Harman Outline 1A Calculus CENG 5131 PDF

September 5, 2013

III. Review of Differentiation

A.Basic Definitions Harman Ch6,P297

Approximations for the Derivative The expression for the derivative of a function $f(t)$ is a fundamental formula in calculus. The definition is

$$f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}, \tag{1}$$

where

$$f'(t) \equiv \frac{df(t)}{dt} \equiv D_t f$$

is the exact derivative at the point t. For computer calculation, h will be a finite value that cannot be taken as zero. Also, roundoff error limits the accuracy of the calculation, even if h is chosen to be very small. This error will be ignored in the following discussion.

Some derivative results are given in Table 1. Here the independent variable is x but any variable $x, t, u, v, \ldots$ can be used. Also, assume that the functions discussed are differentiable functions at the points of interest. This requires that the function be continuous at the point the derivative is taken. However, a continuous function does not necessarily have a derivative everywhere- think of $f(x) = x$ on an interval containing the origin.

Function	*Derivative*	*Name*
x^n	nx^{n-1}	Power
$f(x)g(x)$	$f(x)g'(x)+g(x)f'(x)$	Product
$f(x)/g(x)$	$\dfrac{g(x)f'(x)-f(x)g'(x)}{g^2(x)}$	Quotient
$f[g(x)]$	$f'[g(x)]\ g'(x)$	Chain Rule
e^x	e^x	Exponential
$\log_e x$	$1/x$	$\ln = \log_e$
$\sin x$	$\cos x$	Sinusoids
$\cos x$	$-\sin x$	
$\tan x$	$\dfrac{1}{\cos^2 x}$	Trig
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$	
a^x	$\ln a \cdot a^x$	Base a

Table 1: Table of Derivatives

```
DIFF   Differentiate.
    DIFF(S) differentiates a symbolic expression S with respect to its
    free variable as determined by SYMVAR.
    DIFF(S,'v') or DIFF(S,sym('v')) differentiates S with respect to v.
    DIFF(S,n), for a positive integer n, differentiates S n times.
    DIFF(S,'v',n) and DIFF(S,n,'v') are also acceptable.

    Examples;
       x = sym('x');
       t = sym('t');
       diff(sin(x^2)) is 2*x*cos(x^2)
       diff(t^6,6) is 720.
    Reference page in Help browser        doc sym/diff

>> diff(tan(x))

ans =  tan(x)^2 + 1          [Also sec(x)^(2)]
```

1. EXAMPLE We can prove the quotient rule by substituting in the basic definition Equation 1.

$$D_x\left[\frac{f(x)}{g(x)}\right] = \lim_{h\to 0}\frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

OF QUOTIENT RULE

$$D_x\left[\frac{f(x)}{g(x)}\right]$$

KNOW

$D_x g(u) = g'(u) D_x u$ } CHAIN

$D\, x^n = n \cdot x^{n-1}$

$D_x u^n = n u^{n-1} D_x u$ } POWER

$$\frac{d}{dx} g(x)^{-1} = (-1) g(x)^{-2} \frac{dg(x)}{dx}$$

chain Rule

$$(x) g^{-1}(x)] = f'(x) g^{-1}(x) - (1) g(x)^{-2} g'(x) f(x$$

$$= \frac{f'(x)}{g(x)} - \frac{g'(x) f(x)}{g^2(x)}$$

$$= \frac{f'(x) g(x) - g'(x) f(x)}{g^2(x)}$$

Figure 1: A simpler Proof

A useful application of the *Chain Rule* involves the case where we associate

$$u(x) = f(x) \text{ and } y = g(u)$$

Then

$$D_x[y] = \frac{dy}{du}\frac{du}{dx}$$

3. EXAMPLE Chain Rule to relate variables that are changing
Let's consider a balloon that is being inflated at a known rate and find the rate at which the balloon volume is increasing.

