10   *DISCRETE SYSTEMS*

PREVIEW_____

This chapter presents various techniques useful to analyze sequences, discrete signals, and discrete systems for which the variables involved are defined only for discrete values of the independent variable. First in this chapter we introduce discrete sequences. In systems described as *discrete systems*, the inputs, outputs, and system characteristic are characterized by such discrete data. As we shall see, MATLAB is well-suited to solve problems involving discrete systems.

Difference equations are introduced in this chapter. We will demonstrate solution methods parallel to those used for continuous differential equations as presented in Chapter 5. We also apply difference equation techniques to the approximation of differential equations.

An important area of study for discrete techniques is the digital signal processing of signals. The chapter presents smoothing and digital filtering as two example applications.

Discrete signals can also be analyzed by the *Z-transform*, which may be regarded as a discrete version of the Laplace transform. After the *Z*-transform and MATLAB commands for discrete systems are introduced, the transform is applied to the solution of difference equations. Another section compares the Laplace and *Z*-transforms.

Discrete systems can be analyzed as to their characteristics and stability by difference equations or by Z-transform techniques. Various examples are presented in the chapter.

10.1 INTRODUCTION TO THE SEQUENCES AND DISCRETE FUNCTIONS

The functions of interest in this chapter can be represented by *sequences*. In these sequences, for each integer n in a set of consecutive integers, there is assigned a number $s(n)$. These numbers can be arranged in order according to the values of n. For example, when the domain is the set of all positive integers, the numbers form an *infinite sequence*

$$s(1), s(2), s(3), \ldots, s(n), s(n+1), \ldots.$$

In general for sequences, n is called the *index*, which can be a positive or negative integer.

A table or an equation could serve to define the correspondence that associates each integer n with the value $s(n)$. Also, the subscript notation s_n is often used to designate a value rather than the usual function notation $s(n)$.

Sequences are often used to describe naturally occurring values such as the Dow-Jones averages of the prices of stocks. A sequence

$$p(1), p(2), \cdots$$

could be used to designate the prices. In this sequence, if $p(1)$ represents the average on the first day of interest, $p(2)$ could represent the value on the second day. Thus, $p(n)$ is used in this sequence to denote the price on the nth day. In this case, there is no known relationship between the terms despite many efforts to derive a mathematical description.

In some cases, a general rule of formation for the sequence is given. Thus, the rule

$$F_n = 2n, \quad n = 1, 2, \ldots$$

designates the even integers. Another example leading to a formula relating the elements of a sequence is given by the compound interest problem as Example 10.1 shows.

☐ **EXAMPLE 10.1**   *Compound Interest Problem*

Suppose money is deposited in a savings account that pays interest at the rate p percent, paid at regular intervals of time. For example, let \$1000.00 be deposited with the interest rate 6% a year and the interest compounded every year. The value when the first interest payment is made after a year will be \$1000 + 0.06 × \$1000 =\$1060.

In a more general case, consider the compound interest equation

$$y(nT) = y(nT - T) + \frac{p}{100}\, y(nT - T) + x(nT) = \left(1 + \frac{p}{100}\right)\, y(nT - T) + x(nT),$$
(10.1)

where $y(nT)$ represents the amount of money in an account at time $t = nT$, $y(nT - T)$ is the money in the account at the time of the previous computation, p is the percent interest paid in the interval of time T, and $x(nT)$ is the amount of money deposited or withdrawn at $t = nT$.

Designating the discrete values as $y(n)$, we define

$$y(n) = y(nT), \quad n = 0, 1, 2, \ldots$$

to form a sequence of discrete values of the bank balance. If there are no extra deposits or withdrawals, $x(nT) = 0$ and Equation 10.1 can be written as

$$y(n) = ay(n - 1),$$
(10.2)

where $a = (1 + p/100)$.

As an numerical example, assume that $x(nT)$ is zero and $p = 6\%$ a year with an initial deposit of $y(0)$ dollars. Applying Equation 10.2 repeatedly yields the equations

$$
\begin{aligned}
y(1) &= (1.06)\, y(0) \\
y(2) &= (1.06)\, y(1) = (1.06)^2\, y(0) \\
&\;\;\vdots \\
y(n) &= (1.06)\, y(n - 1) = (1.06)^n\, y(0).
\end{aligned}
$$

It appears that the solution to the equation $y(n) = a\, y(n - 1)$ is

$$y(n) = a^n\, y(0).$$
(10.3)

The accompanying MATLAB script performs the numerical calculations for Equation 10.3 each year for the period of five years with an initial deposit of \$1000. The result shows that the account contains \$1338.20 at the end of five years.

∎

MATLAB Script ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

```
Example 10.1
% EX10_1.M MATLAB solution of the compound interest equation
%    y(nT)=y(nT-T) + (p/100)*y(nT-T)
%     for p = 6 percent and initial deposit y0 = $ 1000
%   y(n) represents the balance after the nth year
clear
```

```
format bank      % Show results as currency
a=1.06;          % Calculation for 5 years at 6% interest
y0 = 1000;       % Initial deposit
for n=1:5
 y(n) = a^(n)*y0
end
y = 1060.00    1123.60    1191.02    1262.48    1338.23
```

WHAT IF? In Example 10.1, we compute that \$1338.23 is in the
account after 5 years if the initial deposit is \$1000. Suppose the interest
is compounded continuously so that the compound interest equation is

$$\frac{dy(t)}{dt} = .06y(t).$$

Compute the new amount after 5 years.

We shall see that the compound interest Equation 10.1 represents a
difference equation that can be analyzed using techniques similar to those
used to solve differential equations. We introduce the techniques in the
next section.

10.2 LINEAR DIFFERENCE EQUATIONS

A *difference equation* expresses a relationship between the terms of a
sequence. In this section, *linear difference equations with constant co-
efficients* are considered. A simple example is the compound interest
Equation 10.1 written in its discrete form

$$y(n) - \left(1 + \frac{p}{100}\right) y(n-1) = x(n).$$

In this equation, x would be termed the input and y the output or solution
using definitions borrowed from the study of differential equations. The
order of the difference equation is the difference between the largest and
smallest indices of y appearing in the equation. In this case, $n-(n-1) = 1$,
so the equation is called first order. When the input x is zero as in
Example 10.1, the equation is called *homogeneous*. Given one initial
value, say $y(n_0)$ in a first-order equation, a numerical value of $y(n)$ could
be found for each $n \geq n_0$.

A general form for an Nth-order linear difference equation with con-
stant coefficients is

$$\begin{aligned}
a_0\, y(n) \quad & + \quad a_1\, y(n-1) + \cdots + a_{N-1}\, y(n-N+1) + a_N\, y(n-N) \\
& = \quad b_0\, x(n) + b_1\, x(n-1) + \cdots + b_M\, x(n-M),
\end{aligned}$$

$$(10.4)$$

where $a_0, \ldots, a_N$ are constant with a_0 and a_N nonzero. The coefficients $b_0, \ldots, b_M$ are also constant with b_0 and b_M nonzero in the general case.

In summation notation, Equation 10.4 becomes

$$\sum_{i=0}^{N} a_i y(n - i) = \sum_{i=0}^{M} b_i x(n - i). \tag{10.5}$$

The coefficients will be assumed to be real. If the b coefficients are nonzero, the x terms are considered the input terms for the current instant n and $M - 1$ previous instants, while the y terms are output values. Notice that when $N > 0$, the current output $y(n)$ depends not only on the inputs but on previous values of the outputs if the a coefficients are nonzero. When this is true, such equations are called *recursive*.

A *solution* to the difference equation is any sequence $y(n)$ for which Equation 10.5 is satisfied for $n = 0, 1, 2, \ldots$. As with differential equations, the solution to Equation 10.5 will consist of a complementary solution and a particular solution. The complementary solution to the homogeneous equation can be formed from the linear combination of N linearly independent solutions of the homogeneous equation which contains N arbitrary constants. If N initial values of y are given such as for $y(0), y(1), \ldots y(N - 1)$, the problem is called the *initial value problem* and the values of the constants can be determined from the initial conditions. To complete the solution, it is necessary to find a particular solution to the inhomogeneous equation. The solution to the initial value problem is unique.

Since Equation 10.5 is linear, the general solution to the difference equation can be expressed as the sum

$$y(n) = y_c(n) + y_p(n) \tag{10.6}$$

where $y_c(n)$ is called the *complementary solution* and $y_p(n)$ is a *particular solution*.

THE COMPLE-
MENTARY
SOLUTION

The solution to the homogeneous equation

$$a_0\, y(n) + a_1\, y(n-1) + \cdots + a_{N-1}\, y(n-N+1) + a_N\, y(n-N) = 0 \tag{10.7}$$

is called the *complementary* or *homogeneous* solution which we will designate as $y(n) = y_c(n)$ in the present discussion.

It can be shown that a solution is of the form

$$y_c(n) = Cr^n \tag{10.8}$$

where C and r are to be determined. Computing each term in Equation 10.7 yields the terms

$$y_c(n) \quad = \quad Cr^n = Cr^{-N}r^N r^n$$

$$y_c(n-1) \quad = \quad Cr^{n-1} = Cr^{-1} r^n = Cr^{-N}r^{N-1}r^n$$

$$\vdots$$

$$y_c(n-N) \quad = \quad Cr^{n-N} = Cr^{-N} r^n \qquad \text{(10.9)}$$

Substituting the terms in Equation 10.7 and taking out the term r^{-N} yields

$$\left(a_0 + a_1 r^{-1} + \cdots + a_{N-1} r^{-(N-1)} + a_N r^{-N}\right) Cr^n$$
$$= \left(a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-1} r + a_N\right)Cr^{-N} r^n = 0.$$

If the solution is nontrivial $(C \neq 0)$, there results the polynomial equation

$$a_0 r^N + a_1 r^{N-1} + \cdots + a_{N-1} r + a_N = 0. \qquad \text{(10.10)}$$

Describing a discrete system, this equation is called the *characteristic equation* for the system.

Distinct Roots   If the roots $r_i, i = 1, 2, \ldots N$ are distinct, the linearity of the homogeneous Equation 10.7 yields a solution in the form

$$y_c(n) = C_1 r_1^n + C_2 r_2^n + \cdots + C_N r_N^n. \qquad \text{(10.11)}$$

The N unknowns $C_1, C_2, \ldots, C_N$ are determined by the general solution and the initial conditions. Typically, one value such as $y(0)$ would be specified for a first-order equation, $y(0)$ and $y(1)$ for a second-order system, and $y(0), y(1), \ldots y(N-1)$ for an Nth-order system.

