Outline Differential Equations I

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First Order Differential Equations Examples

First-order equations with an initial condition in the form

$$\frac{dy(t)}{dt} = ky(t) \qquad \text{with } y(0) = Y_0 \tag{1}$$

can be written as

$$\frac{dy}{y} = k \text{ or } \int \frac{dy}{y} = \int k\ dt.$$

and have solutions of the form $\ln y = kt + c$ if a is constant. Taking the exponent and rewriting this, the result is

$$y(t) = \exp{(at + c)} = e^c e^{kt} = Y_0 e^{kt}. \tag{2}$$

Always check the result to see that the solution solves the equation and satisfies the initial condition. Although $t = 0$ was used as the time in the example, the initial condition could be given at any time such as the value $y(t_0)$.

In Equation 1 if $y(t)$ represents the amount of a substance, the rate of change of $y(t)$ is proportional to the amount of substance y at time t. If k is positive, the equation represents growth. If k is negative, the equations solution gives the time rate of decay in the quantity of the substance. For example, for decay of Radium, $k = -1.4 \times 10^{-11}\ sec^{-1}$.

The integration method just shown is called *Separation of Variables* and will work if $k = k(t)$. The *integrating factor* method described later as in Equation 7 will also work for such equations with nonconstant coefficients (Harman p. 212).

Solution for Homogeneous Linear Equation with Constant Coefficients

When the coefficients of a first-order linear differential equation are constant, the solution to the homogeneous equation (right-hand side equal to zero)

$$\frac{dy(t)}{dt} + ay(t) = 0 \qquad \text{with } y(0) = Y_0$$

is of a standard form

$$y_c(t) = Ce^{\lambda t} \qquad \text{with } C = Y_0.$$

This solution called the *complementary* or *homogeneous* solution obviously satisfies the equation and the initial condition as shown be substuting y_c into the differential equation. The exponent λ is found by assuming the solution is of the form $Ce^{\lambda t}$ and realizing that the exponental term cannot be zero for all t and cancelling it so that

$$\lambda Ce^{\lambda t} + aCe^{\lambda t} = 0 \quad \text{or } \lambda + a = 0$$

The equation $\lambda + a = 0$ is the *characteristic equation* and determines the exponent since the solution requires $\lambda = -a$ and $y(t) = Y_0\ e^{-at}$.

First Order Differential Equations with a Forcing Function

A first-order differential equation of the form

$$\frac{dT(t)}{dt} = k(T(t) - T_A) \tag{3}$$

might represent the cooling of a body if $k < 0$ according to Newton's Law of Cooling. Suppose the initial temperature of the body is T_1 and the final temperature is the ambient temperature T_A. The equation can be solved by separation of variables as described before or by a method called *Undetermined coefficients*. The latter is the better method as long as k is constant. Rewriting the equation,

$$\frac{dT(t)}{dt} - kT(t) = -kT_A, \quad k < 0$$

the complementary (homogeneous) solution is $T_c(t) = Ce^{kt}$ or

$$T_c(t) = Ce^{-|k|t}$$

to emphasize that the temperature would be decreasing from the initial temperature. However, there is another solution to the differential equation that is caused by the constant function $-kT_A$ which can be viewed as an *input* to the system described by the differential equation. This solution is called the *particular solution*.

Sometimes good sense can lead to solutions without excessive mathematical effort. Note that as t goes to infinity, the final temperature should be stable at T_A. This means $dy/dt = 0$ in Equation 3 and

$$T_{final} = T_A.$$

We should expect a solution of the form

$$T(t) = Ce^{-|k|t} + K.$$

since the complementary solution goes to zero eventually. Thus, the *undetermined constant* K is determined by assuming that the particular solution is also a constant. Substituting K as the solution of Equation 3 yields the value of $K = T_A$.

