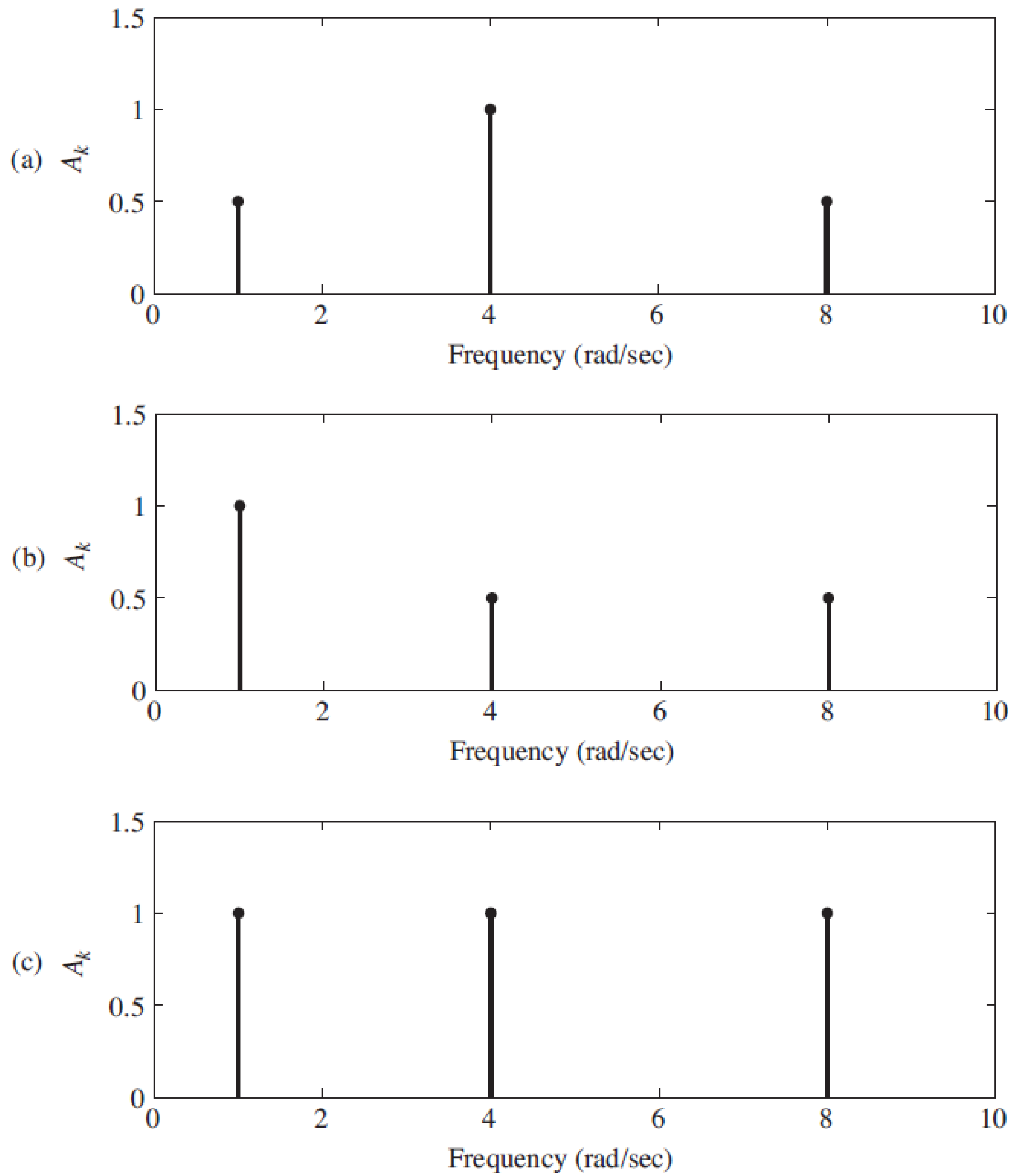
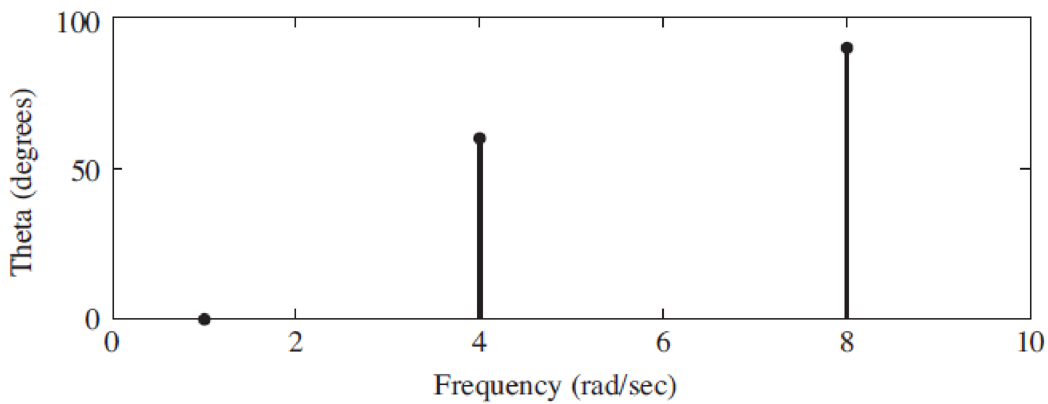


