Fourier Transforms

Section 3.4-3.7

Kamen and Heck

And

Harman

3.4 Fourier Transform

- Definition (Equation 3.30)
  - Exists if integral converges (Equation 3.31)
- Example 3.7 **Constant Signal**
  - Does not have a Fourier transform in the ordinary sense. Violates Condition Eq. 3.31
- Example 3.8 **Exponential Signal**
  - For $b \leq 0$ does not have a Fourier transform in the ordinary sense.
  - Figure 3.12– b= 10.

$x(t) = e^{-bt}$ u(t)   X(jω) = $\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}$ dt   P 115 Ex 3.8

The Fourier transform of the impulse response of a first-order LTI differential equation $\frac{dy}{dt}$ +by(t) = $\delta(t)$ is

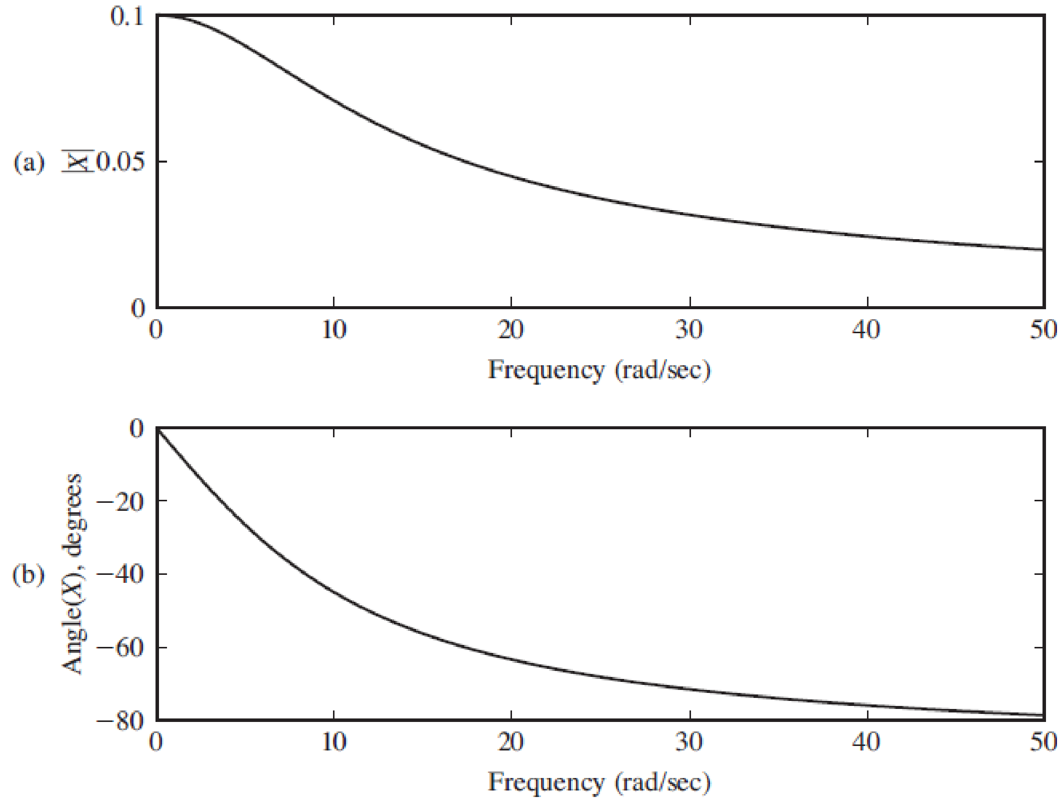
|X(jω)| = $\frac{1}{\sqrt{b^2+\omega^2}}$ Note that at $\omega = b\,\frac{r}{s}$ value is $\frac{1}{\sqrt{2}}$*X(0)

In Example 3.8 MATLAB and plot b=10.