Applying the initial condition $T(0) = C + K = T_1$ yields the result that the constant $C = T_1 - T_A$ and thus

$$T(t) = (T_1 - T_A)e^{-|k|t} + T_0 \quad \text{with} \quad T_1 > T_A.$$

If $T_1 = 66^\circ$ F and $T_A = 32^\circ$ F, the solution is

$$T(t) = 34e^{-|k|t} + 32 \quad {}^\circ F.$$

If the temperature drops 3 degrees in 2 hours to 63° F , solving the equation yields $k = -0.046$. This is a slow cooling off that might occur in a house with the heat turned off and outside temperature of 32° F.

Generalize Thus, we can generalize and say that all equations of the form

$$\frac{dy(t)}{dt} + ay(t) = k \qquad \text{with } y(0) = Y_0 \tag{4}$$

have the solution

$$y(t) = Ce^{-at} + K$$

where C is determined by initial conditions and K is called the particular solution. We have only to find the *undetermined coefficient* K which is independent of initial conditions and then apply the initial conditions to find C. Thus,

$$y_p(t) = K \quad \text{ so that } 0 + aK = k \quad \text{ or } \; K = \frac{k}{a}.$$

Thus, applying the initial condition $y(0) = Y_0 = C + \frac{k}{a}$, the solution is

$$y(t) = \left(Y_0 - \frac{k}{a}\right) e^{-at} + \frac{k}{a}.$$

Considering the equation

$$\frac{dy(t)}{dt} + ay(t) = f(t) \qquad \text{with } y(0) = Y_0,$$

the solution of this ordinary differential equation with constant coefficient consists of the sum of a complementary solution and a particular solution as

$$y(t) = y_c(t) + y_p(t)$$

where the particular solution is determined by $f(t)$ and the complementary solution is of the form $Ce^{\lambda t}$ with $\lambda = a$ the solution of the characteristic equation and C determined by the initial condition.

Two examples in Class

1. RC circuit discharging

2. RC circuit with applied potential

The first case represents energy decay with the resistor and capacitor in series. Current flows to discharge the capacitor. The form of the current will be

$$i(t) = I(0)e^{-t/RC}.$$

The second case, represents a constant voltage applied to the circuit at $t = 0$. The capacitor with no inital charge storage (voltage) will charge up to the imput voltage V_{in} as

$$V_c = V_{in}(1 - e^{-t/RC}).$$

This is called the *step response* of the RC circuit.

For the passive circuits, the homogeneous solution represent the current or voltage decay as current flows throuch the resistor and energy is dissipated as heat. This solution must go to zero as time increases and is often called the *transient solution*. The "forcing function" $f(t)$ determines the particular solution and hence the time response of the system over time after the transient solution reaches zero for all practical purposes. The particular solution is also called the *steady state solution*. The two solutions are independent but the *complete* solution $y(t)$ represents the sum of the two solutions.

Exponential Inputs to Differential Equations We know that the particular solution to a differential equation with constant coefficients that has a constant forcing function $f(t) = K$ as in Equation 4, has a particular solution that is also a constant, $y_p(t) = K/a$.

The other important case in which the solution is of the same form as the forcing function is if the forcing function is an exponential input of the form $f(t) = e^{\beta t}$ as long as the exponent is not equal to $-a$ in the equation

$$\frac{dy(t)}{dt} + ay(t) = e^{\beta t} \qquad \text{with } y(0) = Y_0.$$

The solution to this equation is then

$$y(t) = Ce^{-at} + De^{\beta t} \quad \text{with} \;\; D = \frac{1}{\beta + a}; \;\; \beta \neq -a.$$

Examples in Class

1. RC circuit $f(t) = e^{i\omega t}$
2. RC circuit with $\sin \omega t$ input.

Special Case for Exponential Inputs

Suppose the solution to the characteristic equation $\lambda = a$ is also the negative of the exponent of the forcing function as

$$\frac{dy(t)}{dt} + ay(t) = ke^{-at} \qquad \text{with } y(0) = Y_0. \tag{5}$$

The equation has the normal homogeneous solution but the particular solution must be changed to yield the complete solution

$$y(t) = Ce^{-at} + K\, te^{-at}$$

with K determined with the particular solution and then C found from the initial condition. In electrical or mechanical terms, the forcing function has the same form as the natural behavior of the system and a phenomenon called *resonance* appears.