Fig 3.2 p99
Vary Ai


(a)
A_k
1.5
1
0.5
0
0
2
4
6
8
10
Frequency (rad/sec)
(b)
A_k
1.5
1
0.5
0
0
2
4
6
8
10
Frequency (rad/sec)
(c)
A_k
1.5
1
0.5
0
0
2
4
6
8
10
Frequency (rad/sec)

Always consider the phase = time shift

$$\frac{\theta}{360} = \frac{tshift}{T}$$



Shift of A2 term is 60 degrees so x shifts 60/360*T in time, or (1/6)* $\frac{\pi}{\pi/2}$ = $\frac{\pi}{12}$ seconds

4t - $\frac{\pi}{12} = 0 \; at \; t = -\frac{\pi}{12}$ as before, a shift LEFT

3.2 Trigonometric Fourier Series

- Let T be a fixed positive real number.
- Let $x(t)$ be a periodic continuous-time signal with period T.

- Then x(t) can be expressed, in general, as an infinite sum of sinusoids.

- $x(t)=a_0+\sum_{k=1,\infty} a_k\cos(k\omega_0 t)+b_k\sin(k\omega_0 t)$

  $\infty < t < \infty$ (Eq. 3.4)

This is amazing and one of the most useful results in all of science and Engineering

Fourier analysis 5,560,000 results

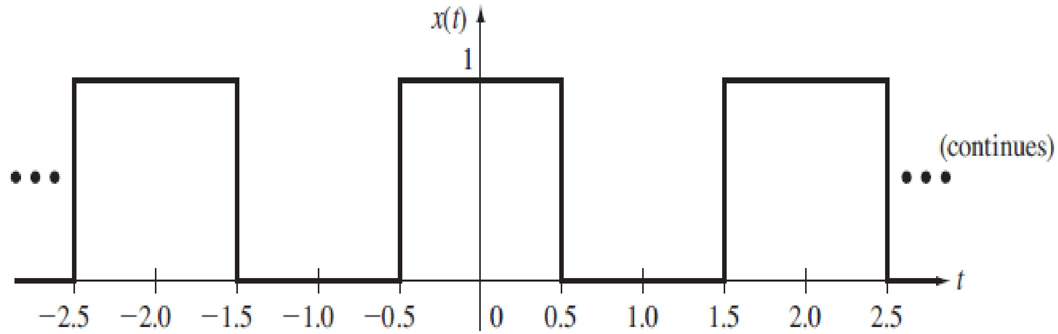
Here is my “Real World” Experience – Short version