For the angle, when $\omega = b\,\frac{r}{s}$ the angle is 45°

$Let\ b = -\ \frac{1}{RC}$ as in Equation 2.76 p. 77

```matlab
% Gives plot for Example 3.8 P117
%  Impulse response of 1st order system   h=exp(-bt)*u(t)
w = 0:0.2:50;
b = 10;
X = (1)./(b+j*w);
clf
subplot(211),plot(w,abs(X)); % plot magnitude of X (2 Panes)
title('Example 3.8')
xlabel('Frequency (rad/sec)')
ylabel('|X|')
subplot(212),plot(w,angle(X)*180/pi);%plot angle of X in deg.
xlabel('Frequency (rad/sec)')
ylabel('Angle(X), degrees')
subplot(111)
```



As $\omega \to \infty, |X(j\omega)| \to \frac{1}{\omega}$ and $\arg[X(j\omega)] \to -90°$

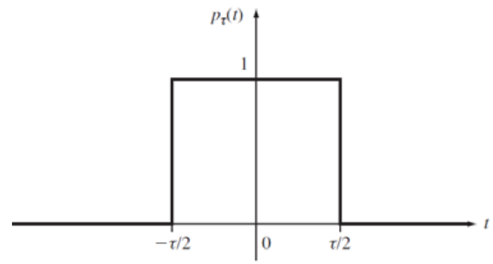
3.4.1 Rectangular and Polar Form

- Use Euler's formula.
- $X(\omega) = R(\omega) + j\, I(\omega)$ (Equation 3.33)
- $X(\omega) = |X(\omega)| \exp[\, j \angle X(\omega)\,]$ (Eq. 3.34)

3.4.2 Signals with Even or Odd Symmetry

- When a signal x(t) is even, the Fourier transform will be purely real.
- When a signal x(t) is odd, the Fourier transform will be purely imaginary.
- In such cases, the transforms can be computed using equations 3.35 and 3.36.

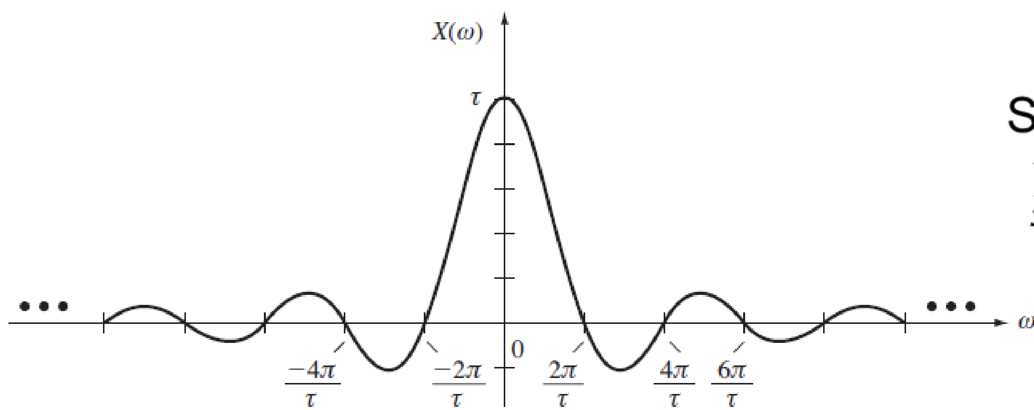
- Example 3.9 **Rectangular Pulse Page 119**