Convolution or Integrating Factor Solution Consider the first-order differential equation with a a scalar,

$$\frac{dy(t)}{dt} + ay(t) = f(t). \tag{6}$$

Assuming that the equation is defined on the interval $t \geq 0$, the integrating factor for the equation is

$$e^{\int_0^t a d\tau} = e^{at},$$

where the variable of integration has been changed to τ to emphasize that the integral is a function of t, the upper limit of integration, not the "dummy" variable τ.

The complete solution then takes the form

$$y(t) = \int_0^t f(\tau)e^{-a(t-\tau)}d\tau + ce^{-at} = e^{-at}\int_0^t f(\tau)e^{a\tau}d\tau + ce^{-at}. \tag{7}$$

This solution also works with equations of the form

$$\frac{dy(t)}{dt} + a(t)y(t) = f(t).$$

for which the coefficients are not constant.

Applications The first-order differential equation is used to *model* a number of phenomena in mathematical terms. Some of the problems treated with first-order equations include

1. Growth and decay,
2. Radioactive half life,
3. Carbon dating,
4. Cooling of a body,
5. Series electrical circuits.

In these applications, the mathematical model is formulated as a first-order differential equation since it has been observed that the rate of change of some variable (e.g., population, mass, temperature, etc.) is proportional to the value of the variable. See Harman P208-215 for more details.

Considering Equation 7 in the first form, it is a convolution of the input function $f(t)$ and the function e^{-at}. Thus, the non-homogeneous differential equation

$$\dot{y}(t) + ay(t) = f(t), \qquad y(0) = 0 \tag{8}$$

where a is a constant has the solution

$$y(t) = \int_0^t f(\tau)e^{-a(t-\tau)}\, d\tau. \tag{9}$$

Assume that $f(t) = \delta(t)$ so that using the sifting property of $\delta(t)$ we find

$$y(t) = \int_0^t \delta(\tau)e^{-a(t-\tau)}\, d\tau = e^{-at} \quad t \geq 0$$

as the impulse response of the system. Using this result shows that Equation 9 is the convolution with $f(t)$ and the impulse response of the system. We associate the impulse response of this system with $h(t)$. More details are given in Harman P 445-448.

Higher Order Differential Equations The nth-order differential equation with constant coefficients is treated in this section. The general solution to the homogeneous equation is easily found in principle. This problem of finding n functions to satisfy the homogeneous equation is reduced to finding the n roots of an algebraic equation.

Consider the nth-order, linear differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = f(t), \tag{10}$$

where the a_i are constants. This equation occasionally will be written in operator form as $L_n[y(t)] = f(t)$ to simplify the notation.

Homogeneous Solution First, we find the general solution to the homogeneous equation. As discussed in Chapter 4, the function $y(t) = e^{\lambda t}$ is an eigenfunction of each derivative term in Equation 10. Then, substitution of $e^{\lambda t}$ in the equation $L_n[e^{\lambda t}] = 0$, where L_n is the nth-order linear differential operator previously defined, yields

$$e^{\lambda t}(\lambda^n + \cdots + a_0) = g(\lambda)e^{\lambda t} = 0.$$

Accordingly, $e^{\lambda t}$ will be a solution provided that λ satisfies the algebraic equation of order n, $g(\lambda) = 0$. The equation $g(\lambda) = 0$ is called the *characteristic equation* associated with Equation 10. The roots of the equation

$$g(\lambda) = (\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0) = 0 \tag{11}$$

define the valid values of λ for which $e^{\lambda t}$ satisfies the homogeneous equation. There are n solutions to this polynomial equation.

Distinct Roots Consider first the case where the roots of the characteristic Equation 11 are distinct; that is, no root is repeated. In this case, the general solution of the homogeneous equation with constant coefficients is simply written down as the superposition of exponential terms with the roots of the characteristic equation as the constant in the exponents. The result can be stated as the following theorem.