☐ **EXAMPLE 10.2**   *Homogeneous Difference Equations*

    a. Consider the first-order difference equation

$$y(n) + y(n-1) = 0$$

with the characteristic equation $r + 1 = 0$. Then, the solution given by Equation 10.8 is $y(n) = C_1(-1)^n$. It is easily verified in this case, since substituting the solution in the equation yields

$$C_1(-1)^n + C_1(-1)^{n-1} = -C_1(-1)^{n-1} + C_1(-1)^{n-1} = 0.$$

The arbitrary constant would be determined if some value of $y(n)$ were known. For example, $C_1 = y(0)$ if $y(0)$ is known.

    b. The second-order equation

$$y(n) - 5y(n-1) + 6y(n-2) = 0$$

has the characteristic equation $r^2 - 5r + 6 = (r-3)(r-2) = 0$ so that

$$y(n) = C_1(2)^n + C_2(3)^n.$$

The constants C_1 and C_2 can be determined from two values of y in the solution set.

The roots of the characteristic equation can be found using the MAT-LAB command **roots**. For the last example, the MATLAB statements

```
>>p = [1 -5 6]
>> r = roots(p)
```

yield the roots.

□

Repeated Roots   If the characteristic Equation 10.10 has roots that are repeated, the form of the solutions to the difference equation must be modified. For each real root r of multiplicity m, the solution can be shown to have the terms $r^n, nr^n, \ldots, n^{m-1} r^n$. This is analogous to the solutions $te^{\lambda t}, t^2 e^{\lambda t}, \ldots$ that arise from the homogeneous solution of differential equations when the characteristic equation has repeated roots.

□ EXAMPLE 10.3   *Homogeneous Difference Equation with Repeated Roots*
Consider the difference equation

$$y(n) + 2y(n-1) + y(n-2) = 0$$

which has the characteristic equation $r^2 + 2r + 1 = (r+1)^2 = 0$ so that $r_1 = r_2 = -1$ is a repeated root. The solution is

$$y(n) = C_1(-1)^n + C_2 n(-1)^n = (C_1 + C_2 n)(-1)^n.$$

In this case, $C_1 = y(0)$ and $C_2 = -C_1 - y(1)$.

□

Second-order Difference Equation   Consider the difference equation

$$y(n) - 2ay(n-1) + y(n-2) = 0 \qquad \textbf{(10.12)}$$

with a a real coefficient. This second-order homogeneous equation has the characteristic equation

$$r^2 - 2ar + 1 = 0$$

which has the roots

$$
\begin{aligned}
r_1 &= a + \sqrt{a^2 - 1} \\
r_2 &= a - \sqrt{a^2 - 1}.
\end{aligned}
\qquad \textbf{(10.13)}
$$

Based on the value of a, we consider the cases:

1. If $|a| > 1$, the roots are real and distinct so that

$$y(n) = C_1(r_1)^n + C_2(r_2)^n. \qquad \textbf{(10.14)}$$

2. If $|a| < 1$, the roots are complex and distinct so

$$
\begin{aligned}
y(n) &= C_1(r_1)^n + C_2(r_2)^n \\
&= C_1(a + i\sqrt{1 - a^2})^n + C_2(a - i\sqrt{1 - a^2})^n. \quad \textbf{(10.15)}
\end{aligned}
$$

This result can be simplified by noting that a complex number can be expressed as $\rho e^{i\theta}$, so we can write

$$
r_1 = \rho_1 e^{i\theta_1}, \quad r_2 = \rho_2 e^{i\theta_2}
$$

where

$$
\rho_1 = \rho_2 = \sqrt{a^2 + (1 - a^2)} = 1, \quad \theta_1 = \tan^{-1} \frac{\sqrt{1 - a^2}}{a} = -\theta_2.
$$

Also, applying Euler's formula as described in Chapter 2

$$
(\rho e^{i\theta})^n = \rho^n (\cos n\theta + i \sin n\theta)
$$

with $\rho = 1$, leads to the result

$$
\begin{aligned}
y(n) &= C_1 \cos n\theta + iC_1 \sin n\theta + C_2 \cos n\theta - iC_2 \sin n\theta \\
&= C_3 \cos n\theta + C_4 \sin n\theta. \quad \textbf{(10.16)}
\end{aligned}
$$

3. If $a = 1$, the roots are real and equal to 1, so that the solution is

$$
y(n) = C_1 + C_2 n. \quad \textbf{(10.17)}
$$

4. If $a = -1$, the roots are real and equal to -1, so that the solution is

$$
y(n) = (C_1 + C_2 n)(-1)^n. \quad \textbf{(10.18)}
$$

□ EXAMPLE 10.4   *Rabbits and Fibonacci*

Suppose one pair of newborn rabbits is left on an island to reproduce and populate the island with their offspring and descendants. Further suppose that a newborn pair produces offspring at 2 months of age but a pair of rabbits is born each month to each adult pair. How will the rabbit population grow each month?

Let $y(n)$ denote the number of pairs of rabbits at the *beginning* of the nth month with $y(0) = 0$ and $y(1) = 1$ pair. We could count the pairs to determine the sequence

$$
y(0) = 0, \; y(1) = 1, \; y(2) = 1, \; y(3) = 2, \; y(4) = 3, \; y(5) = 5, \ldots
$$

As a difference equation, the number at the beginning of a month $y(n)$ is the total number from the last month $y(n - 1)$ (no rabbits pass away), plus the number of newborns. Each pair that is two months or older produces a new

pair so the number of fertile pairs in month n is $y(n-2)$. Thus, we find that $y(n)$ is defined by the difference equation

$$y(n) = y(n-1) + y(n-2), \quad n = 2, 3, 4, \ldots \qquad \textbf{(10.19)}$$

with $y(0) = 0$, $y(1) = 1$. The characteristic equation is $r^2 - r - 1 = 0$ with the roots

$$r_1 = \frac{1 + \sqrt{5}}{2}, \quad r_2 = \frac{1 - \sqrt{5}}{2}.$$

The solution of Equation 10.19 is

$$y(n) = C_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + C_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n \qquad \textbf{(10.20)}$$

with the constants determined by the conditions $y(0) = 0$ and $y(1) = 1$. You are asked to show in Problem 10.3 that the solution becomes

$$y(n) = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n. \qquad \textbf{(10.21)}$$

The *Fibonacci sequence* introduced in Chapter 6 can be described by the difference equation derived for the rabbits. Every Fibonacci number after the first two is the sum of its two predecessors.

$\square$

Summary of Homogeneous Solutions Table 10.1 summarizes the solutions to a linear homogeneous difference equation with constant coefficients based on the form of the roots of the characteristic equation. For combinations, use superposition to create the solution.

TABLE 10.1 *Homogeneous solutions*

Root of characteristic equation	*Choice for $y_c(n)$*
Real and distinct r_i	$C r_i^n$
Complex conjugate $a \pm ib = \rho e^{\pm i\theta}$	$[C_1 \cos(n\theta) + C_2 \sin(n\theta)]\rho^n$
Real repeated root r^m	$(K_0 + K_1 n + \cdots + K_m n^m) r^n$
Complex, repeated $[\rho e^{i\theta}]^m$	$(C_0 + C_1 n + \cdots + C_m n^m)\cos(n\theta) r^n$
	$+ (D_0 + D_1 n + \cdots + D_m n^m)\sin(n\theta) r^n$

We now turn to the task of finding a particular solution to the difference equation

$$a_0\, y(n) + a_1\, y(n-1) + \cdots + a_{N-1}\, y(n-N-1) + a_N\, y(n-N) = x(n).$$
(10.22)

This is a special case of Equation 10.5 with $b_0 = 1$ and $M = 0$. As with differential equations, a particular solution sometimes can be determined by the method of undetermined coefficients if the form of the forcing function $x(n)$ is simple as shown in Table 10.2.

TABLE 10.2 *Undetermined coefficient solutions*

$x(n)$	*Choice for* $y_p(n)$
C	C_1 (C and C_1 constant)
n^q	$K_q n^q + K_{q-1} n^{q-1} + \cdots + K_1 n + K_0$
a^n	$C a^n$ if a is not a root of the characteristic equation.
	$(C_1 + C_2 n)a^n$ if a is a distinct root of the characteristic equation.
	$(C_1 + C_2 n + \cdots + C_m n^m)a^n$ if a is a multiple root of the characteristic equation repeated m times.

☐ **EXAMPLE 10.5**   *General Solution to a Difference Equation*

The difference equation

$$y(n) + y(n-1) = n - 1$$

has the characteristic equation $r + 1 = 0$ so that the homogeneous solution is

$$y_c(n) = C(-1)^n.$$

A particular solution of the form $y_p(n) = K_1 n + K_0$ is assumed for the input $x(n) = n - 1$ as found in Table 10.2 by combining the solutions for the input function n^q with $q = 1$ and a constant input. Substituting the choice for $y_p(n)$ yields

$$K_1(n) + K_0 + K_1(n-1) + K_0 = n - 1.$$

substituting the assumed particular solution for two points, say $n = 0$ and $n = 1$, yields

$$
\begin{aligned}
2K_0 - K_1 &= -1 \\
2K_0 + K_1 &= 0
\end{aligned}
$$

and by solving this system we find $K_0 = -1/4$ and $K_1 = -2K_0 = 1/2$. Thus, The general solution is

$$y(n) = C(-1)^n + \frac{2n - 1}{4}$$

as you should check by substituting the solution into the difference equation.

□

General Solutions In a manner similar to that for differential equations, the general solution for the linear nonhomogeneous difference equation can be found by first finding the general solution of the associated homogeneous equation and then finding a particular solution of the nonhomogeneous equation. The complete solution is the sum of the two solutions. In addition to the techniques discussed already in this chapter, there are also approaches to solving the difference equation analogous to the variation of parameters method described in Chapter 5. O'Neil's textbook and other references listed in the Annotated Bibliography for this chapter presents the method. However, in this textbook, we will use Z-transform techniques to find the general solutions as shown in a later section.

10.3 APPROXIMATION TO DIFFERENTIAL EQUATIONS

It is often convenient to make an association between continuous quantities and discrete quantities. In particular, a linear differential equation with constant coefficients can be discretized, resulting in a difference equation that can be solved to approximate the solution of the differential equation.

