Summary Fourier, Laplace CENG 4331

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Fourier Series Page 101

Assuming that $x(t)$ is defined on the interval $-T/2 \le t \le T/2$ and is periodic with period T seconds, the coefficients a_k and b_k of the Fourier Series of $x(t)$ can be computed by the formulas

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt, \\ a_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(\frac{2k\pi t}{T}\right)\,dt, \\ b_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin\left(\frac{2k\pi t}{T}\right)\,dt, \end{aligned} \tag{1}$$

where $k = 1, 2, \ldots$ is any positive integer and if T is in seconds,

$$\omega_0 = \frac{2\pi}{T} = 2\pi f_0 \qquad \text{radians/second.}$$

The Fourier series on the interval $[\,-T/2, T/2\,]$ is thus

$$f(t) = a_0 + \sum_{k=1}^{\infty}\left[a_k\cos\left(\frac{2k\pi t}{T}\right) + b_k\sin\left(\frac{2k\pi t}{T}\right)\right]. \tag{2}$$

In terms of the fundamental radian frequency w_0, the shifted cosine series can be written

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[A_k\cos(\omega_0 kt + \theta_k)], \tag{3}$$

in which the numerical value of A_k is the *amplitude* and the angle θ_k is the *phase* of the kth harmonic.

The complex series (Page 108) is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{i2\pi k f_0 t},$$

has coefficients c_n that are in general complex and c_k and c_{-k} are complex conjugates.

Complex Series The Fourier trigonometric series contains a series of sines and cosines and thus involves real functions. It is often convenient to write the series for a function $x(t)$ with period T as a sum of exponential functions in the form

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t}, \tag{4}$$

where $\omega_0 = 2\pi/T$ as before and the coefficients c_k are the complex Fourier coefficients.

By substituting the identities

$$\begin{aligned} \cos(n\omega_0 t) &= \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}, \\ \sin(n\omega_0 t) &= \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}, \end{aligned} \tag{5}$$

in the trigonometric form of the series (K and H Eq 3.4), the relationship between the trigonometric and exponential coefficients is found to be

$$\begin{aligned} c_0 &= a_0, \\ c_k &= \frac{a_k - ib_k}{2} \quad \text{for } k > 0 \ , \\ c_{-k} &= \frac{a_k + ib_k}{2}. \end{aligned} \tag{6}$$

Notice that c_{-k} is the complex conjugate of the term c_k. Thus, the series in Equation 4 becomes

$$x(t) = c_0 + \sum_{k=1}^{\infty} [c_k e^{ik\omega_0 t} + c_{-k} e^{-ik\omega_0 t}]. \tag{7}$$

Orthogonality To find the coefficients in Equation 4, each side is multiplied by $e^{-ik\omega_0 t}$ and integrated over the period to yield

$$\int_{-T/2}^{T/2} x(t) e^{-ik\omega_0 t}\, dt = \sum_{k=-\infty}^{\infty} \alpha_k \int_{-T/2}^{T/2} e^{i(n-m)\omega_0 t}\, dt. \tag{8}$$

Since the terms with different exponents are orthogonal, all terms but that for which $m = k$ are zero for the integral on the right-hand side. Thus,

$$\int_{-T/2}^{T/2} x(t) e^{-ik\omega_0 t}\, dt = \int_{-T/2}^{T/2} e^{-ik\omega_0 t} e^{ik\omega_0 t}\, dt = c_n T,$$

so that dividing both sides T yields the coefficients

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-ik\omega_0 t}\, dt. \tag{9}$$

Even Pulse Example Page 103 Example 3.2

Complex Series Square Wave Example

Consider an odd square wave and the complex Fourier coefficients

$$c_k = \frac{1}{T}\int_{-T/2}^{0} (-A)e^{-ik\omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2} (A)e^{-ik\omega_0 t}\,dt, \tag{10}$$

which leads to the series

$$f(t) = \frac{2A}{i\pi}\sum_{k=-\infty}^{\infty} \frac{e^{i(2k-1)\omega_0 t}}{(2k-1)}, \tag{11}$$

as defined in Equation 4.

This form contains complex coefficients, but the series can be written in terms of sine waves by combining the corresponding terms for positive and negative arguments. To determine the coefficients, the amount of difficulty is about the same for the trigonometric series and the complex series. However, the complex series perhaps has an advantage when the magnitude of the coefficients are of interest.

Each coefficient has the form

$$c_k = \frac{2A}{ik\pi} = \frac{2A}{k\pi}e^{-i\pi/2}, \qquad k = \pm 1, \pm 3, \ldots,$$

and the coefficients for even values, $k = 0, \pm 2, \ldots$, are zero. Notice that the coefficients decrease as the index k increases. The use of these coefficients to compute the *frequency spectrum* of $f(t)$ is considered later.

