Harman Outline 1A1 Integral Calculus CENG 5131

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III. Review of Integration

A.Basic Definitions Harman Ch14,P642

Fundamental Theorem of Calculus The fundamental theorem of calculus shows the intimate relationship between the derivative and the integral. This theorem allows the evaluation of integrals without computing $\int_a^b f(x)\,dx$ from its definition as the limit of a sum. To apply the theorem, the *antiderivative* of $f(x)$ is defined as a function $F(x)$ such that $F'(x) = f(x)$.

Fundamental Theorem of Calculus *If $f(x)$ is a continuous function in the closed interval $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is an antiderivative of f. Assuming the function $f(x)$ exists, this theorem implies that

$$\frac{dF(x)}{dx} = f(x) \quad a \leq x \leq b.$$

Integration by Parts Consider the formula for the derivative of a product of functions,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Integrating both sides of the product equation yields

$$f(x)g(x) = \int f(x)g'(x)\,dx + \int g(x)f'(x)\,dx,$$

or

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx. \tag{1}$$

This is the formula for *integration by parts*. It is usually written in condensed form by letting

$$\begin{aligned} u &= f(x), & du &= f'(x)\,dx, \\ v &= g(x), & dv &= g'(x)\,dx, \end{aligned}$$

so that the formula for integration by parts becomes

$$\int u\,dv = uv - \int v\,du. \tag{2}$$

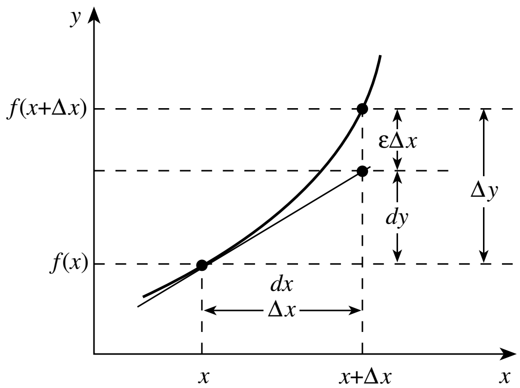

Figure 1: Caption for F12_4_new

Consider the arc length in the figure. The length of a small segment is

$$ds^2 = (dx)^2 + (dy)^2 = \left[1 + \frac{(dy)^2}{(dx)^2}\right](dx)^2$$

Thus, the length of the curve is the sum of the small line segment lengths. If the curve is continuous, the integral of ds over the length in x is

$$s = \int_a^b \sqrt{1 + y'^2}\, dx$$

The integral is used in various ways in every field of science and engineering. Some examples are given in the table assuming that the function $f(x)$ is continuous in the interval considered. (Harman P. 649)

Function	*Integral*	*Name*
Area defined by $y = f(x)$	$A = \int_a^b f(x)\,dx,\ f(x) \geq 0$	Area
Average of $f(x)$	$\bar{f} = \frac{1}{b-a}\int_a^b f(x)dx$	Average on $[a, b]$
Length of $y = f(x)$	$s = \int_a^b \sqrt{1 + y'^2}\,dx$	Arc Length

Table 1: Table of Integration

EXAMPLE of Integration by Parts To evaluate the integral

$$\int_0^\pi x \sin x\,dx,$$

we apply the integration by parts formula of Equation 2. There are two possible choices for the variables as

$$\begin{aligned} & u = x, \qquad dv = \sin x\,dx \\ \text{or} \quad & u = \sin x, \qquad dv = x\,dx. \end{aligned}$$

The first choice simplifies the integral to be evaluated by parts, with the result

$$\begin{aligned} \int_0^\pi x \sin x\,dx &= -x\cos x\Big|_0^\pi - \int_0^\pi (-\cos x)\,dx \\ &= -\pi\cos\pi + 0\cos 0 + \sin\pi - \sin 0 = \pi. \end{aligned}$$

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 INT    Integrate   (Symbolic Matlab)
    INT(S,v) is the indefinite integral of S with respect to v. v is a
      scalar SYM.
    INT(S,a,b) is the definite integral of S with respect to its
      symbolic variable from a to b. a and b are each double or
      symbolic scalars.
    INT(S,v,a,b) is the definite integral of S with respect to v
      from a to b.
     Example:
>> syms x
>> int(x*sin(x))
ans =  sin(x) - x*cos(x)
>> int(x*sin(x),0,pi)
ans =pi   (As Expected)
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Prehaps the most important integral operations in mathematical physics and engineering are the *Fourier Transform*, *Laplace transform*, and the operation of *convolution*

Function	*Integral*	*Name*
Fourier	$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$	$\omega = 2\pi f$ rad/sec
Laplace	$F(s) = \int_{0}^{\infty} f(t)e^{-ist}\, dt$	$s = \alpha + i\omega$
Convolution	$y(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau$	Convolve $f(t)$ and $h(t)$

Table 2: Table of Transforms and Operations

See Harman P396 Ch8, P417 Ch9, and P444.

