Linear Systems

April 9, 2014

Consider a system defined by a differential equation:

Figure 1: SystemDifferentialEq Time $h(t)$

Consider a Linear Time Invariant (LTI) system defined by a Transfer Function:

Figure 2: SystemTransferFunction Laplace $H(s)$ or Fourier $H(j\omega)$ or $H(z)$.

System Relationships Analog LTI System No I.C.s

Input	*System*	*Output*
$x(t)$	$h(t)$	$y(t) = \int_0^t x(\tau)h(t-\tau)\, d\tau.$
$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$	$H(j\omega)$	$Y(j\omega) = X(j\omega)H(j\omega)$
$X(s) = \int_0^{\infty} x(t)e^{-st}\, dt \quad s = \sigma + j\omega$	$H(s)$	$Y(s) = H(s)X(s)$

To make this more concrete, consider an input $x(t)$ with output $y(t)$ in the time domain and $Y(s) = H(s)X(s)$ in the Laplace domain.

Assuming that $x(\tau) = 0$ for $\tau < 0$, adding the response to all the past inputs leads to the integral

$$y(t) = \int_0^t x(\tau)h(t-\tau)\, d\tau. \tag{1}$$

This integral is called the *convolution* or *superposition* integral and the operation is said to be the convolution of x, the input, and h, the impulse response of the system. The convolution is often written $x * h$.

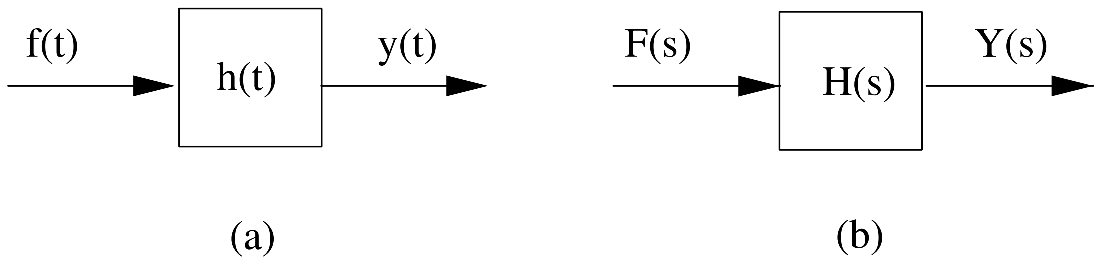

Figure 3: System views in time and Lplace variable s

Let the Laplace Transform of $f(t)$ be $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[h(t)] = H(s)$. Then,

$$\mathcal{L}[f * h] = \mathcal{L}\left[\int_0^t f(\tau)h(t-\tau)\, d\tau\right] = F(s)H(s). \tag{2}$$

In words, the Laplace transform of the convolution $f(t) * h(t)$ is the product $F(s)H(s)$. (Reference Harman page 444 and K&H pages 76, 323)

If the Fourier Transform of $f(t)$ exists, we can use the Laplace Transform to find

$$F(j\omega) = F(s)|_{s=j\omega}$$

System Relationships Digital

Input	*System*	*Output*
$x[n]$	$h[n]$	$\sum_{i=0}^{n} h[i]x[n-i]$
$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$		DFT
$X(Z) = \sum_{n=0}^{\infty} x[n]z^{-n}$	$H(Z)$	$Y(Z) = X(Z)H(Z)$

Here is a diagram of a closed loop system:

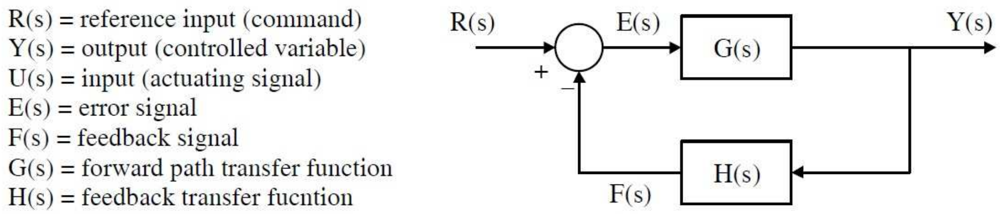

Figure 4: ClosedLoopwithNotes

In this figure $U(s) = E(s)$ since there is no compensator in the loop.