One method of associating a differential equation with a difference equation approximates the derivatives with differences. For example, letting $\hat{y}(t)$ be a continuous function, the first derivative can be approximated as

$$\left.\frac{d\hat{y}(t)}{dt}\right|_{t=nT} \approx \frac{\hat{y}(nT+T) - \hat{y}(nT)}{T} \tag{10.23}$$

as we have seen previously in Chapter 6 using the Euler approximation. The second derivative can be approximated as

$$\left.\frac{d^2\hat{y}(t)}{dt^2}\right|_{t=nT} \approx \frac{\dfrac{d\hat{y}(nT+T)}{dt} - \dfrac{d\hat{y}(nT)}{dt}}{T}$$

$$= \frac{\hat{y}(nT+2T) - 2\hat{y}(nT+T) + \hat{y}(nT)}{T^2}. \tag{10.24}$$

From these approximations, a difference equation can be formed from a given differential equation.

First Order Equations

Consider the first-order differential equation

$$\frac{d\hat{y}(t)}{dt} + a\hat{y}(t) = b\hat{x}(t) \tag{10.25}$$

where a and b are constants with initial condition $\hat{y}(0)$ given. Using the approximation of Equation 10.23, this differential equation can be associated with the difference equation

$$\frac{\hat{y}(nT + T) - \hat{y}(nT)}{T} + a\hat{y}(nT) = b\hat{x}(nT).$$

Multiplying by T and replacing n by $n - 1$ yields the difference equation

$$y(n) - (1 - aT)y(n - 1) = bTx(n - 1). \tag{10.26}$$

Since the characteristic equation is $r - (1 - aT) = 0$, the homogeneous solution to the difference equation is

$$y(n) = (1 - aT)^n y(0). \tag{10.27}$$

Note that the homogeneous solution to the original differential equation is

$$\hat{y}(t) = e^{-at}\hat{y}(0), \qquad t \geq 0. \tag{10.28}$$

Problem 10.6 asks you to compare the difference equation solution with this continuous solution.

□

10.4 SMOOTHING AND DIGITAL FILTERS

As an example of the use of difference equations, we will consider the *filtering* of a signal that is sampled to form a discrete signal. In this case, filtering is the process of modifying the frequency characteristics of a signal. The primary objective is to remove unwanted signal energy (noise) and enhance the signal information of interest. The filtered signal is the output of the filter which is generally a *smoothed* version of the input in time when noise removal is the objective of filtering.

If the signals are continuous in time, the input $\hat{x}(t)$ and output $\hat{y}(t)$ can be related by a differential equation representing the filter as an analog system. When the filtering is done by a discrete system, the continuous signals must be converted to a discrete format. This is typically accomplished by *sampling* the input signal at discrete instants of time to yield a *discrete signal*. The discrete system is often called a *digital filter* in such applications.

In this section, a *digital filter* is represented by a difference equation and the input is the signal to be filtered. The output signal is the solution to the difference equation. Since the independent variable is most often time, the signals are called discrete-time signals.

DISCRETE TIME SIGNALS

A *discrete-time signal* is a signal defined only at discrete points of time. The signal is defined by a sequence that defines the amplitude at the discrete points in time. Although some signals are inherently discrete, most discrete-time signals are obtained from continuous-time signals by *sampling*. The continuous signal $\hat{f}(t)$, for example, can be sampled at the times $0, T_s, 2T_s, \ldots$ to produce the *discrete time* signal

$$\hat{f}(nT_s), \qquad n = 0, 1, \ldots$$

for which there are various notations such as

$$\hat{f}(nT_s) = \left. \hat{f}(t) \right|_{t=nT_s} = f(n) = f_n$$

to denote the amplitude of the sample at the time $t = nT_s$.[1]

For computer applications, the amplitude will also be a discrete value that can only assume certain defined values, due to the character of analog-to-digital converters and the necessity to store the sampled values in a memory with a limited number of bits. Discrete-time signals with discrete amplitude values are called *digital signals*.

LOWPASS FILTERS

We begin by introducing a particular type of filter called a *lowpass* or *smoothing* filter. This filter is used to remove noise from a signal by limiting the output variations of the filtered signal, and thus produce an output that is smoother than the input signal. After giving an example of a smoothing filter, we present the frequency response of filters described by difference equations.

□ **EXAMPLE 10.7** **First-Order Smoothing Filter**

Consider the discrete-time signal represented by the sequence

$$x(0), x(1), x(2), \cdots,$$

which we assume is a signal corrupted by random noise. One method of reducing the unwanted fluctuations is to compute a smoothed version of the signal according to the rule

$$y(n) = a\, y(n - 1) + (1 - a)\, x(n) \qquad \textbf{(10.29)}$$

where $0 < a < 1$ and $y(-1) = 0$.[2] At each value n, the output $y(n)$ is formed as a weighted average of the new input $x(n)$ and the output $y(n - 1)$ at the preceding time instant $n - 1$. If the constant a is almost zero, the output is almost equal to the input. The closer a is to 1, the more the preceding output is weighted and the "smoother" is the output.

---

[1] Some authors, particularly those describing signals and linear systems, use the notation $f[n]$ with square brackets to designate discrete values of $\hat{f}(t)$.

[2] It is assumed that the output $y(n)$ cannot begin before the input $x(0)$. Systems that do not respond before being stimulated are called causal.

For a given input sequence $x(n)$, the output $y(n)$ is easily calculated using MATLAB. In the accompanying script, the input signal is a sampled sine wave corrupted by random noise. The command **rand** generates random values from 0 to 1 in amplitude. Here the values are scaled to values between -0.05 and $+0.05$.

Since MATLAB vectors are indexed from 1 to N, the first values of x and y are x(1) and y(1) respectively in the script. Figure 10.1 shows the input signal and the smoothed result for the value of a defined in the title. Notice the change in amplitude and the shift of the peak value as well as the smoothness of the filtered signal as compared to the unfiltered version.

MATLAB Script ⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯⎯

```
Example 10.7
% SMOOTHER.M A smoothing filter defined as
%   y(n) = a*y(n-1) +(1-a)*x(n) , y(-1)=0
%   x(n) is input signal, y(n) is smoothed output
%
% Test signal is sin(w*t) with random noise
%  INPUT:  Weighing factor a
%  OUTPUT: Plot of x and y
%
clear, clf
w=2*pi/5;
t = linspace(0,10,100);     % Time steps
s = sin(w*t);               % Noiseless signal
% Add random noise
len=size(t);
na = 0.1;                   % Noise amplitude
noise = na*(rand(len)-.5);  % (-.05 to +.05)
x = s + noise;
%
% Weighing factor
a = input('Weighing factor a= ')
%
y(1)=(1-a)*x(1);
for I=2:100
 y(I) = a*y(I-1) + (1-a)*x(I);  % Digital Filter
end
plot(t,x,'--',t,y,'-')
xlabel('Time'), ylabel('Signals')
title(['Effect of Smoothing Filter, a = ', num2str(a)])
legend('Input x','Output y')
```
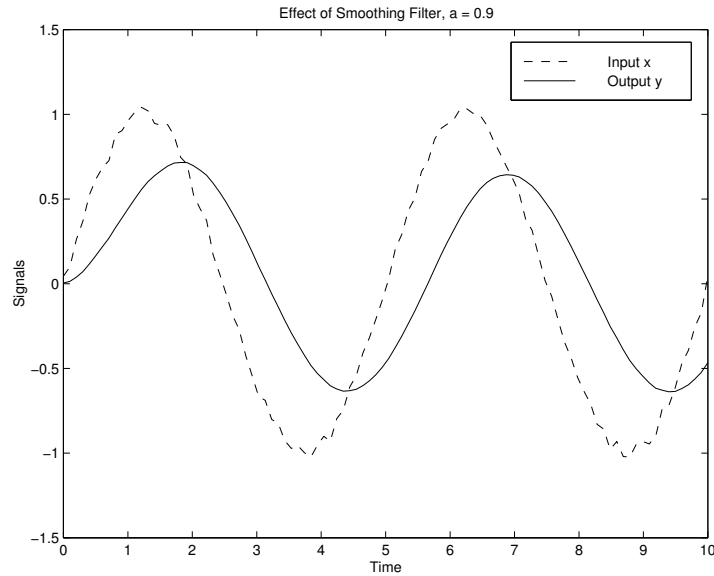
FIGURE 10.1 *Signal and Smoothed Version*

In terms of the frequency spectrum of the input signal, the difference equation describes a *lowpass filter* when a is nonzero since higher frequencies are removed from the input signal. The frequency characteristics of the smoothing filter will be presented in Example 10.8.

□

FREQUENCY
RESPONSE OF
DIGITAL
FILTERS

As explained in Chapter 8, for continuous-time systems that are linear and time invariant, the frequency response is determined by assuming a sinusoidal signal as the input to the system. The steady state output is also a sinusoid of the same frequency, perhaps with an amplitude scaling and a phase shift compared to the input signal. In the differential equation model for the system with zero initial conditions, the ratio of the output sinusoidal signal to the input sinusoidal signal is called the *transfer function* of the system. The transfer function computed or plotted for a range of input frequencies represents the frequency response of the system.

In an analogous way, the frequency response of a linear, discrete-time system that is time-independent is determined by using a complex sampled sinusoid

$$x(n) = \hat{x}(nT_s) = Ae^{i\omega nT_s}$$

as the input. The value A is a real constant and T_s is the time between samples of the input signal. The ratio of the output $y(n)$ to $x(n)$ is the

transfer function for the discrete system. By varying the input frequency, the frequency response of the discrete system is determined.

Often the linear frequency variable (hertz) is normalized to yield the *digital frequency*

$$F = \frac{f}{f_s} = fT_s. \tag{10.30}$$

Also, $\Omega = 2\pi F = \omega T_s$ is the digital radian frequency. The analog frequency $f = 1/T_s$ corresponds to the digital frequency $F = 1$ or $\Omega = 2\pi$.

Difference Equations   To determine the frequency response of a system described by the difference equation

$$y(n) + \sum_{m=1}^{N} a_m\, y(n - m) = \sum_{m=0}^{M} b_m\, x(n - m), \tag{10.31}$$

we apply a unit sinusoid

$$x(n) = e^{i\Omega n}$$

as the discrete input and assume the solution to the difference equation has the form

$$y(n) = He^{i\Omega n}.$$

Thus, H is the transfer function for the system.