Theorem (5.3) Solution of $L_n[y(t)] = 0$. *If the n roots of the characteristic equation are all distinct and the roots are $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the solution of the nth-order homogeneous differential equation defined by Equation 10 with $f(t) = 0$ is*

$$y_c(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \cdots + c_n e^{\lambda_n t}, \tag{12}$$

where the designation $y_c(t)$ is used for this complementary solution.

Repeated Roots Suppose it is not possible to find n distinct roots of the characteristic Equation 11. In this case, it can be shown that the solutions of the differential equations for a root λ_k of multiplicity m are of the form

$$e^{\lambda_k t}, \quad te^{\lambda_k t}, \quad t^2 e^{\lambda_k t}, \cdots t^{m-1} e^{\lambda_k t}.$$

To show the result for a root of multiplicity two, we first assume that two roots of the characteristic equation are distinct and then let the roots approach each other to become one root of multiplicity two. Let λ_1 and λ_2 be two distinct real roots of the characteristic equation. Then, the function

$$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2} \tag{13}$$

is a solution of the homogeneous differential equations with constant coefficients. Now assume that the coefficients of the characteristic equation change so that λ_2 tends to λ_1. Equation 13 can be written

$$\lim_{\lambda_2 \to \lambda_1} \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}. \tag{14}$$

Remembering the definition of the derivative from calculus, this limit is the derivative of $e^{\lambda_1 t}$ with respect to λ_1. Therefore, one of the solutions of the differential equation is

$$\frac{d}{d\lambda_1} e^{\lambda_1 t} = te^{\lambda_1 t}$$

when the characteristic equation yields repeated roots.

Summary of Homogeneous Solutions The Table lists the various solutions according to the type of roots of the characteristic equation as real or complex numbers. In any case, there will be a total of n functions $y_1(t), y_2(t), \ldots, y_n(t)$ thus obtained. The general solution of $L[y_c(t)] = 0$ will be

$$y_c(t) = c_1 y_1(t) + c_2 y_2(t) + \cdots + c_n y_n(t). \tag{15}$$

Roots for Homogeneous	*Solutions $y_i(t)$*
Simple real root λ	$e^{\lambda t}$
Simple complex root $a \pm ib$	$e^{at}\cos bt, e^{at}\sin bt$
Real root λ_k of multiplicity k	$e^{\lambda_k t}, te^{\lambda_k t}, \ldots, t^{k-1}e^{\lambda_k t}$
Complex root of multiplicity k	$e^{at}\cos bt, e^{at}\sin bt, \ldots$
	$t^{k-1}e^{at}\cos bt, t^{k-1}e^{at}\sin bt$

Examples:

1. Suppose we have the growth ($k > 0$) or decay ($k < 0$) equation given as

$$Q'(t) = kQ(t) \quad \text{with } Q(0) = Q_0.$$

This homogeneous differential equation can be solved in the form

$$Q'(t) - kQ(t) = 0 \quad \text{with } Q(0) = Q_0.$$

Since the characteristic equation has a simple real root, the homogeneous solution is given by $Q(t) = Ae^{\lambda t}$ where $\lambda - k = 0$ or $\lambda = k$. Thus

$$Q(0) = Q_0 \quad \text{with the result} \quad Q(t) = Q_0 e^{\lambda t}.$$

2. The solution of the equation

$$\frac{d^2 y(t)}{dt^2} + 6\frac{dy(t)}{dt} + 9y(t) = 0 \tag{16}$$

with initial conditions $y(0) = 2$ and $dy(0)/dt = 0$ is $y(t) = 2(1 + 3t)e^{-3t}$. To show this, assume solutions of the form $Ae^{\lambda t}$ and form the characteristic equation

$$\lambda^2 + 6\lambda + 9 = 0 \quad \text{which has roots} \quad -3, -3.$$

Since the roots are repeated with multiplicity 2, the solution becomes

$$y(t) = (C_1 + C_2 t)e^{-3t} \quad \text{with} \quad y(0) = C_1 = 2, \quad \frac{dy(0)}{dt} = -3C_1 + C_2 = 0,$$

with the result

$$y(t) = 2(1 + 3t)e^{-3t}.$$

You should check the result by plugging in the original equation to see this is the solution and also check the initial conditions.