EXAMPLE 3

4

EXAMPLE
Ed+P
p 123

A spherical BALLOON HAS VOLUME

$$V = \tfrac{4}{3}\pi r^3 \quad cm^3$$

Suppose r increases by 0.2 cm/sec when $r = 5cm$. So $\left.\frac{dr}{dt}\right|_5 = 0.2$ cm/sec

A. $$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$

$$= 4\pi(5)^2(0.2) = 63 \text{ cm}^3/\text{sec}$$

CHECK DIMENSIONS

$$4\pi(5cm)^2\cdot(0.2\ cm/sec) = x\ cm^3/sec \checkmark$$

See ALSO HARMAN P 554

Figure 2: Blow up My Baloon

B. Maximum and Minimum

If $f'(a)$ exists, the *tangent line* to $f(x)$ at a in the $x - y$ plane is

$$y - f(a) = f'(a)(x - a).$$

This is a line $y = mx+b$ with the slope given by $f'(a)$. There are many examples of the use of the tangent line to a function but an important theorem states that the tangent line is *horizontal* at a point of local maximum or minimum of the curve $f(x)$. See Harman Page 579 for more details.

For a differentiable function $y = f(x)$ defined in an *open interval*, a local or relative minimum or maximum occurs only at a point where the derivative is zero so that

$$f'(x) = 0.$$

Theorem: *If* $f'(x_0) = 0$ *and* $f''(x_0) > 0$*, then* $f(x)$ *has a relative minimum at* x_0.

EXAMPLE Laser Light Path

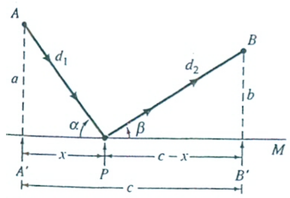

Fig. 3.6.13 Reflection at P of a light ray by a mirror M (Example 6)

EXAMPLE 6 We consider the reflection of a ray of light by a mirror M as in Fig. 3.6.13, which shows a ray traveling from point A to point B via reflection off M at the point P. We assume that the location of the point of reflection is such that the total distance $d_1 + d_2$ traveled by the light ray will be minimized. This is an application of *Fermat's principle of least time* for the propagation of light. The problem is to find P.

Solution Drop perpendiculars from A and B to the plane of the mirror M. Denote the feet of these perpendiculars by A' and B' (Fig. 3.6.13). Let a, b, c, and x denote the lengths of the segments AA', BB', $A'B'$, and $A'P$, respectively. Then $c - x$ is the length of the segment PB'. By the Pythagorean theorem, the distance to be minimized is then

$$d_1 + d_2 = f(x) = \sqrt{a^2 + x^2} + \sqrt{b^2 + (c - x)^2}. \tag{10}$$

We may choose as the domain of f the interval $[0, c]$, because the minimum of f must occur somewhere within that interval. (To see why, examine the picture you get if x is *not* in that interval.)

Then

$$f'(x) = \frac{x}{\sqrt{a^2 + x^2}} + \frac{(c - x)(-1)}{\sqrt{b^2 + (c - x)^2}}. \tag{11}$$

Because

$$f'(x) = \frac{x}{d_1} - \frac{c - x}{d_2}, \qquad f''(x) = \frac{1}{d_1} + \frac{1}{d_2} > 0 \tag{12}$$

MIN

we find that any horizontal tangent to the graph of f must occur over the point x determined by the equation

$$\frac{x}{d_1} = \frac{c - x}{d_2}. \tag{13}$$

At such a point, $\cos \alpha = \cos \beta$, where α is the angle of the incident light ray and β is the angle of the reflected ray (Fig. 3.6.13). Both α and β lie between 0 and $\pi/2$, and thus we find that $\alpha = \beta$. In other words, the angle of incidence is equal to the angle of reflection, a familiar principle from physics.

Figure 3: Angle of Incidence = Angle of Reflection

C. Approximations to Derivatives and Differential Equations

Numerical methods for differential equations and Taylor expansion of derivatives are introduced in Harman (Section 6.4, P297). Using the approximation discussed there

$$f'(t) = \frac{f(t+h) - f(t)}{h} - \frac{h}{2!} f''(t_1), \tag{2}$$

where $t \leq t_1 \leq t + h$ in the remainder term, differential equations of the form

$$\frac{dy(t)}{dt} = f[t, y(t)], \qquad a \leq t \leq b,$$

with initial condition $y(a) = c$ using the Taylor series approximation to the derivative derived in Equation 2.