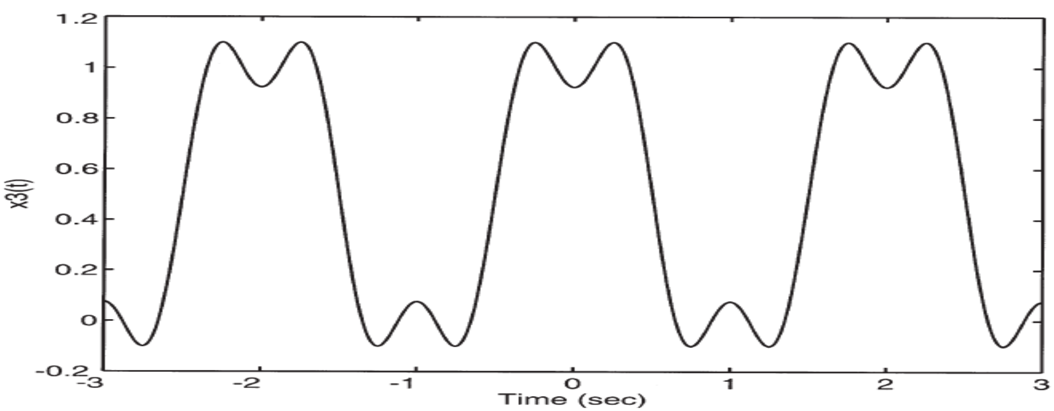
1. Propagation of signal in nerve fiber
2. NASA Vibration and Acoustic Test Facility Building 49
3. Analysis of the periodicity of the Price of Gold
4. Patent (with others) with anti-flicker frequency protection
5. Harmonic analysis of products to determine Power Quality
6. Image processing application using FFT
7. Spectrometer Biosensor (Rice) – Spectral Analysis of Breath
8. Piezoelectric ultrasonic motor rpm vs frequency input
9. Analysis of pollution concentration- frequency sweep of concentrations

3.2 Trig Fourier Series (p.101)

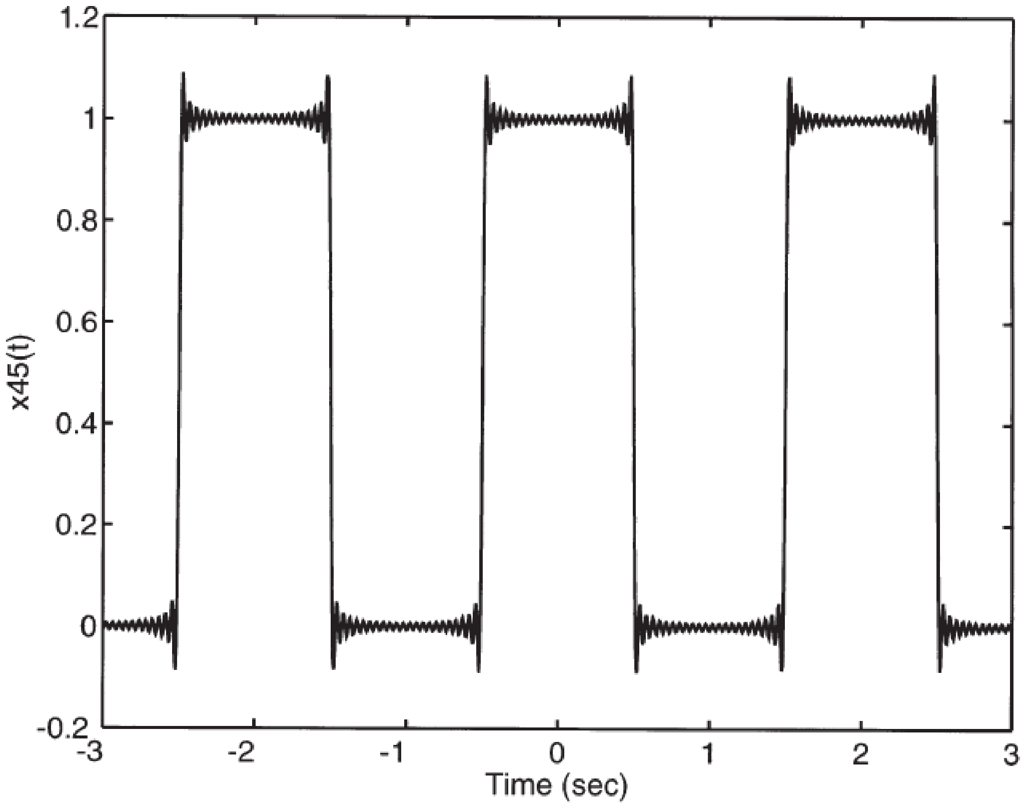
- a_0, a_k, b_k are real numbers.
- ω_0 is the fundamental frequency (rad/sec)
- $\omega_0 = 2\pi/T$, where T is the fundamental period.
- $a_k = 2/T \int_0^T x(t) \cos(k\omega_0 t)\, dt$, k=1,2,… (3.5)
- $b_k = 2/T \int_0^T x(t) \sin(k\omega_0 t)\, dt$, k=1,2,... (3.6)
- $a_0 = 1/T \int_0^T x(t)\, dt$, k = 1,2,… (3.7)
- The “with phase form”—3.8,3.9,3.10.