Substituting the assumed response into Equation 10.31 yields

$$He^{i\Omega n} + \sum_{m=1}^{N} a_m He^{i\Omega(n-m)} = \sum_{m=0}^{M} b_m e^{i\Omega(n-m)}.$$

Dividing out $e^{i\Omega n}$ and solving for H shows that the frequency response is

$$H = \frac{\displaystyle\sum_{m=0}^{M} b_m e^{-i\Omega m}}{\displaystyle 1 + \sum_{m=1}^{N} a_m e^{-i\Omega m}}. \tag{10.32}$$

Although H is a complex constant for each digital frequency, H will vary as the input frequency changes and it is customary to write the transfer function as $H(e^{i\Omega})$ to emphasize the dependence on $e^{i\Omega}$ in Equation 10.32. Notice that the discrete-time system frequency response is periodic with the sampling frequency

$$\omega_s = \frac{2\pi}{T_s}$$

since $H(e^{i(\omega + k\omega_s)T_s}) = H(e^{i\omega T_s})$ where k is an integer.

Chapter 10 ■ DISCRETE SYSTEMS

□ **EXAMPLE 10.8**   ***Frequency Response of First-Order System***
     The frequency response of a first-order discrete system will be investigated. Consider the smoothing filter of Example 10.7

$$y(n) = a\,y(n-1) + (1-a)\,x(n)$$

where $0 < a < 1$. Using Equation 10.32 with $a_1 = -a$ and $b_0 = (1-a)$,

$$
\begin{aligned}
H &= \frac{1-a}{1 - ae^{-i\Omega}} = \frac{1-a}{1 - ae^{-i2\pi F}} \\
&= \frac{1-a}{1 - a\cos 2\pi F + ia\sin 2\pi F}
\end{aligned}
\tag{10.33}
$$

since $\Omega = 2\pi F$. Writing H in polar form $H(e^{2\pi iF}) = A(F)e^{i\phi(F)}$ yields the amplitude and phase response of the system as

$$
\begin{aligned}
A(F) &= \frac{1-a}{\sqrt{1 + a^2 - 2a\cos 2\pi F}} \\
\phi(F) &= -\tan^{-1}\frac{a\sin 2\pi F}{1 - a\cos 2\pi F}.
\end{aligned}
\tag{10.34}
$$

Notice that the *dc gain* $H(0) = 1$ and that H is periodic. Figure 10.2 displays the amplitude and phase in degrees for three values of a.
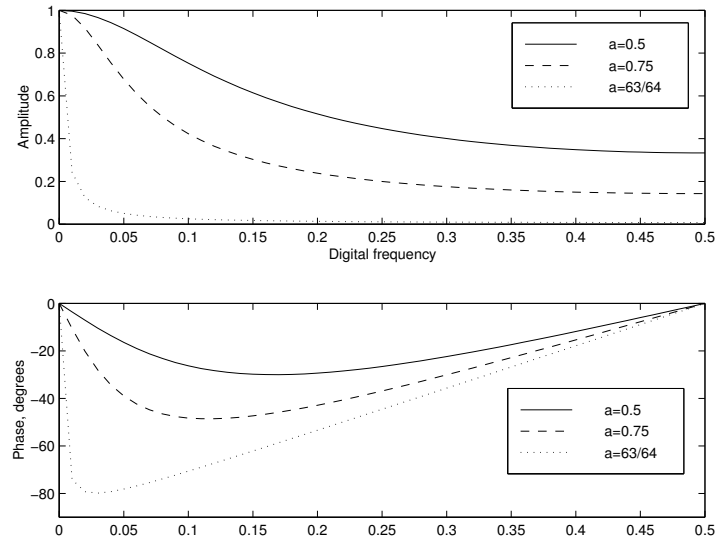

FIGURE 10.2 *Amplitude and phase plots*

     From Figure 10.2, we see that this form of a digital filter acts as a low-pass filter since it attenuates the higher frequencies in an input signal relative to the lower frequencies. If a continuous input signal is sampled at time intervals T_s to form $x(n)$, Equation 10.30 shows that the sample interval T_s as well as the parameter a determine the frequency characteristic of the discrete system considered as a digital filter.

□

Digital signal processing (DSP) is the general name of a set of techniques used to analyze and modify digital signals, especially for the purpose of spectral analysis and filtering. The DFT and FFT introduced in Chapter 11 are methods used to determine the frequency spectrum of a signal. Digital filtering can be performed with software algorithms or with hardware using discrete-digital components or DSP microprocessors.

The *Signal Processing Toolbox* is a collection of M-files that solve problems in digital signal processing. The toolbox is included with the Student Edition of MATLAB but must be purchased separately from the professional version of MATLAB.

10.5 INTRODUCTION TO z TRANSFORMS

In preceding chapters, we have studied differential equations by direct solution as well as by using Fourier and Laplace transform techniques. We now turn to a transform that is called the *Z-transform* due to the use of the complex variable $z = x + iy = \rho e^{i\Omega}$ in the transform. The Z-transform plays the same role for discrete systems as the Laplace transform does for continuous systems. Table 10.3 summarizes some of the applications of the Z-transform.

TABLE 10.3 *Applications of Z-transform*

Area	Application
Definition	The Z-transform is defined as a sum that transforms discrete signals to the complex frequency (Z) domain.
System analysis	The Z-transform converts convolutions to a product and difference equations to algebraic equations.
Stability	Stability of a discrete linear system can be determined by analyzing the transfer function $H(z)$ given by the Z-transform.
Frequency response	The transfer function $H(z)$ can be evaluated to determine the frequency response of a discrete system.
Digital filters	Digital filters can be analyzed and designed using the Z-transform.
Control	Digital control systems can be analyzed and designed using Z-transforms.

In this section, we follow generally the development of Chapter 9 on Laplace transforms. Indeed, we shall see that there is a close connection

between the Z and Laplace transforms. The Z-transform is first intro-duced and its properties defined, after which we discuss the computation of the transform and inverse transform. MATLAB commands for the transform are presented. Then, we study applications of the Z-transform to difference equation and discrete systems. The chapter ends with a summary of properties of the Z-transform.

LAPLACE AND Z-TRANSFORMS   In this section, the term *sampling* refers to replacement of a function $f(t)$ by the function $f(nT)$. Here we consider *sampled-data* systems in which input and output functions are considered at only discrete values of t, usually at values $nT, n = 0, 1, 2, \ldots$, where T is a positive constant.

By sampling the continuous function $f(t)$ at every T_s seconds, we obtain the discrete function $f_d(t)$ with values defined at $t = 0, T_s, 2T_s, \ldots$. This discrete function can be written in terms of the unit impulse function

$$\delta(t - nT_s) = \begin{cases} 1, & t = nT_s \\ 0, & t \neq nT_s \end{cases} \tag{10.35}$$

where $n = 0, 1, \ldots$ in the following manner:

$$f_d(t) = \sum_{n=0}^{\infty} f(nT_s)\delta(t - nT_s). \tag{10.36}$$

The discrete time function $f_d(t)$ has as its Laplace transform

$$\begin{aligned}
\mathcal{L}[f_d(t)] &= \int_0^{\infty} \sum_{n=0}^{\infty} f(nT_s)\delta(t - nT_s)e^{-st}\, dt \\
&= \sum_{n=0}^{\infty} f(nT_s)e^{-nT_s s} \tag{10.37}
\end{aligned}$$

using the definition of the one-sided Laplace transform and the properties of the impulse function.

Defining a new complex variable

$$z = e^{T_s s}$$

leads to the definition of the Z-transform as the Laplace transform of the discrete function $f(nT_s)$

$$\mathcal{L}[f(nT_s)] = \sum_{n=0}^{\infty} f(nT_s)z^{-n} = F(z). \tag{10.38}$$

Before presenting applications of the Z-transform, the mathematical properties of the transform are considered. Also, a number of examples are presented in this section.

Mathematically, the Z-transform is a rule by which a sequence of numbers is transformed into a function of the complex variable $z = x + iy$, which can be written

$$z = \rho \exp(i2\pi F) = \rho \exp(i\Omega).$$

If $\{f(n)\}$ is a sequence, then we write

$$\mathcal{Z}[f(n)] = F(z) \quad \text{and} \quad f(n) = \mathcal{Z}^{-1}[F(z)]. \tag{10.39}$$

The Z-transform is defined by the series

$$
\begin{aligned}
\mathcal{Z}[f(n)] &= \sum_{n=0}^{\infty} f(n)z^{-n} \\
&= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots. \tag{10.40}
\end{aligned}
$$

This transform is typically used to analyze sequences for which $f(n) = 0$ for $n < 0$ so this transform is called the *unilateral*, or *single-sided Z* transform since the sum begins at zero. .

Region of Convergence The transform of Equation 10.40 may only converge for a restricted set of values of z called the *region of convergence* in the complex plane. Thus, for each Z-transform, the region of convergence should be investigated in certain cases. The ratio test can be applied to the series of Equation 10.40 to test for convergence as in Problem 10.8. This book considers convergence of real functions in Chapter 6. Kaplan's textbook listed in the Annotated Bibliography for this chapter treats complex series.

For the one-sided Z-transform, the infinite series will converge exterior to a circle in the z plane that is larger than the largest pole magnitude of $F(z)$. The cases studied in Example 10.9 illustrate this property.

Finite Sequences For finite sequences, the Z-transform may be written as a polynomial in z. Thus, in computer applications, the question of convergence need not arise except in cases when $z = 0$ or $z = \infty$. Elsewhere in the z-plane, the series in z will converge.

□ **EXAMPLE 10.9** *Z-transform Computation*

a. Consider the sequence $f(n) = a^n$ for $n = 0, 1, \dots$ where a is a constant. The Z transform of $f(n)$ is

$$F(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n. \tag{10.41}$$

This is a geometric series, discussed for the real case in Chapter 6. As in the real case, the series converges with $r = az^{-1}$ to $1/(1-r)$ since

$$1 + r + r^2 + \cdots = \sum_{m=0}^{\infty} r^m = \frac{1}{1-r} \tag{10.42}$$

provided that $|r| < 1$. Equation 10.41 thus becomes

$$F(z) = \mathcal{Z}[a^n] = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \tag{10.43}$$

if $|az^{-1}| < 1$ or $|z| > |a|$. In the z-plane, the transform is a function defined for z outside a circle of radius a centered at the origin.

b. The discrete-time unit-step sequence is defined as

$$u(n) = \begin{cases} 1, & n = 0, 1, 2, \ldots \\ 0, & n < 0. \end{cases}$$

The Z-transform of the discrete-time unit-step sequence is

$$\begin{aligned} U(z) &= \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + \cdots \\ &= \frac{1}{1 - z^{-1}} = \frac{z}{z - 1} \end{aligned} \tag{10.44}$$

$\square$

PROPERTIES OF THE Z TRANSFORM   As with the Laplace transform, a number of the properties of the Z transform can be used to simplify calculations of the transform or the inverse transform.