Notice that integration by parts will often be used to compute the Fourier and Laplace transforms. Convolution requires a proper selection of the interval of integration.

EXAMPLE of Convolution Using the definition of the *Unit Step Function*,

$$U(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

we wish to convolve $h(t) = e^{-\alpha t}U(t+3)$ and $f(t) = e^{-\alpha t}U(t-1)$. The important thing here is to recognize that $x(t)$ is zero before $t = -3$ and $h(t)$ is zero until $t = 1$. However, in the convolution formula

$$f(\tau) = e^{-\alpha\tau}U(\tau - 1) \text{ as before but } h(t-\tau) = e^{-\alpha t - \tau}U(t - \tau + 3)$$

The convolution integral becomes

$$y(t) = \int_{-\infty}^{\infty} [e^{-\alpha\tau}U(\tau - 1)][e^{-\alpha(t-\tau)}U(t - \tau + 3)]d\tau$$

Since $e^{-\alpha t}$ is not a function of τ, we can write

$$y(t) = e^{-\alpha t}\int_{-\infty}^{\infty} [U(\tau - 1)][U(t - \tau + 3)]d\tau$$

The lower limit is $t = 1$ and the upper is $t - \tau + 3 = 0$ or $\tau = t + 3$ and thus

$$y(t) = e^{-\alpha t}\int_{1}^{t+3} d\tau = e^{-\alpha t}[t + 3 - 1] = (t + 2)e^{-\alpha t}U(t + 2)$$

where the restriction of the range of $y(t)$ comes from the limits $t + 3 \geq 1$ or $t \geq -2$.

3. EXAMPLE Physical Applications of Integrals In physics and engineering, derivatives (rate of change) and integrals define many important properties of physical systems. Derivative forms of physical relationships are given in the table.

Function	*Derivative*	*Name*
Electrical	$L\frac{di}{dt}, C\frac{dV}{dt}, i = \frac{dq}{dt}$	L and C constant
E field	$E = -\frac{dV}{dx}$	Electric field strength (volts/m) is rate of change of voltage with distance
Force	$F = \frac{d(mv)}{dt}$	Newton's 2nd Law
Power	$P = \frac{dW}{dt}$	If power in kilowatts, work W in kilowatt-hrs
Conservative force	$F(x) = -\frac{dU(x)}{dx}$	Force equals change in potential energy
Heat Conduction	$\frac{dQ}{dt} = -kA\frac{dT}{dx}$	Heat flow across area A due to temperature gradient k is thermal conductivity

Table 3: Table of Physical Relationships

Some comments on the entries in the table are as follows:

1. The electrical equations are accurate when the values L and C are constant. The $\mathbf{E}$ field is a vector in general.

2. The net force on a particle is equal to the time rate of change of its linear momentum: $F = d(mv)/dt$. If m is constant then the familiar form $F = ma$ is used where $\mathbf{F}$ is a vector in general.

3. The work done on a body by a force is equal to the change in kinetic energy of the body. Power is the rate at which work is done.

3. EXAMPLE Physical Applications of Integrals In physics and engineering, the integral form of equations is used when a "summation" of a variable is needed.

Function	*Relationship*	*Name*
velocity	$v(t) = \int_a^b a(t)\,dt$	$a(t)$ is acceleration
Voltage	$V = -\int_a^b E(x)\,dx$	
Work and Force	$W = \int_a^b F(x)\,dx$	Work moving object from a to b
Heat (Btu)	$Q = m\int_{T_i}^{T_F} c(T)\,dT$	Heat to mass m to increase temperature from T_i to T_f c is specific heat
Gas Volume V at pressure P	$W = \int_{V_i}^{V_f} P\,dV$	Work from expansion

Table 4: Table of Physical Relationships

Some comments on the entries in the table are as follows:

1. Velocity and acceleration and the electric **E** field are vectors generally.
2. If the forces or fields are *conservative* meaning no loss of energy if an object is moved through a closed path, the work is said to be *independent of path.*