The trigonometric series is derived from the complex series by expanding the complex series of Equation 11 as

$$\begin{aligned} f(t) &= \sum_{k=-\infty}^{\infty} c_k e^{ik\omega_0 t} \\ &= \cdots - \frac{2A}{3\pi i}e^{-i3\omega_0 t} - \frac{2A}{\pi i}e^{-i\omega_0 t} + \frac{2A}{\pi i}e^{i\omega_0 t} + \frac{2A}{3\pi i}e^{i3\omega_0 t} + \cdots \end{aligned}$$

and recognizing the sum of negative and positive terms for each k as $2\sin(k\omega_0 t)$. The trigonometric series becomes

$$f(t) = \frac{4A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right) = \frac{4A}{\pi}\sum_{k=1}^{\infty} \frac{\sin[(2k-1)\omega_0 t]}{(2k-1)},$$

which is the result of Example 8.4.

Fourier Series Sine Wave of Period T and Power

Find the Fourier series of the function

$$f(t) = A \sin \omega t \qquad 0 \leq t \leq T, \qquad T = \frac{2\pi}{\omega} = \frac{2\pi}{2\pi f} = \frac{1}{f}$$

with the period T seconds. We observe the following since the function $\sin \omega t$ is an odd functions (K&H Page 101):

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-T/2}^{T/2} A \sin \omega t \, dt = 0, \\ a_n &= \frac{2}{T} \int_{-T/2}^{T/2} A \sin \omega t \cos \left(\frac{2n\pi t}{T} \right) \, dt = 0. \end{aligned} \tag{12}$$

Thus, the possible nonzero components are

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} A \sin \omega t \, \sin \left(\frac{2n\pi t}{T} \right) \, dt,$$

and this integral is nonzero only when

$$\omega = \frac{2n\pi}{T} = \frac{2n\pi\omega}{2\pi} = n\omega.$$

This requires that $n = 1$. The integral of $A(\sin \omega t)^2$ is $2A/T$ making $b_1 = A$ so the Fourier series for $f(t) = A \sin \omega t$ is actually

$$f(t) = A \sin \omega t.$$

Taking the standard ac wave with $f = 60$ Hz,$\omega \approx 377$ rad/sec, we find $T = 1/f \approx 16.67$ ms. If $A = 170$ volts is the amplitude of the sine wave, the power in the sine wave is proportional to the value

$$P = \frac{1}{T} \int_{-T/2}^{T/2} (A \sin \omega t)^2 \, dt = \frac{A^2}{2}.$$

A constant voltage signal with the same power is the rms (root mean square) value given by $\sqrt{P} = A/\sqrt{2}) = 170/\sqrt{2} = 120$ volts rms. Thus, the voltage supplying a common household appliance is about 120 volts rms or $2 \times 170 = 340$ volts peak to peak. See K&H Page 113- Parseval's Theorem, Power is the same value in the time or frequency domain.

Fourier Transform If the Fourier transform of $f(t)$ exists, it is defined as

$$\mathcal{F}[f(t)] = F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt. \tag{13}$$

The transform $F(i\omega)$ represents the *frequency spectrum* of $f(t)$, and it may be complex even though $f(t)$ is real. The magnitude $|F(i\omega)|$ is called the amplitude spectrum of $F(i\omega)$. See K&H Section 3.4.

Definition of DFT Assume that a function $f(t)$ is defined at a set of N points, $f(nT_s)$ for $n = 0, \ldots, N-1$ values. The DFT yields the frequency spectrum at N points by the formula

$$F_k = F\left(\frac{k}{NT_s}\right) = \sum_{n=0}^{N-1} f[nT_s]e^{-i2\pi nk/N} \tag{14}$$

for $k = 0, \ldots, N-1$. Thus, N sample points of the signal in time lead to N frequency components in the discrete spectrum spaced at intervals $f_s = 1/(NT_s)$.