■ **THEOREM 10.1**   ***Linearity of the Z-transform***
Assume that $\mathcal{Z}[f(n)] = F(z)$ and $\mathcal{Z}[g(n)] = G(z)$. Then, with α and β constant,
$$\mathcal{Z}[\alpha f(n) + \beta g(n)] = \alpha F(z) + \beta G(z).$$

The proof is obtained directly from the defining expression for the Z transform.

■ **THEOREM 10.2**   ***Right Shifting of $f(n)$***
Suppose that $\mathcal{Z}[f(n)] = F(z)$, then
$$\mathcal{Z}[f(n-N)] = z^{-N}F(z) + f(-N) + f(-N+1)z^{-1} + \cdots + f(-1)z^{-N+1} \tag{10.45}$$

The proof is straightforward using the change of variable $m = n - N$ in the Z-transform with the result

$$\begin{aligned} \mathcal{Z}[f(n-N)] &= \sum_{n=0}^{\infty} f(n-N)z^{-n} = \sum_{m=-N}^{\infty} f(m)z^{-(m+N)} \\ &= z^{-N} \sum_{m=-N}^{\infty} f(m)z^{-m} \end{aligned}$$

$$
\begin{aligned}
&= \quad z^{-N}[f(-N)z^N + \cdots + f(-1)z + f(0) + f(1)z^{-1} + \cdots] \\
&= \quad f(-N) + f(-N+1)z^{-1} + \cdots + f(-1)z^{-N+1} \\
&\qquad + z^{-N}[f(0) + f(1)z^{-1} + \cdots]
\end{aligned}
$$

Since the term in brackets is the Z transform of $f(n)$, the theorem is proved.

If the values of $f(n)$ for negative n are not zero, the values must be included in the shifted result as described in the theorem. For the Z transform, a right shift of N samples brings samples for $n < 0$ into the range $n \geq 0$. For the case that $f(n) = 0, n < 0$, the right-shift theorem becomes

$$
\mathcal{Z}[f(n-N)] = z^{-N} F(z) \qquad f(n) = 0, n < 0. \qquad \textbf{(10.46)}
$$

The interpretation of $f(n-1)$ as a sequence is the sequence $f(n)$ shifted to the right by one sampling interval, called a *delay* of one unit. Thus, If $N = 1$ in Equation 10.46, the multiplier z^{-1} for $F(z)$ corresponds to a *unit delay* of $f(n)$.

■ THEOREM 10.3   **Left Shifting of** $f(n)$
       Suppose that $\mathcal{Z}[f(n)] = F(z)$, then

$$
\mathcal{Z}[f(n+N)] = z^N F(z) - f(0)z^N - f(1)z^{N-1} - \cdots - f(N-1)z^{-N+1} \quad \textbf{(10.47)}
$$

                                                         ■

The right-shift and left-shift properties are useful when using the Z-transform to solve difference equations.

■ THEOREM 10.4   **Multiplication by** n
       If $\mathcal{Z}[f(n)] = F(z)$,

$$
\mathcal{Z}[nf(n)] = -z\frac{dF(z)}{dz}. \qquad \textbf{(10.48)}
$$

                                                         ■

Thus, multiplying $f(n)$ by n results in (negative) differentiation of the transform.

☐ EXAMPLE 10.10   **Examples Using Z Transform Properties**

     a. The Z transform of

$$
f(n) = \cos n\omega = \frac{1}{2}e^{in\omega} + \frac{1}{2}e^{-in\omega} \qquad n \geq 0,
$$

     can be computed as

$$
\begin{aligned}
F(z) &= \quad \frac{1}{2}\mathcal{Z}[e^{in\omega}] + \frac{1}{2}\mathcal{Z}[e^{-in\omega}] \\
&= \quad \frac{1/2}{1 - e^{i\omega}z^{-1}} + \frac{1/2}{1 - e^{-i\omega}z^{-1}}
\end{aligned}
$$

$$= \frac{1 - \cos \omega z^{-1}}{1 - 2 \cos \omega z^{-1} + z^{-2}}$$

$$= \frac{z^2 - z \cos \omega}{z^2 - 2z \cos \omega + 1} \qquad |z| > 1$$

using the linearity property of the Z transform and the result of Equation 10.43 with $a = e^{\pm i \omega}$.

b. Consider the discrete pulse of width N points, $f(n) = u(n) - u(n - N)$. From Example 10.9 and the shifting property

$$\mathcal{Z}[u(n) - u(n - N)] = \frac{z}{z - 1} - z^{-N} \frac{z}{z - 1} = \frac{z(1 - z^{-N})}{z - 1}.$$

Since the Z-transform $F(z) = 1 + z^{-1} + \cdots + z^{-(N-1)}$ is a rational function, the region of convergence is the entire z-plane except $z = 0$. If $z = 1$, the result is $F(z) = N$.

$\square$

Table 10.4 presents a few Z-transforms which can be computed using the examples and properties presented in this section. The region of convergence is also indicated for each transform.

TABLE 10.4 *Example Z Transforms*

$f(n)$	$F(z)$	$f(n)$	$F(z)$
$u(n)$	$\dfrac{z}{z - 1}, |z| > 1$	$\cos(n\omega)u(n)$	$\dfrac{z^2 - z \cos \omega}{z^2 - 2z \cos \omega + 1}, |z| > 1$
$a^n u(n)$	$\dfrac{z}{z - a}, |z| > a$	$\sin(n\omega)u(n)$	$\dfrac{z \sin \omega}{z^2 - 2z \cos \omega + 1}, |z| > 1$
$nu(n)$	$\dfrac{z}{(z - 1)^2}, |z| > 1$	$na^n u(n)$	$\dfrac{za}{(z - a)^2}, |z| > a$

INVERSE Z TRANSFORM

For the inverse Z transform, we write

$$\mathcal{Z}^{-1}[F(z)] = f(n)$$

to designate the inversion of $F(z)$ to obtain the discrete function $f(n)$. There are many methods to invert a given complex function, assumed to be the Z-transform of a sequence $f(n)$. Here we present the method of *partial fraction expansion* for inverting $F(z)$. As with the method applied to invert certain Laplace transforms, we write

$$F(z) = F_1(z) + F_2(z) + \cdots$$

when $F(z)$ can be written in terms of the expansion of simpler functions whose Z transforms are known.

Note: It is suggested that the reader review the sections of Chapter 9 covering Laplace transforms that deal with the method of partial fraction expansion to compute the inverse Laplace transform. The techniques discussed there apply directly to the expansion of a Z-transform that is a ratio of polynomials in z. In many cases, $F(z)/z$ is a proper rational fraction that can be expanded into simpler terms. When the expansion is multiplied by z, terms arise such as those included in Table 10.4.

☐ **EXAMPLE 10.11**   *Inverse Z-transform by Partial Fraction Expansion and Division*

   a. Given

$$F(z) = \frac{3z^2}{z^2 + z - 2},$$ **(10.49)**

   we write

$$F(z) = z\frac{3z}{z^2 + z - 2} = z\frac{3z}{(z-1)(z+2)}.$$

   Now $F(z)/z$ is a proper rational fraction in z. The partial fraction expansion becomes

$$\frac{F(z)}{z} = \frac{3z}{(z-1)(z+2)} = \frac{1}{z-1} + \frac{2}{z+2}$$

   so that

$$F(z) = \frac{z}{z-1} + \frac{2z}{z+2}.$$

   Using Table 10.4, the inverse transform is thus

$$f(n) = [1 + 2(-2)^n]u(n).$$

   b. Since we can write

$$F(z) = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots,$$

   one way to determine the coefficients of $f(n)$ is by long division. In the case of Equation 10.49 , divide $z^2 + z - 2$ into $3z^2$ and use powers of z^{-1} for terms after the second. The result is

$$f(0) = 3, \ f(1) = -3, \ f(2) = 9, \ f(3) = -15, \ \cdots$$

   since the quotient is the series

$$F(z) = 3 - 3z^{-1} + 9z^{-2} - 15z^{-3} + \cdots.$$

■

WHAT IF?   When a closed form expression for $f(n)$ cannot be obtained, $F(z)$ can be expanded in a power series in z^{-1} to determine the values $f(i), i = 1, 2, \ldots n$. An alternative is direct long division as just described in Example 10.11. Verify these results.

10.6 MATLAB COMMANDS FOR DISCRETE SYSTEMS

MATLAB has commands that can be used to help solve difference equations and to compute Z-transforms and their inverses. The *Symbolic Math Toolbox* and the *Signal Processing Toolbox* have additional commands for this purpose.[3] Table 10.5 lists commands that are useful for analysis of discrete systems.

TABLE 10.5 *MATLAB commands for Discrete Analysis*

Command	Result
Numerical operations:	
conv	Convolution and polynomial multiplication.
roots	Roots of a polynomial.
residue	Partial fraction expansion.
Symbolic Math Toolbox operations:	
ztrans	Z transform.
iztrans	Inverse Z transform.
Signal Processing Toolbox operations:	
freqz	Frequency response of discrete system.
residuez	Partial-fraction expansion.

□ EXAMPLE 10.12   **MATLAB Solution for Inverse Z-transform**

Consider the Z-transform

$$Y(z) = \frac{0.1z^2}{z^2 - 1.9z + 0.9}.$$

The accompanying MATLAB script uses the command **residue** to compute the coefficients in the partial fraction expansion of $Y(z)/z$. These are given in the vector K in the program. The vector Z holds the corresponding roots of the denominator. The fact that p is a null vector indicates that a proper rational fraction was expanded. The resulting Z-transform for simple factors becomes

$$\frac{Y(z)}{z} = \frac{\texttt{K(1)}}{z - \texttt{Z(1)}} + \frac{\texttt{K(2)}}{z - \texttt{Z(2)}}.$$

---

[3]These toolboxes are included in the student version of MATLAB. They must be purchased separately from the professional version of MATLAB.

```
Example 10.12
%EX10_12.M Compute the inverse z-transform by partial fraction
% expansion for Y(z)/z = 0.1z/(z^2-1.9z+0.9)
%
clear
num=[0.1 0]             % Numerator    0.1z
den=[1.0 -1.9 0.9]      % Denominator z^2-1.9z+0.9
[K,Z,p]=residue(num,den)  % K coefficients, Z roots
%
% Results (edited)
%
K = 1.0000
   -0.9000

Z = 1.0000
    0.9000

p = []
```

From the MATLAB result for $Y(z)/z$, we conclude that

$$Y(z) = \frac{z}{z-1} - \frac{0.9z}{z-0.9}.$$

Thus, the sequence becomes

$$y(n) = 1 - (0.9)^{n+1} \quad n \geq 0$$

as determined by choosing the appropriate terms for $y(n)$ from Table 10.4 given the partial fraction expansion of $Y(z)$.