Letting the approximate values be designated $y_i(t_i) = y_i$, and using the approximation for the derivative, the differential equation becomes a *difference equation*

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n). \tag{3}$$

Solving for y_{n+1} yields the recursion formula

$$y_{n+1} = y_n + hf(t_n, y_n), \tag{4}$$

subject to $y_0 = c$, a constant. This formulation is sometimes called *Euler's method.*

Although no restrictions have been put on $f(t_n, y_n)$ in Equation 3, we will solve a very simple example, so that the errors will become evident. The equation

$$\frac{dy(t)}{dt} = y(t), \qquad y(0) = 1, \tag{5}$$

has the exact solution $y(t) = e^t$. The approximation of Equation 4 leads to the relationship

$$y_{n+1} = y_n + hy_n = (1+h)y_n$$

with $y_0 = 1$.

5. EXAMPLE Euler's Method and ex6_7.m

```
% ex6_7.m Test Euler method on Dy(t)=y(t) t=[0 1].
%  Compare exact value with Euler solution
%  y(n+1)=y(n)+hy(n)=(1+h)y(n); h=0.1
clear
n=10;                     % Number of intervals
y(1)=1;                   % Initial value
T(1)=0;                   % Initial time
yexact(1)=1;
yerror(1)=0;
h=0.1                    % Fixed step size
for I=1:n;
 T(I+1)=I*h;
 y(I+1)=y(I)*(1+h);
 yexact(I+1)=exp(I*h);   % Exact value
 yerror(I+1)=y(I+1)-yexact(I+1);
end;
format short             % Show four places
test=[T' y' yexact' yerror'];
disp('      t       yn        exp     error')
disp(' ')
disp(test)
pause
clf
plot(T,yexact,'-',T, y,'x')
title('First Order Equation, Euler (-x-) and exact solution')
xlabel('Time')
ylabel('y(t)')
legend('Exp', 'Euler')
```