$Height\ A, width\ \tau\ seconds$

$X(\omega) = A\tau\, sinc(f\tau)$

Since $\omega = 2\pi f$



Show that

$\lim_{x\to 0}[(SINC(x)]) = 1$

This is the SINC function $\frac{\sin(\pi x)}{\pi x}$

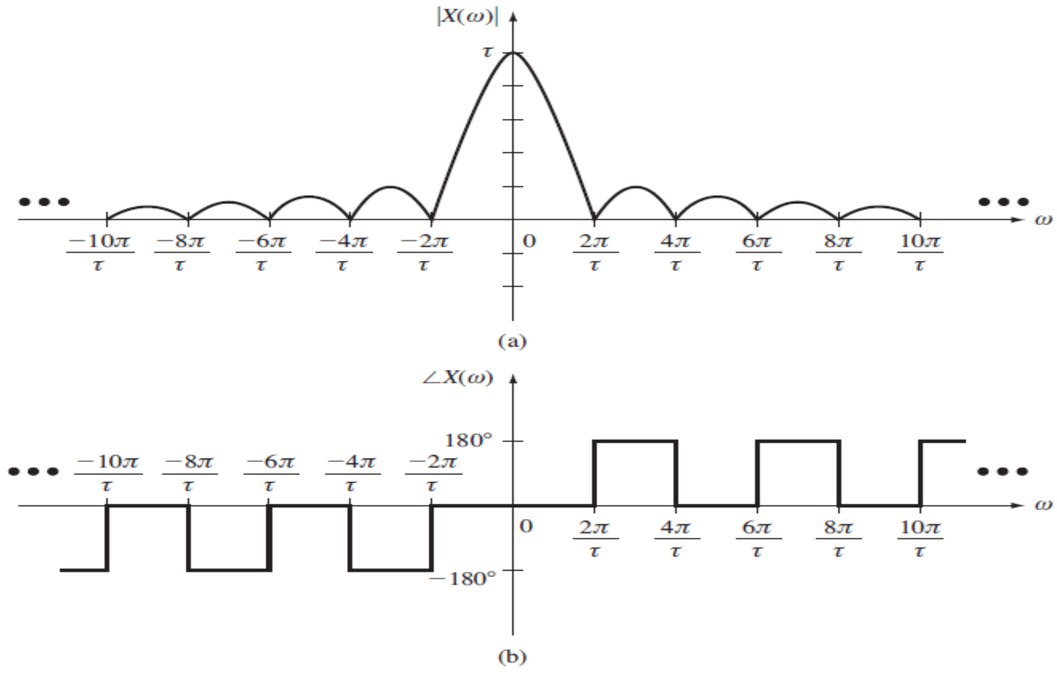
K&H Page 119-122

3.4.3 Bandlimited Signals

- A signal is said to be bandlimited if its Fourier transform $X(\omega)$ is zero for all $\omega > B$, where B is some positive number, called the bandwidth of the signal.
- Bandlimited signals cannot be time limited; that is x(t) is time limited if x(t) = 0 for all t<-T and t>T, for some positive T.
- Time-limited signals cannot be bandlimited.
- In practice, it is always possible to assume that a time-limited signal is bandlimited for a large enough B.

$\Delta\omega * \Delta t = K$ This is the Uncertainty Relationship

- Example 3.10, P121—Frequency Spectrum (of a pulse)—sidelobes get smaller and smaller, so eventually it can be approximated as being bandlimited.



Note the PHASE!

3.4.4 Inverse Fourier Transform

- The equation for the inverse Fourier transform is given by equation 3.38,P 122.
- In general, a transform pair is denoted as
  - $x(t) \leftrightarrow X(\omega)$
- One of the most fundamental transform pairs is for the pulse (Example 3.9)
  - $p_T(t) \leftrightarrow \tau \text{ sinc } (\tau\omega/2\pi)$ (Equation 3.9)

