Z-transform Homework 6 CENG 5131 Due October 13

Review parts of Chapter 10 in the textbook.

A *geometric series* is a series with each term after the first being a fixed multiple of the preceding term. The multiplier is a real number r, called the *ratio*, so that $a_{n+1} = ra_n$. If the sum is taken from $n = 0$, the geometric series is represented as

$$\sum_{n=0}^{\infty} ar^n = a + ar + \cdots \quad (a \neq 0). \tag{1}$$

The series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \tag{2}$$

if $-1 < r < 1$ but diverges if $|r| > 1$.

The nth partial sum for the geometric series is found by subtracting the terms

$$\begin{aligned} S_n - rS_n &= a + ar + \cdots + ar^n - \left(ar + ar^2 + \cdots + ar^{n+1}\right) \\ &= a - ar^{n+1}, \end{aligned}$$

so that $S_n - rS_n = a(1 - r^{n+1})$. Thus, solving for S_n leads to the result

$$S_n = \frac{a(1 - r^{n+1})}{1-r}$$

for the sum of the first $n + 1$ terms. Taking the limit as n goes to infinity with $|r| < 1$ shows that the sum of the series is $a/(1 - r)$, as shown in Equation 2.

Power series are used extensively for computing or approximating values of functions. Suppose that a power series converges to the value of $f(x)$, so that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots. \tag{3}$$

Such a series is called a *power series representation*, or a *power series expansion*, of $f(x)$. Alternatively, the function is said to be *represented*, or *expressed*, by the series. Although many functions of interest can be represented by a power series, there are many functions that cannot. If $f(x)$ can be represented by a power series in an interval, the function is termed *analytic* in the interval. The Taylor series is an important example of a power series.

Mathematically, the Z-transform is a rule by which a sequence of numbers is transformed into a function of the complex variable $z = x + iy$, which can be written

$$z = \rho \exp(i2\pi F) = \rho \exp(i\Omega).$$

If $\{f(n)\}$ is a sequence, then we write

$$\mathcal{Z}[f(n)] = F(z) \quad \text{and} \quad f(n) = \mathcal{Z}^{-1}[F(z)]. \tag{4}$$

The Z-transform is defined by the series

$$\begin{aligned} \mathcal{Z}[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\ &= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots \qquad (5) \\ &= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots. \qquad (6) \end{aligned}$$

This transform is typically used to analyze sequences for which $f(n) = 0$ for $n < 0$ so this transform is called the *unilateral*, or *single-sided* Z transform since the sum begins at zero.

Do the problems by hand unless otherwise indicated and **check** the results whenever possible. However, you may wish to verify your results with MATLAB solutions to the problems when appropriate. You can use symbolic MATLAB to check results.

Problem 1 10 Points
Sum the following series

(a) $1 + \frac{1}{2} + \frac{1}{4} + \cdots$

(b) $\pi + \frac{\pi}{\sqrt{2}} + \cdots + \frac{\pi}{\sqrt{2^n}} + \cdots$

Problem 2 15 Points

(a) Find $x[n]$ by inverting the z-transform

$$X(x) = \frac{8z^3 + 2z^2 - 5z}{z^3 - 1.75z + .75}$$

(b) Check the result using MATLAB "iztrans".

(c) Use the command

x=filter(num,den,[1 zeros(1,9)])

to print the first 10 values of $x[n]$. This is the impulse response.

Problem 3 15 Points
Find the first 5 values of $e(k)$ from the inverse z-transform by long division.

$$E(z) = \frac{z}{z^2 - 3z + 2}.$$

Check your answer with MATLAB.

Problem 4 30 Points
Solve the following difference equation using z-transforms:

$$y[n] - 0.5y[n-1] = 4u[n]; \qquad y[-1] = 0;$$

(a) Use the z-transform method to determine $y[n]$. That is, take the z-transform of the equation and invert it to get $y[n]$.

(b) Compute 11 terms of $y[n]$ by programming in MATLAB directly and plot with stem.

(c) Determine 11 terms of $y[n]$ using the "filter" command. In this case, you do not have to use any initial conditions. Plot and compare to the result in Part b.

Problem 5 30 Points
Sinusoidal response. Consider the transfer function

$$H(z) = \frac{0.1z}{z - 0.9}.$$

Find $H(z)$ for the following cases:

(a) $z = e^{i0.01}$.

(b) $z = e^{i3}$.

(c) Using MATLAB, plot the digital frequency response

$$H(z), \quad z = e^{i\Omega} \text{ for } -\pi \leq \Omega \leq \pi$$

using increments of $2\pi/300$ in the plot.

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The MATLAB Symbolic Toolbox can be a great help.
First, get help  >> help symbolic or check out the demonstrations.

 sym     - Create symbolic object.
 syms    - Construct several symbolic objects (= sym for multiple variables).

 simplify    - Simplify.
    expand      - Expand.
    factor      - Factor.
    collect     - Collect.
    simple      - Search for shortest form.
    pretty(S)   - prints the symbolic expression S in a "nice" format.


 Integral Transforms.
    fourier     - Fourier transform.
    laplace     - Laplace transform.
    ztrans      - Z transform.
    ifourier    - Inverse Fourier transform.
    ilaplace    - Inverse Laplace transform.
    iztrans     - Inverse Z transform.

 dirac       - Delta function.
 heaviside   - Step function.
 ezplot      - Easy to use function and curve plotter.

For numerical results,
residue Partial-fraction expansion (residues).
    [R,P,K] = residue(B,A) finds the residues, poles and direct term of
    a partial fraction expansion of the ratio of two polynomials B(s)/A(s).
    If there are no multiple roots,
       B(s)       R(1)       R(2)             R(n)
       ----  =  -------- + -------- + ... + -------- + K(s)
       A(s)     s - P(1)   s - P(2)         s - P(n)
    Vectors B and A specify the coefficients of the numerator and
    denominator polynomials in descending powers of s.  The residues
    are returned in the column vector R, the pole locations in column
    vector P, and the direct terms in row vector K.  The number of
    poles is n = length(A)-1 = length(R) = length(P). The direct term
    coefficient vector is empty if length(B) < length(A), otherwise
    length(K) = length(B)-length(A)+1.
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