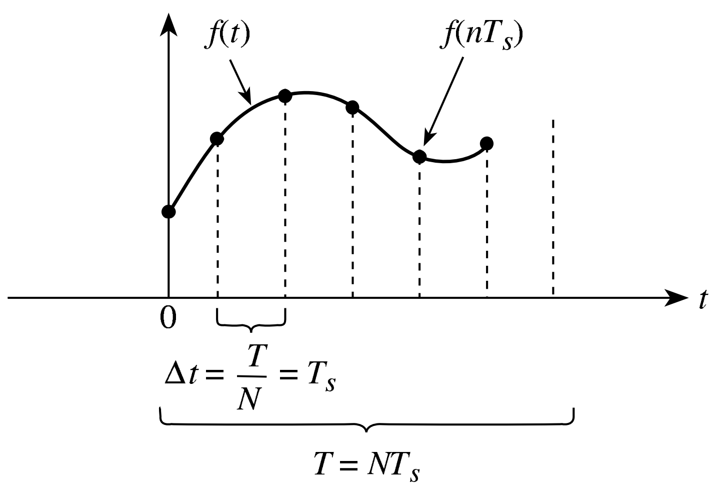

Figure 1: Sampled Signal

If the bandwidth of the sampled signals in radians/sec is $\omega_b = 2\pi f_b$, the sampling rate in samples/second should be

$$\omega_s = 2\pi f_s = 2\omega_b.$$

The DFT approximates the Fourier Transform when the DFT is multiplied by T_s if ωT_s is small. See K&H Section 5.4.

Laplace Transform Chapter 6

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt. \tag{15}$$

$f(t)$	$F(s)$	$F(i\omega)$
$\delta(t)$	1	1
$U(t)$	$\dfrac{1}{s}$	$\dfrac{1}{i\omega} + \pi\delta(\omega)$
$e^{-at}\,U(t)$	$\dfrac{1}{s+a}$	$\dfrac{1}{i\omega + a}$
$t\,e^{-at}\,U(t)$	$\dfrac{1}{(s+a)^2}$	$\dfrac{1}{(i\omega + a)^2}$

Table 1: Laplace and Fourier Transform Pairs

Use of Fourier Transform in Differential Equations *Assume that $f(t)$ is piecewise continuous and that $f(t)$ and its derivative $f'(t)$ have absolutely convergent integrals for all t. Then,*

$$\mathcal{F}[\frac{df}{dt}] = (i\omega)F(i\omega).$$

By extension, the nth *derivative $f^{(n)}(t)$ has the Fourier transform*

$$\mathcal{F}[f^{(n)}(t)] = (i\omega)^n F(i\omega)$$

.

Laplace Transforms of Derivatives The notation for the derivative of $f(t)$ with respect to t will be $\dot{f}$ in this section.

Suppose that $f(t)$ is continuous for all $t \geq 0$ and that $f(t)$ is of exponential order, and its derivative $\dot{f}(t)$ is piecewise continuous on every interval. Then, for a function $f(t)$ with Laplace transform $F(s)$

$$\mathcal{L}\left[\frac{df}{dt}\right] = \mathcal{L}\left[\dot{f}(t)\right] = sF(s) - f(0). \tag{16}$$

Applying this to the second derivative $d^2f/dt^2 \equiv \ddot{f}(t)$ yields

$$\begin{aligned} \mathcal{L}\left[\ddot{f}(t)\right] &= s\mathcal{L}[\dot{f}(t)] - \dot{f}(0) \\ &= s\{s\mathcal{L}[f(t)] - f(0)\} - \dot{f}(0) \\ &= s^2\mathcal{L}[f(t)] - sf(0) - \dot{f}(0). \end{aligned} \tag{17}$$

In general, the nth derivative $f^{(n)}(t)$ has the Laplace transform

$$\mathcal{L}[f^{(n)}(t)] = s^n\mathcal{L}[f(t)] - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - f^{(n-1)}(0) \tag{18}$$

See Examples in Section 6.4.

Z-transforms Chapter 7 The one-sided Z-transform is defined by the series

$$\mathcal{Z}[f[n]] = \sum_{n=0}^{\infty} f[n]z^{-n} = f[0] + \frac{f[1]}{z} + \frac{f[2]}{z^2} + \cdots. \qquad (19)$$

$f[n]$	$F(z)$
$\delta[n]$	1
$u[n]$	$\frac{z}{z-1}$
$a^n\ u[n]$	$\frac{z}{z-a} = \frac{1}{1-az^{-1}}$
$n\ u[n]$	$\frac{z}{(z-1)^2}$
$x[n-m]$	$z^{-m}\ X(z)$ if $x(n<0)=0$
$x[n-1]$	$z^{-1}\ X(z) + x[-1]$
$x[n+1]$	$z\ X(z) - x[0]z$
$x[n+2]$	$z^2\ X(z) - x[0]z^2 - x[1]z$

Table 2: Z-transforms One-Sided

Solution of Difference Equations Difference equations can be solved by recursion (Section 2.3) or by using z-transforms (Section7.4). They can also be solved by the *classical* method of finding the homogeneous and particular solutions. When inverting $F(z)$, it is usually better to use partial fraction expansion on

$$\frac{F(z)}{z}.$$