$\square$

10.7 Z TRANSFORM SOLUTION OF DIFFERENCE EQUATIONS

Consider the Nth order linear difference equation with constant coefficients

$$a_0 \, y(n) \;\; + \;\; a_1 \, y(n-1) + \cdots + a_{N-1} \, y(n-N+1) + a_N \, y(n-N)$$
$$= \;\; b_0 \, x(n) + b_1 \, x(n-1) + \cdots + b_M \, x(n-M) \qquad \textbf{(10.50)}$$

where $a_0, \ldots, a_N$ and $b_0, \ldots, b_M$ are constant with a_0 nonzero.

ZERO INITIAL CONDITIONS

Assuming that $y(n)$ and $x(n)$ are zero for $n < 0$, taking the Z transform of the difference equation of Equation 10.50 yields

$$a_0 Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$
$$-[a_1 z^{-1} Y(z) + \cdots + a_N z^{-N} Y(z)]. \qquad \textbf{(10.51)}$$

Solving for $Y(z)$ results in the form

$$Y(z) = \frac{[b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}]X(z)}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}. \qquad \textbf{(10.52)}$$

If $X(z)$ exists and the transform can be inverted, the inverse $y(n)$ is the solution to the difference equations with zero initial conditions.

☐ EXAMPLE 10.13   ***Difference Equation***

To determine the response of the system described by the difference equation

$$y(n) = 0.5y(n-1) + x(n), \qquad y(-1) = 0 \qquad \textbf{(10.53)}$$

to the discrete-time step input $x(n) = 4u(n)$, we take the Z transform with the result

$$Y(z) - 0.5[z^{-1} Y(z) + y(-1)] = 4U(z).$$

With zero initial condition,

$$Y(z) = \frac{1}{1 - 0.5z^{-1}} \frac{4z}{z - 1}$$
$$= \frac{4z^2}{(z - 0.5)(z - 1)}.$$

Expanding $Y(z)/z =$ by partial fractions yields

$$\frac{Y(z)}{z} = \frac{4z}{(z - 0.5)(z - 1)} = \frac{8}{z - 1} - \frac{4}{z - 0.5}$$

with the inverse of $Y(z)$

$$y(n) = 8u(n) - 4(0.5)^n u(n)$$

as the solution of the original difference Equation 10.53.

☐

10.8 APPLICATIONS OF Z-TRANSFORMS TO LINEAR DISCRETE SYSTEMS

◾

In this section, we present the application of Z-transforms to discrete linear systems. We generally follow the outline of the application of Laplace transforms to linear systems in Chapter 9 and we assume the reader is familiar with that chapter. For linear time-invariant discrete-time systems with no initial energy, the response to an arbitrary input can be computed from the *unit-pulse response* of the system by *convolution*.

The *stability* of a discrete system can be determined by examining the transfer function of the system which is the Z-transform of the system's unit pulse response. The position of the poles of this transfer function determine the characteristics of the system.

CONVOLUTION AND UNIT PULSE RESPONSE

Consider a system with output response $h(n)$ when the input $x(n)$ is the discrete-time unit pulse defined as

$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases} \qquad \textbf{(10.54)}$$

The discrete-time unit pulse is a counterpart of the Dirac delta (impulse) function, although $\delta(n)$ should not be considered a sampled version of the impulse function $\delta(t)$.

If we assume the system is *causal*, the output cannot depend on future inputs so that $h(n) = 0$ for $n < 0$. Otherwise, there could be a response before an input is applied. Also, it is assumed that there is no initial energy in the system so that $y(i) = 0$ for $i = -1, -2, \ldots, -N$.

For an arbitrary $x(n)$ with $x(n) = 0$ for $n = -1, -2, \ldots$, $x(n)$ can be expressed as

$$\begin{aligned} x(n) &= x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \cdots \\ &= \sum_{i=0}^{\infty} x(i)\delta(n-i), \quad n = 0, 1, 2, \ldots. \end{aligned} \qquad \textbf{(10.55)}$$

To compute $y(n)$ for the input $x(n)$, we note that if the system is time-invariant, the response to $\delta(n-i)$ must be equal to $h(n-i)$. By the homogeneous property of linear systems with no initial energy, the response to any input $af(t)$ with scalar a is equal to a times the response to $f(t)$. Thus, the response to $x(i)\delta(n-i)$ is given by

$$y_i(n) = x(i)h(n-i)$$

associating $x(i)$ with a in the definition of the homogeneous property. Using the *additive* property of linear systems allows us to sum the responses

to each term in Equation 10.55 to yield the result

$$y(n) = \sum_{i=0}^{\infty} x(i)h(n-i), \quad n = 0, 1, 2, \ldots. \qquad \textbf{(10.56)}$$

as the response to $x(n)$. Note that since the system is causal, the sum in Equation 10.56 is finite for each n.

The *convolution* of two sequences $x(n)$ and $h(n)$ is the sequence defined by the expression

$$\sum_{i=0}^{\infty} x(i)h(n-i), \quad n = 0, 1, 2, \ldots \qquad \textbf{(10.57)}$$

denoted as $x(n) * h(n)$. Thus, the output $y(n)$ due to input $x(n)$ to a system with no initial energy with unit pulse response $h(n)$ is given as

$$y(n) = x(n) * h(n)$$

according to Equation 10.56. This sum for finite length discrete functions can be carried out by the MATLAB command **conv**.

☐ EXAMPLE 10.14   *Convolution Solution*

Let $h(n) = 0.9^n$ be the unit pulse response for a system with unit step input $u(n)$. Using the convolution sum of Equation 10.56, the response is

$$y(n) = \sum_{i=0}^{\infty} (0.9)^i.$$

For a finite sum,

$$y(n) = \sum_{i=0}^{n} (0.9)^i = \frac{1 - (0.9)^{n+1}}{1 - 0.9} = 10(1 - (0.9)^{n+1}).$$

Problem 10.22 asks you to compute the MATLAB solution using **conv** and compare it to this sum for a finite number of terms.

☐

Convolution and Z-transform Products   The convolution theorem relates the product of Z-transforms to the convolution of the corresponding sequences. This theorem is often useful to determine the response of a discrete system.

■ THEOREM 10.5   *Convolution and Z-transforms*

Let $F_1(z)$ and $F_2(z)$ be, respectively, the Z-transforms of $f_1(n)$ and $f_2(n)$. Then, $F_3(z) = F_1(z)F_2(z)$ is the Z-transform of the convolution of $f_1(n)$ and $f_2(n)$ given by

$$f_3(n) = \sum_{k=0}^{\infty} f_1(k)f_2(n-k).$$

Convolution Theorem Example

For the difference equation

$$y(n) = ay(n-1) + (1-a)x(n)$$

the unit pulse response is

$$h(n) = (1-a)a^n \quad n \geq 0$$

as is easily shown by letting $x(0) = \delta(0)$ and computing the terms in $h(n)$ as $h(0) = 1 - a$, $h(1) = a(1-a)$, ..., $h(n) = (1-a)a^n$.

Let $x(n) = u(n)$, the unit step sequence. Then, the step response can be computed as the convolution

$$s(n) = h(n) * u(n) = \sum_{m=0}^{\infty} h(n-m). \tag{10.58}$$

Letting $k = n - m$, yields

$$s(n) = \sum_{k=-\infty}^{n} h(k).$$

Thus, the step response is the running sum of the unit pulse response. Note also that $s(n) - s(n-1) = h(n)$ so the unit pulse response can be determined by differencing the step response.

In the present case, the step response becomes

$$
\begin{aligned}
s(n) &= \sum_{m=0}^{n} h(m) = (1-a) \sum_{m=0}^{n} a^m \\
&= (1-a)(1 + a + a^2 + \cdots + a^n) \\
&= 1 + a + a^2 + \cdots + a^n - a - a^2 - \cdots - a^n - a^{n+1} \\
&= 1 - a^{n+1} \quad n \geq 0.
\end{aligned}
$$

Taking the Z-transform of $h(n)$ and $u(n)$ yields

$$H(z) = \frac{(1-a)z}{z-a} \quad \text{and} \quad U(z) = \frac{z}{z-1}.$$

According to the convolution theorem as applied to Equation 10.58, the step response of the system can be computed as the inverse Z transform of the product

$$S(z) = H(z)U(z) = \frac{(1-a)z^2}{(z-a)(z-1)}. \tag{10.59}$$

The partial fraction expansion of $S(z)/z$ leads to the terms

$$S(z) = \frac{-az}{z-a} + \frac{z}{z-1}$$

with the inverse transform

$$s(n) = (1 - aa^n)u(n) = (1 - a^{n+1})u(n)$$

as previously determined by the convolution sum.

$\blacksquare$

STABILITY

In this section, we consider stability criteria for the Nth order linear difference equation with constant coefficients

$$
\begin{aligned}
a_0\, y(n) \quad + \quad & a_1\, y(n-1) + \cdots + a_{N-1}\, y(n-N+1) + a_N\, y(n-N) \\
= \quad & b_0\, x(n) + b_1\, x(n-1) + \cdots + b_M\, x(n-M) \qquad \textbf{(10.60)}
\end{aligned}
$$

where $a_0, \ldots, a_N$ and $b_0, \ldots, b_M$ are constant with a_0 nonzero. In addition, it is assumed that the initial conditions are zero for the system described by Equation 10.60. Furthermore, the unit pulse response will be designated as $h(n)$.

The stability criterion called *bounded-input bounded-output* (BIBO) will be presented here. This definition of stability is based on the response to a bounded input sequence. We say that a sequence $x(n)$ is *bounded* if its terms do not get arbitrarily large. Thus, for a bounded sequence there exists a constant M_1 such that

$$
|x(n)| \leq M_1 < \infty
$$

for all $n \geq 0$. In terms of the output function $y(n)$, BIBO stability requires that the output $y(n)$ of a linear time-invariant discrete system with no initial energy is a bounded sequence whenever the input $x(n)$ is a bounded sequence. It can be shown that this criterion is equivalent to the statement that the unit pulse response of the system is absolutely summable.