- Example 3.2 Rectangular Pulse Train





3 terms



N=45

PULSE TRAIN Figure 3.9 Page 108

```
% Consider a pulse train – Figure 3.5 page101  Period =1s A=1
% computes trigonometric Fourier series for Example 3.2
% used to generate Figures 3.6 - 3.9-  See Equation 3.25 Page 109
t = -3:6/1000:3;   % 1001 points
N = input('Number of harmonics ');
c0 = 0.5;   % Average value
w0 = pi;    % T=2 seconds
xN = c0*ones(1,length(t));   % dc component
for k=1:2:N,                 % even harmonics are zero
  theta = ((-1)^((k-1)/2) - 1)*pi/2  %
  xN = xN + 2/k/pi*cos(k*w0*t + theta);
end
plot(t,xN)
title(['Example 3.4, N = ',num2str(N)])
xlabel('Time (sec)')
ylabel(['x',num2str(N),'(t)'])

%Number of harmonics 9
%theta =     0      n=1
%theta =   -3.1416  n=3
%theta =     0      n=5
%theta =   -3.1416  n=7
%theta =     0      n=9
```

LET'S WATCH A MOVIE – SEE HANDOUT

I DO A SQUARE WAVE a_0 =0

As a proof of the Fourier coefficient equations –

Integrate both sides of Equation 3.4 (Page 101) over a period

$$\int_0^T x(t)dt = \int_0^T a_o \mathrm{dt} + \int_0^T \{Sum\, of\, cos + \sin terms\}dt$$

The integral of any sinusoid over a period or multiple periods is 0.
Therefore, only the term a_0 T is left on the right-hand side.
Equation 3.7 results. The is the average of the function over a period.

The terms a_k and b_k are twice the average of x(t) multiplied by the cos and sin terms respectively.

Why is only a_k left as in Equation 3.5 and all the other terms are o??

3.2 Trig Fourier Series (p.102)

- Conditions for Existence (Dirichlet)
  - 1. $x(t)$ is absolutely integrable over any period.
  - 2. $x(t)$ has only a finite number of maxima and minima over any period.
  - 3. $x(t)$ has only a finite number of discontinuities over any period.

3.2 Trig Fourier Series

- 3.2.1 Even or Odd Symmetry Pg 104
  - Equations become 3.13-3.18.
  - Example 3.3 Use of Symmetry (Pulse Train)
  - Save a lot of work.
- 3.2.2 Gibbs Phenomenon
  - As terms are added (to improve the approximation) the overshoot remains approximately 9%.
  - Fig. 3.6, 3.7, 3.8

3.3 Complex Exponential Series

- $x(t) = \sum_{k=-\infty,\infty} c_k \exp(j\omega_0 t), \; -\infty < t < \infty$ (3.19)
- Equations for complex valued coefficients —3.20 – 3.24.
- Example 3.4 Rectangular Pulse Train
- 3.3.1Line Spectra
  – The magnitude and phase angle of the complex valued coefficients can be plotted vs.the frequency.
  – Examples 3.5 and 3.6 Line Spectra

3.3 Complex Exponential Series (p.2)

- 3.3.2 Truncated Complex Fourier Series
  - As with trigonometric Fourier Series, a truncated version of the complex Fourier Series can be computed.
  - 3.3.3 Parseval’s Theorem
    - The average power P, of a signal x(t), can be computed as the sum of the magnitude squared of the coefficients.

I would do this:

Prove the shifted cosine relations Eq. 3.8-3.10

Prove the relationships Eq. 3.20-3.24 p 108 Relate trig and vectors

Show result Eq 3.25 p 109 is the same as Eq 3.12 p104

Try the Symbolic solutions on Page 110

Understand in detail Example 3.6 P112.

Compute the power in x(t) = cos t + 0.5 cos (4t+pi/3)   Ex3.5 P111
using Parseval's theorem in time and with the c_k
the answer is ½ +0.25/2 watts if x(t) is voltage across R=1 ohm.
Compute the rms value of x(t)=A cos t   ans is $\frac{A}{\sqrt{2}}$ using Eq 3.28