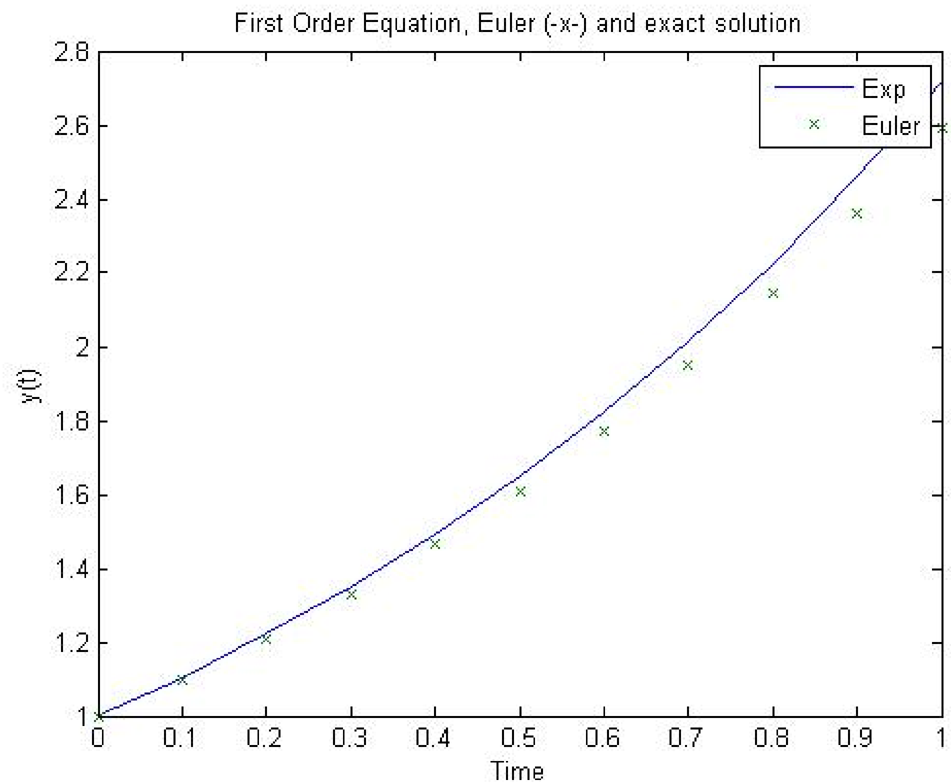

Figure 4: Caption for Harman ex6_7

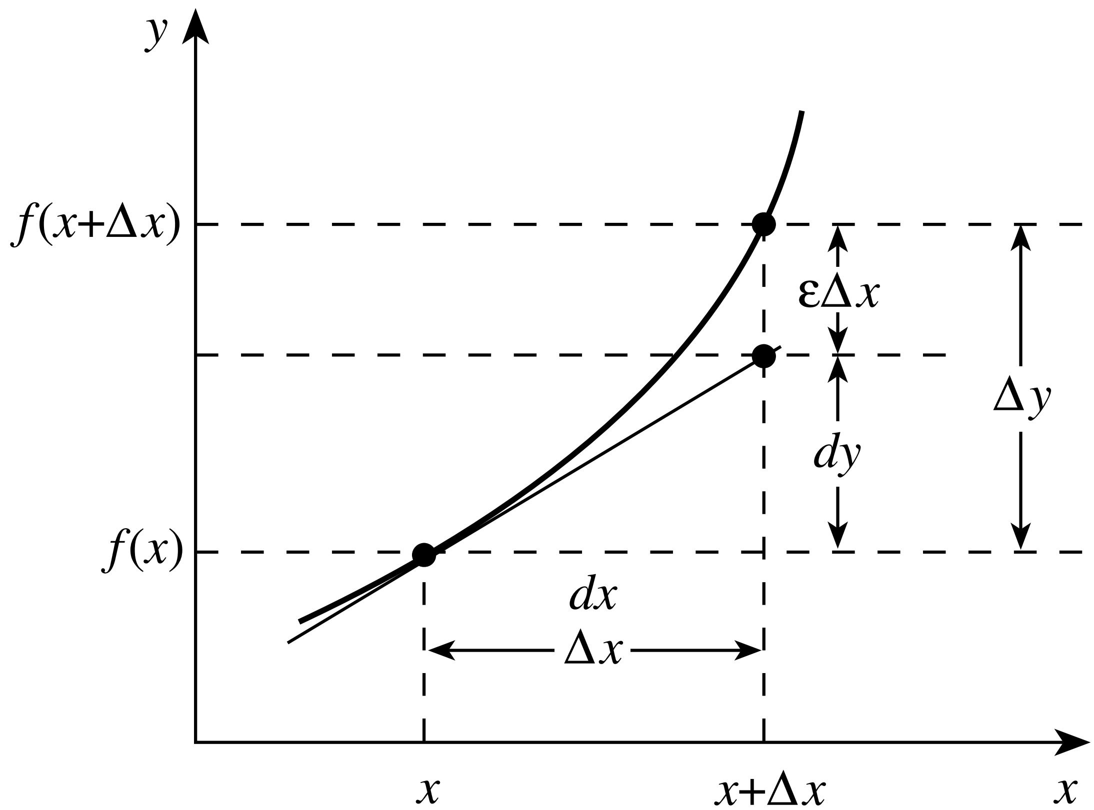

Figure 5: Caption for Harman F12_4_new

D. Differentials and Linear Approximations (12.3, P560)

The *differential* of a function $f(x)$ at the point x is defined by the formula

$$dy = f'(x)\,dx, \tag{6}$$

where dx is an arbitrary increment of the independent variable x. As shown in Figure 6, dy can be viewed as the change in height of a point that moves along the tangent line at the point $[x, f(x)]$ rather than along the curve $f(x)$. If $\Delta x = dx$ is an increment in x, then

$$\Delta y = f(x + \Delta x) - f(x) \tag{7}$$

according to the definitions of the variables in Figure 6.