■ THEOREM 10.6   **Stability in the time-domain**

A system with unit pulse response $h(n), n = 0, 1, 2, \ldots$ is BIBO stable if and only if $h(n)$ is absolutely summable, so that

$$
\sum_{n=0}^{\infty} |h(n)| \leq M < \infty.
$$

Problem 10.18 asks you to prove this theorem.

POLE-ZERO
ANALYSIS

The transfer function of the system described by Equation 10.60 is found by taking the Z-transform of the time-domain equation and solving for $H(z) = Y(z)/X(z)$ to yield

$$
\begin{aligned}
H(z) \quad &= \quad \frac{[b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}]}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} \\
&= \quad K\frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}. \qquad \textbf{(10.61)}
\end{aligned}
$$

Assuming there are no common factors in the numerator and denominator, the poles, $p_i, i = 1, \ldots N$, determine the response $h(n)$ as follows:

1. if a pole p is real, $h(n)$ contains a term $C \times p^n$;

2. if a pole is complex $p = \rho e^{i\alpha}$, $h(n)$ contains a term $C \times \rho^n e^{i\alpha}$;

3. if a pole is repeated m times, $h(n)$ contains terms such as

$$C \times (n)^b p^n, \qquad 0 \le b \le m - 1,$$

where C is a constant in all the cases. In each case, for BIBO stability, it is necessary that

$$|p| < 1$$

so that $h(n)$ converges to zero and is thus absolutely summable. This requirement that all the poles have magnitude strictly less than 1 means that all the poles must lie *within* the unit circle in the z-plane. Systems with simple poles on the unit circle may or may not be stable depending on the input sequence. Such systems are sometimes called *marginally stable*. In conclusion, we state a theorem that characterizes BIBO stability in the z-domain.

■ THEOREM 10.7   **Stability in the z-domain**

A linear time-invariant discrete system with no initial energy is BIBO stable if and only if every pole of $H(z)$ has a magnitude less than 1. Equivalently, for stability, every pole must lie within the unit circle of the z-plane.

□ EXAMPLE 10.16   **BIBO Stability Example**

a. Consider a system with the unit pulse response $h(n) = a^n$ with $h(0) = 0$. Then, summing $h(n)$ shows that

$$\sum_{n=0}^{\infty} |h(n)| = \sum_{n=1}^{\infty} |a|^n = \begin{cases} \infty, & \text{if } |a| \ge 1 \\ \dfrac{|a|}{1 - |a|}, & \text{if } |a| < 1. \end{cases}$$

According to Theorem 10.6, the system is stable only if $|a| < 1$.

b. The difference equation

$$y(n) - ay(n - 1) = x(n - 1)$$

has the unit pulse response solution $a^n, n = 1, 2, \ldots$ and the transfer function

$$H(z) = \frac{Y(z)}{U(z)} = \frac{a}{z - a}$$

has one pole at $z = a$. Theorem 10.7 implies that the system is BIBO stable if and only if $|a| < 1$. This agrees with the previous conclusion arrived at in Part a by analyzing the unit pulse response.

$\square$

As with Laplace transform analysis of continuous systems, stability can be analyzed in the time domain or in the transform domain for discrete systems. One method of analysis in the z-domain is to plot the poles of $H(z)$. This method will be discussed after the frequency response of discrete systems is considered.

10.9 Z-TRANSFORMS AND FREQUENCY RESPONSE

The frequency response of a first-order discrete system was presented in Example 10.8 using the time-domain output signal $y(n)$. In this section, we wish to determine the frequency response from the transfer function given by the Z-transform of $y(n)$.

If the transfer function $H(z)$ is evaluated for values of

$$z = \exp(i2\pi F) = \exp(i\Omega)$$

we obtain the *frequency response*, $H(i2\pi F)$, of the system. This is equivalent to evaluating $H(z)$ on the unit circle in the z-plane. Note that the function $H(i2\pi F)$ is periodic with period 1 since $\exp(i2\pi F)$ is periodic with period 1. According to Equation 10.30 the *digital frequency*

$$F = \frac{f}{f_s} = fT_s,$$

where T_s is the sampling time or time between samples. The analog frequency $f = 1/T_s$ corresponds to the digital frequency $F = 1$ or $\Omega = 2\pi$.

$\square$ **EXAMPLE 10.17**   ***z-plane and Frequency Response***
    The accompanying MATLAB script and figures show the pole-zero plot and frequency response for a system with transfer function

$$H(z) = \frac{z}{z - 0.9}.$$

After defining the numerator and denominator of $H(z)$, the **subplot** command in the script reduces the size of the pole-zero plot but leaves the labels full size. The command **zplane** produces the plot of Figure 10.3. Then, the command **freqz** plots the magnitude in dB and the phase of the frequency response.

```
Example 10.17
% ZEX.M Show the Z-plane and frequency response of the function
%    H(z)=z/(z-.9)
clear,clf
num=[1 0]; den=[1 -0.9]; % Define numerator and denominator of H(z)
subplot(2,2,1)            % Keep plot small
figure(1)
zplane(num,den)  % Draw the z-plane
figure(2)
freqz(num,den)            % Plot frequency response
```
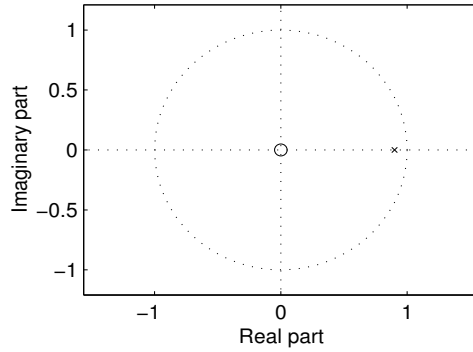

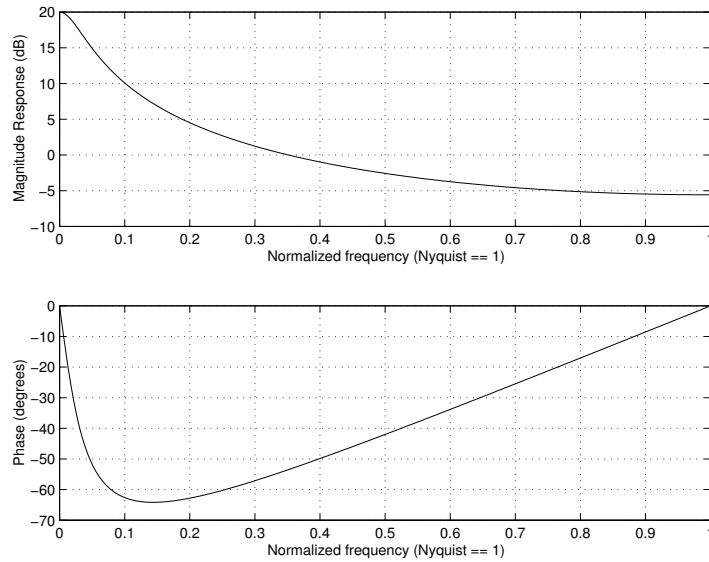**FIGURE 10.3** *z-plane plot of $H(z) = z/(z - 0.9)$*

FIGURE 10.4 *Amplitude and phase plots of $H(i2\pi F)$*

Notice that the highest frequency plotted is normalized to the *Nyquist frequency* which is *one half the sampling frequency*. Here $T_s = 1$ so the actual frequency corresponding to the Nyquist frequency is 0.5 hertz. The command **freqz** can produce other types of plots. In the present plots, the normalized or Nyquist frequency value of 1 in Figure 10.4 corresponds to a digital frequency of 0.5 cycles/sample (π radians/sample).

□

10.10 SUMMARY OF Z-TRANSFORM PROPERTIES

Table 10.6 summarizes some of the properties of the Z-transform. In the table, assume that $\mathcal{Z}[f(n)] = F(z)$ and $\mathcal{Z}[g(n)] = G(z)$ and α and β constants.

TABLE 10.6 *Properties of the Z-transform*

Property	$f(n)$	$\Longleftrightarrow$	$F(z)$
Linearity	$\alpha f(n) + \beta g(n)$	$\Longleftrightarrow$	$\alpha F(z) + \beta G(z)$
Delay	$f(n - N)$	$\Longleftrightarrow$	$z^{-N}F(z) - \sum\limits_{n=-N}^{-1} z^{-(N+n)}f(n)$
Times n	$nf(n)$	$\Longleftrightarrow$	$-z \, dF(z)/dz$
Difference	$f(n) - f(n-1)$	$\Longleftrightarrow$	$(1 - z^{-1})F(z)$
Convolution	$f(n)g(n)$	$\Longleftrightarrow$	$F(z) * G(z)$
	$f(n) * g(n)$	$\Longleftrightarrow$	$F(z)G(z)$

The delay (right-shift) property applied to causal sequences $f(n)$ ($f(n) = 0, n < 0$) leads to the result $f(n - N) \Longleftrightarrow z^{-N}F(z)$.

INITIAL AND FINAL VALUE THEOREMS

If the Z-transform of a sequence $f(n)$ exists, there are several theorems that describe the limiting behavior of $f(n)$ and the transform $F(z)$. The initial value and final value theorems describe the limiting behavior of $f(n)$ from the limiting behavior of $F(z)$.

■ THEOREM 10.8   ***Initial value*** $f(0)$
If the indicated limits exist, then

$$f(0) = \lim_{z \to \infty} F(z).$$

■

For a proof, consider the definition

$$F(z) = f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots$$

and take the limit as $z \to \infty$. This theorem also states that a valid transform $F(z)$ approaches $f(0)$ as z approaches infinity.

Successive values of $f(n)$ can be obtained in a similar way by multiplying $F(z)$ by powers of z and rearranging the series before taking the limit. Specifically,

$$f(1) = \lim_{z \to \infty} z[F(z) - f(0)].$$

■ THEOREM 10.9   ***Final value*** $f(\infty)$
If the indicated limits exist, then

$$\lim_{n \to \infty} f(n) = \lim_{z \to 1}(z - 1)F(z).$$

■

Problem 10.20 asks you to prove the assertion of this theorem.

10.11 REINFORCEMENT EXERCISES AND EXPLORATION PROBLEMS

■

REINFORCEMENT EXERCISES

P10.1. Properties of discrete systems   Determine if the system

$$y(n) = x(n)/2 + x(n - 1)$$

is linear.

P10.2. Homogeneous difference equations   Find the homogeneous solution to the difference equations

   a. $y(n) - y(n - 1) = 2y(n - 2), \quad y(0) = 1, y(1) = 3..$
   b. $y(n + 1) + 2y(n) + 4y(n - 1) = 0.$

Write the result in Part b in terms of sinusoids.