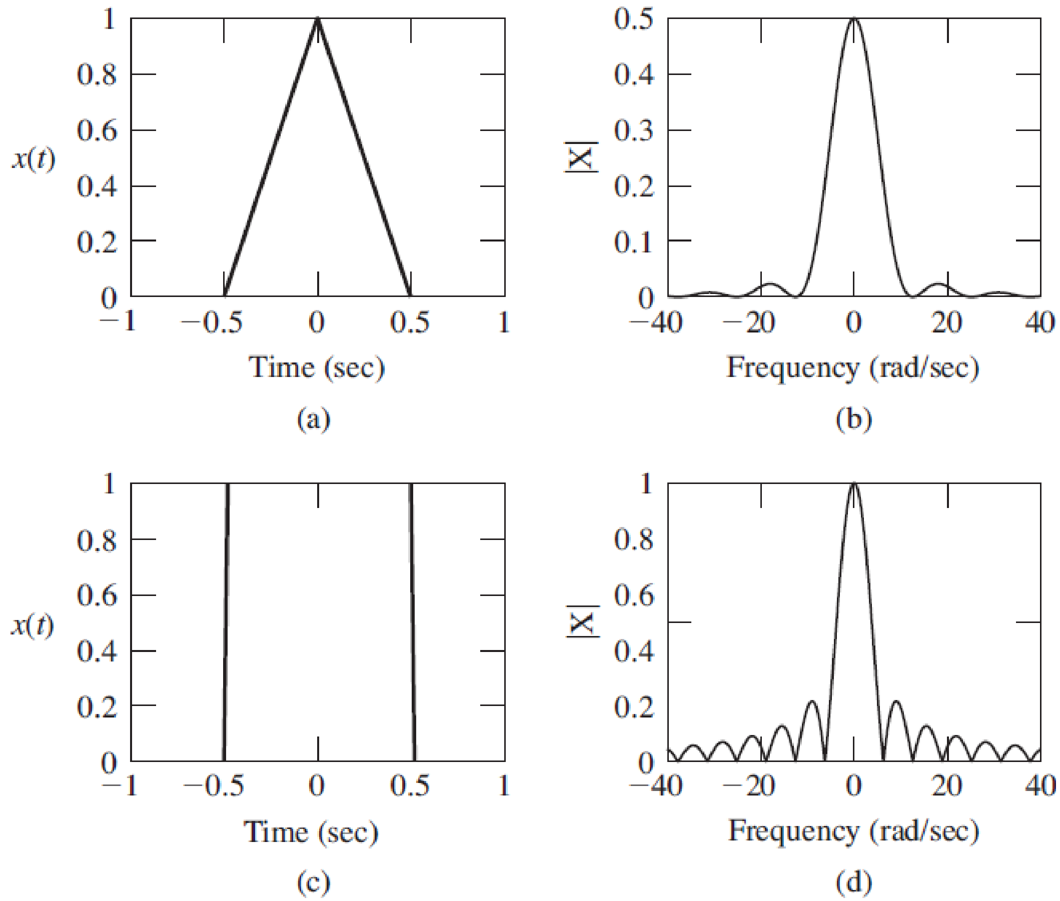
3.5 Spectral Content of Common Signals

- This section uses the MATLAB Symbolic Math Toolbox to compute the Fourier transform of several common signals so that their spectral component can be compared.
  - fourier(f) where f is a symbolic object.
  - ifourier(F) where F is a symbolic object.
  - The command int is actually used.

Spectral Content Examples

- Example 3.11 Triangular Pulse
  - When compared to the pulse, the faster transitions in the time domain result in higher frequencies in the frequency domain.
  - The results in the decaying exponential illustrate a similar result--as b gets larger, the time transition is faster and the spectrum is wider.

Figure 121 Page 121



```matlab
% Example 3.11 P 123
% Fourier Transform of triangular pulse
clear,clf
syms x t w X1 Xrect
tau = 1;
X1 = int((1-2*abs(t)/tau)*exp(-i*w*t),t,-tau/2,tau/2);
simplify(X1)
% results in -4*(cos(1/2*w)-1)/w^2
% defined nonsymbolic terms for plotting
tp1 = -tau:.02:-tau/2; tp2 = -tau/2+0.02:0.02:tau/2;
tp3 = tau/2+.02:.02:tau;
xp = [zeros(size(tp1)),(1-2*abs(tp2)/tau),zeros(size(tp3))];
wp = -40:.07:40;
Xp = -4*(cos(1/2*wp)-1)./wp.^2;
subplot(222),plot(wp,abs(Xp))
xlabel('Frequency (rad/sec)')
ylabel('|X|')
subplot(221), plot([tp1, tp2, tp3],xp)
xlabel('x(t)')
ylabel('Time (sec)')

% compare to the rectangular pulse
Xrect = int(exp(-i*w*t),t,-tau/2,tau/2)
simplify(Xrect)
% results in 2*sin(1/2*w)/w = sinc(1*w/2*pi)
xp = [zeros(size(tp1)),ones(size(tp2)),zeros(size(tp3))];
Xp = 2*sin(1/2*wp)./wp;
subplot(224),plot(wp,abs(Xp))
xlabel('Frequency (rad/sec)')
ylabel('|X|')
subplot(223), plot([tp1, tp2, tp3],xp)
xlabel('x(t)')
ylabel('Time (sec)')
subplot(111)
title('Compare triangle and rectangle X(\omega) P121')
```