Using the definition of the derivative,

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x},$$

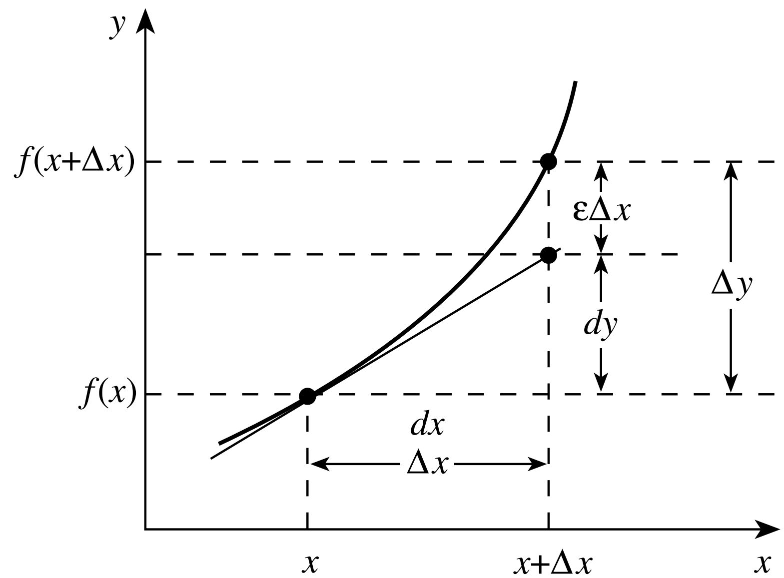

Figure 6: The estimate of increment Δy by the differential dy

and the properties of limits, this expression can be written in the form

$$\frac{\Delta y}{\Delta x} = f'(x) + \epsilon, \tag{8}$$

where $\lim_{\Delta x \to 0} \epsilon = 0$. Multiplying by Δx yields the result

$$\Delta y = f'(x)\Delta x + \epsilon \Delta x. \tag{9}$$

Substituting the expression for dy from Equation 6 with $\Delta x = dx$ leads to a relationship between the increment Δy and the differential dy,

$$\Delta y = dy + \epsilon \Delta x,$$

with $\lim \epsilon = 0$ as $\Delta x \to 0$. The conclusion is that, for small values of Δx, the increment Δy is a good approximation to the differential dy and $\Delta y \approx f'(x)\Delta x$. Considering the definition of Δy in Equation 7, the function f at the point $x+\Delta x$ can be approximated as

$$f(x + \Delta x) \approx f(x) + \Delta y = f(x) + f'(x)\Delta x. \tag{10}$$

This is called the *linear approximation* to $f(x + \Delta x)$ because the change in f depends linearly on Δx.

The linear approximation to $f(x)$ near a point x_0 is usually written by replacing x with x_0 in Equation 10 and letting $\Delta x = x - x_0$ so that

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0). \tag{11}$$

This linear expression represents the first two terms of the Taylor series expansion of function near the point x_0, as treated in Chapter 6. The approximation will also be discussed in more detail later. First, we wish to define the differential for a function of many variables.

6. EXAMPLES Use of Differentials A few examples

$$f(x) = \sqrt{1+x} \approx 1 + \frac{1}{2}x$$

then

$$\sqrt{1.1} = \sqrt{1+.1} \approx 1.05,$$

close to the true value $1.0488\cdots$, but $\sqrt{3} \approx 1 + .5 \times 2 = 2$, not too close to value $1.732\cdots$. A little better approximation would be

$$\sqrt{3} = \sqrt{2(1+1/2)} = 1.41 \times 1.25 = 1.7625$$

good to only one decimal place in the fraction assuming you know $\sqrt{2}$.

Another example shows error in measurement. Suppose the length of a cubic box is measured as $4 \pm .05$ inches. The error in the volume is then

$$dV = 3x^2 dx = 3(4)^2(\pm.05) = \pm 2.4 in^3$$

so the volume is $V = 64 \pm 2.4 in^3$.

For a function of two variables, consider the ideal gas law $pV = cT$, in which p is pressure, V is the volume, T is temperature, and c is a constant. Then, the volume can be written in the form

$$V = V(p, T) = \frac{cT}{p},$$

and the differential of V is

$$dV = \frac{\partial V}{\partial p}\, dp + \frac{\partial V}{\partial T}\, dT.$$

Calculating the partial derivatives yields

$$dV = -\frac{cT}{p^2} dp + \frac{c}{p} dT.$$

From this formula, we can calculate the change in volume due to “small” changes in pressure or temperature or both.