P10.3. Rabbit population   Show that the growth of the rabbit population in Example 10.4 can be written as

$$y(n) = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

P10.4. Comparison of solution methods   Solve the difference equation

$$y(n) - \alpha y(n-1) = C, \qquad y(-1) = 0$$

by the following methods:

      a. the iterative method as in Example 10.1;

      b. by finding the homogeneous and particular solution.

P10.5. General solutions   Find the general solution to the difference equations

      a. $y(n+2) + 3y(n+1) + 2y(n) = n^2$.

      b. $y(n) - 2ay(n-1) + y(n-2) = 3^n$ for $a = -1$.

P10.6. First-order differential equation   Compare the solution to the differential equation $\dot{y}(t) + ay(t) = 0$ of Example 10.6 with the difference equation solution by expanding $y(t)$ in an appropriate Taylor series. Under what condition is the discrete result a good approximation?

P10.7. Frequency response of second-order system   Determine the frequency response for the second-order system

$$y(n) + \alpha y(n-2) = x(n) - x(n-2)$$

for values of $\alpha = 0.5, 0.75, 63/64$.

P10.8. Convergence of Z transforms   Determine the conditions for which the series of Equation 10.40 converges. Apply the result to the Z-transform series for $f(n) = a^n, n > 0$.

P10.9. Z-transforms   Find the Z-transform of the following functions

      a. 2^n.

      b. e^{anT}.

Express the transform as a fraction in z.

P10.10. Z-transforms   Show that the Z-transform of the sequence $\{A \cos \beta n\}$ is

$$F(z) = \frac{Az(z - \cos \beta)}{z^2 - 2z \cos \beta + 1}.$$

P10.11. Difference equation   Find the solution to the difference equation

$$y(n) + 2y(n-1) = x(n) - x(n-1), \quad x(n) = n^2, \; y(0) = 1$$

in the following ways:

      a. Solving directly ;

      b. Solve using Z-transforms.

Hint: For Part 2, write the equation in terms of $y(n+1)$ and use the MATLAB command **residue** to help invert the transform using the partial fraction expansion of $Y(z)/z$.

P10.12. MATLAB comparison   Plot the solutions for the difference equation in Problem 10.6 for various values of the sampling interval T and compare the results to the solution of the continuous differential equation.

P10.13. MATLAB rabbit population Plot the values the growth of the rabbit population in Example 10.4 for the first twelve months.

P10.14. MATLAB first-order smoother Plot the frequency spectrum for the input signal and output signal for the first-order smoother of Example 10.7 defined by the difference equation

$$y(n) = a\,y(n-1) + (1-a)\,x(n).$$

Use the input signal and the smoothing value a defined in the MATLAB script for the example. *Hint:* Use the MATLAB command **fft**.

EXPLORATION PROBLEMS

P10.15. Homogeneous solution Given the first-order difference equation with nonconstant coefficients

$$y(n+1) - a(n)y(n) = 0, \quad y(0) = y_0,$$

show that the solution can be written

$$y(n) = \left[\prod_{i=0}^{n-1} a(i)\right] y_0.$$

Suppose that $a(n) = a$, a constant. Show that the assertion of Equation 10.8 is correct.

P10.16. Relation to differential equations There is an interesting relationship between homogeneous differential and difference equations with constant coefficients. If the solution to the Nth order differential equation is differentiated N times and the independent variable is set equal to zero, the solution to the equivalent difference equation results! The equivalent difference equation is considered the differential equation with the Nth derivative replaced by the discrete $y(n+N)$. Show this result for the equations

$$\frac{d^2 y(t)}{dt^2} - 3\frac{dy(t)}{dt}(t) + 2y(t) = 0$$

and

$$y(n+2) - 3y(n+1) + 2y(n) = 0.$$

P10.17. Frequency spectrum For the signal

$$f(n) = (0.9)^n \cos n\pi, \qquad n = 0, 1, 2, \ldots$$

  a. Find the Z-transform;

  b. Plot the frequency spectrum (amplitude and phase) for the frequency range $\omega = [0, 2\pi]$;

  c. Compare the results in Part 2 with the result using the MATLAB command **freqz**.

Hint: For Part 2, the frequency response can be computed by replacing $z = e^{i\omega}$ in the Z-transform of the signal.

P10.18. Stability of a discrete system Prove the propositions of Theorem 10.6.

P10.19. Stability of discrete systems Discuss the stability of the systems with the transfer functions $H(z) = Y(z)/X(z)$ as follows:

  a. $H(z) = z/(z-3)$;

  b. $H(z) = 1/[z(z-1)^2]$;

c. $H(z) = z/(z-3)$ if $X(z)$ contains a pole at $z = 1$;

P10.20. Final value of $f(n)$ Prove the assertion of Theorem 10.9.

P10.21. $f(n)$ from initial values   Given

$$F(z) = \frac{z}{z+1},$$

find $f(n)$ using the initial value theorem.

P10.22. MATLAB convolution solution   Given the system of Example 10.14 with $h(n) = 0.9^n$ the unit pulse response and unit step input $u(n)$, use MATLAB to determine the response by convolution. Also, compute the result using the closed-form solution for $N = 10$ terms and compare with the solution determined by the command **conv**.

P10.23. MATLAB frequency response   In a sampled data system, let $t_s = 10^{-4}$ seconds so the Nyquist frequency is 5000 Hz. Compute and plot the amplitude response of the system

$$H(z) = \frac{z}{z - 0.9}$$

for $f = 0$ to $f = 5000$ Hz in steps of 100 Hz.

10.12 ANNOTATED BIBLIOGRAPHY

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10.13 ANSWERS

10.1   **Properties of discrete systems**   Considering the system

$$y(n) = x(n)/2 + x(n-1)$$

let

$$\begin{aligned}
y_1(n) &= x_1(n)/2 + x_1(n-1) \\
y_2(n) &= x_2(n)/2 + x_2(n-1).
\end{aligned}$$

If the input is $\alpha x_1 + \beta x_2$, the result is

$$\begin{aligned}
y(n) &= [\alpha x_1(n) + \beta x_2(n)]/2 + \alpha x_1(n-1) + \beta x_2(n-1) \\
&= \alpha[x_1(n)/2 + x_1(n-1)] + \beta[x_2(n)/2 + x_2(n-1)] \\
&= \alpha y_1(n) + \beta y_2(n)
\end{aligned}$$

so the system is linear.

10.3   **Rabbit population**   Using the initial conditions for the rabbit population model

$$\begin{aligned}
y(0) &= 0 = C_1 + C_2 \\
y(1) &= 1 = c_1\, r_1 + c_2\, r_2.
\end{aligned}$$

where

$$r_1 = \left(\frac{1+\sqrt{5}}{2}\right) \qquad r_2 = \left(\frac{1-\sqrt{5}}{2}\right).$$

Solving the system of linear equations without substituting the numerical value of $\sqrt{5}$ yields the result as

$$y(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

10.5   **General solutions**   Find the general solution to the difference equations

a. For the equation $y(n+2) + 3y(n+1) + 2y(n) = n^2$, the assumed solution is

$$y(n) = c_1(-1)^n + c_2(-2)^n + an^2 + bn + c$$

where a, b, and c determine the particular solution. Equating these coefficients to the equivalent coefficients of the forcing function yields the system of equations

$$\begin{pmatrix} 6 & 0 & 0 \\ 10 & 6 & 0 \\ 7 & 5 & 6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solving for the coefficients leads to the solution

$$y(n) = c_1(-1)^n + c_2(-2)^n + \frac{1}{6}n^2 - \frac{5}{18}n + \frac{1}{27}.$$

---

b. The equation $y(n) - 2ay(n-1) + y(n-2) = 3^n$ for $a = -1$ has the characteristic equation

$$r^2 + 2r + 1 = 0$$

with repeated roots -1. This leads to a complementary solution

$$y_c(n) = (c_1 + c_2 n)(-1)^n.$$

The particular solution is of the form $C3^n$, so substituting in the difference equation yields

$$C3^n + 2C3^{n-1} + C3^{n-2} = 3^n$$

with the result $C = 9/16$. The general solution is thus

$$y(n) = (c_1 + c_2 n)(-1)^n + \frac{9}{16}3^n.$$

10.7   Frequency response of second-order system   The frequency response for the second-order system

$$y(n) + \alpha y(n-2) = x(n) - x(n-2)$$

is

$$H_2 = \frac{1 - e^{-i2\Omega}}{1 + \alpha e^{-i2\Omega}}. \tag{10.62}$$

10.9 Z-transforms   The Z-transform of the functions

a. $\mathcal{Z}[2^n] = \dfrac{z}{z-2}$, $|z| > 2$.

b. $\mathcal{Z}[e^{anT}] = \dfrac{z}{z - e^{aT}}$, $|z| > |e^{aT}|$.

10.11   Difference equation

a. For the equation $y(n) + 2y(n-1) = x(n) - x(n-1) = 2n - 1$, with initial conditions $y(0) = 1$, the characteristic equation is

$$1 + 2r^{-1} = 0$$

with solution $r = -2$. Thus, the homogeneous solution is

$$y_h(n) = C(-2)^n.$$

The particular solution is assumed to be $y_p(n) = C_1 n + C_2$ so that

$$C_1 n + C_2 + 2[C_1(n-1) + C_2] = 2n - 1$$

with the result $C_1 = 2/3$ and $C_2 = 1/9$. Thus,

$$y(n) = \frac{2}{3}n + \frac{1}{9} + C(-2)^n.$$

Using this result at $n = 0$ with $y(0) = 1$ yields $C = 8/9$ for the arbitrary constant. The complete solution is thus

$$y(n) = \frac{8}{9}(-2)^n + \frac{2}{3}n + \frac{1}{9}.$$

10.13 Answers

b. To apply the Z-transform method, write

$$y(n+1) + 2y(n) = 2n + 1.$$

Taking the transform yields

$$zY(z) - zy(0) + 2Y(z) = \frac{2z}{(z-1)^2} + \frac{z}{z-1}$$

or

$$Y(z) = \frac{2z}{(z+2)(z-1)^2} + \frac{z}{(z+2)(z-1)} + \frac{z}{z+2}.$$

Analytically, $y(z)/z$ could be written in terms of a partial fraction expansion and the resulting terms inverted to yield $y(n)$. The equation is

$$Y(z)/z = \frac{z^2 - z + 2}{(z+2)(z-1)^2}.$$