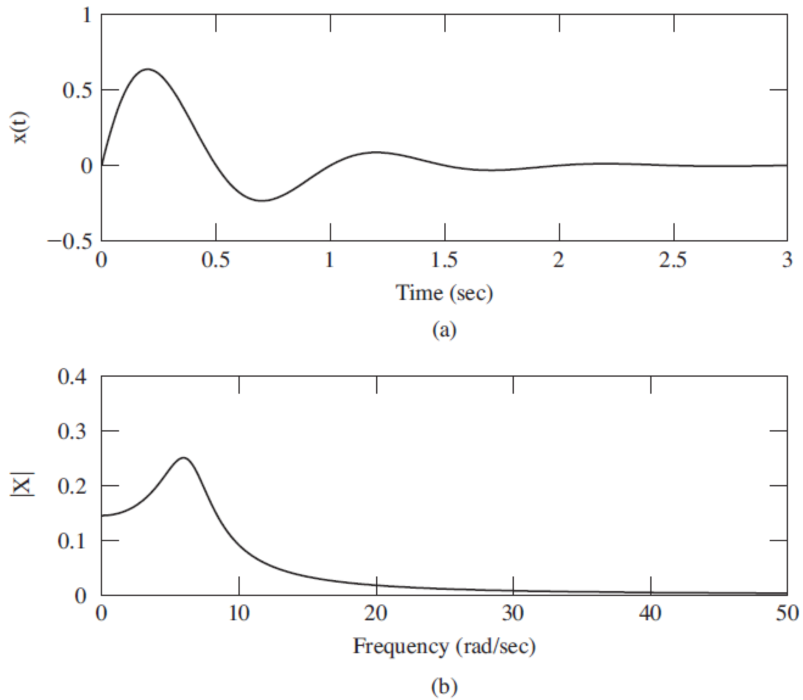
$If\ x(t) = e^{-at}$u(t), X(jω) $= \frac{1}{a+j\omega}$ P 115

- Example 3.12 P 124 Decaying Sinusoid
  - Exponential decay rate is a = 2.
  - The two sinusoidal frequencies are $b\pi = 2\pi$ and 10π.

$X_1(\omega) = \frac{1}{a+j\omega}$ $X(\omega) = \frac{j}{2}\,[X_1(\omega+\omega_0) - X_1(\omega-\omega_0)]$

According to Eq. 3.51 p. 133 and Table 3.1 p. 141

- Multiplication by a Sinusoid



$$X(\omega) = \frac{2\pi}{4-\omega^2+4\pi^2+4j\omega} \quad if\ a = 2,\ \ \omega_0 \ = 2\pi$$

Page 124 Ex. 3.12

3.6 Properties of the Fourier Transform

- 3.6.1 Linearity
- 3.6.2 Left or Right Shift in Time
- 3.6.3 Time Scaling
- 3.6.4 Time Reversal
- 3.6.5 Multiplication by a Power of t
- 3.6.6 Multiplication by a Complex Exponential

3.6 Properties (cont.)

- 3.6.7 Multiplication by a Sinusoid
- 3.6.8 Differentiation in the Time Domain
- 3.6.9 Integration in the Time Domain
- 3.6.10 Convolution in the Time Domain
- 3.6.11 Multiplication in the Time Domain
- 3.6.12 Parseval's Theorem
- 3.6.13 Duality
- Table 3.1 List of the Properties

3.7 Generalized Fourier Transform

- First, consider the transform of $\delta(t)$:
  - $\delta(t) \leftrightarrow 1$
- Consider x(t) =1 and apply the duality property:
  - $x(t) = 1$ (all t) $\leftrightarrow$ $2\pi\delta(\omega)$ (Eq. 3.71)
- Using Eq. 3.71 and the modulation property , the generalized Fourier transform for a sinusoid is
  - $\cos(\omega_0 t) \leftrightarrow j\pi\,[\delta(\omega + \omega_0) + \delta(\omega + \omega_0)]$
  - $\sin(\omega_0 t) \leftrightarrow j\pi\,[\delta(\omega + \omega_0) - \delta(\omega + \omega_0)]$