Second-Order Example

Consider the second-order homogeneous differential equation

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0, \tag{17}$$

where b and c are real constants. Forming the characteristic equation

$$\lambda^2 + b\lambda + c = 0 \tag{18}$$

leads to the solutions

$$\begin{aligned} \lambda_1 &= -\frac{1}{2}b + \frac{1}{2}\sqrt{b^2 - 4c}, \\ \lambda_2 &= -\frac{1}{2}b - \frac{1}{2}\sqrt{b^2 - 4c}. \end{aligned} \tag{19}$$

The type of solution can be described in terms of the value of the *discriminant*, $b^2 - 4c$. If the discriminant is zero, the roots are real and equal with value $-b/2$. If the discriminant is positive, the roots are real and distinct. Otherwise, the roots are complex conjugate pairs.

The complete solution to the homogeneous differential equation with constant coefficients can be written

$$\begin{aligned} y(t) &= c_1 e^{-\frac{b}{2}t} \times \exp(+\frac{1}{2}\sqrt{b^2 - 4c}\,t) \\ &\quad + c_2 e^{-\frac{b}{2}t} \times \exp(-\frac{1}{2}\sqrt{b^2 - 4c}\,t). \end{aligned} \tag{20}$$

Equation 20 shows that the form of the homogeneous solution to the second-order differential equation with constant coefficients is determined by the relationship between the coefficients b and c when the initial conditions are nonzero.

We will assume that $b > 0$ and $c > 0$ corresponding to a physical system with passive elements.[1] There are three cases to consider based on the ratio of b^2/c, as follows:

$$\begin{array}{lcll} b^2 & > & 4c & \text{overdamped;} \\ b^2 & = & 4c & \text{critically damped;} \\ b^2 & < & 4c & \text{underdamped.} \end{array} \tag{21}$$

The terms overdamped, critically damped and underdamped refer to the motion of an object modeled by the differential equation of motion previously discussed as Equation **??**.

When $b^2 > 4c$, the solution decays from any initial value to zero with the form

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

[1] It is possible to design physical systems with negative values of the coefficients. An example is an oscillator using positive feedback.

since $\lambda_1 < 0$ and $\lambda_2 < 0$ because the coefficients of the equation are positive. This solution is called the overdamped solution, compared to the solution that occurs when $b^2 = 4c$. Physically, all oscillations in the solution are damped out, so only exponential decay is exhibited.

When the coefficients of the equation are such that $b^2 = 4c$, the solution represents a *critically damped* solution, and the graph of $y(t)$ represents eventual exponential decay. The characteristic equation then has two real equal roots leading to a solution of the form

$$y(t) = (c_1 + c_2 t)e^{-bt/2}.$$

If the coefficient b is reduced at all, the solution will contain oscillations.

The term *underdamped* is applied to the solution that allows an oscillatory solution, but the solution is eventually "damped" and approaches zero. The solution to the second-order homogeneous equation in the underdamped case can be written

$$y(t) = Ce^{-bt/2}\sin(\omega t + \phi),$$

where $\omega = \sqrt{4c - b^2}/2$ and the constants C and ϕ are determined by the initial conditions. In the extreme case that $b = 0$, the system is said to be undamped, and the solution is pure sinusoidal oscillation called harmonic oscillation. This will be studied in a later example.

In summary, the second-order homogeneous differential equation with positive coefficients and nonzero initial conditions has two distinct types of solution depending on the relationship between the coefficients b and c. If $b^2 < 4c$, an underdamped solution results and the solution displays damped sinusoidal oscillation. If $b^2 > 4c$, the solution exhibits exponential decay. The critically damped solution that divides the two types of solutions occurs when $b^2 = 4c$.

Special Case With No Damping Consider the second-order equation with $b = 0$

$$\frac{d^2y}{dt^2} + cy = 0, \tag{22}$$

where c is a real constant. Forming the characteristic equation

$$\lambda^2 + c = 0 \tag{23}$$

leads to the solutions

$$\begin{aligned} \lambda_1 &= +i\sqrt{c}, \\ \lambda_2 &= -i\sqrt{c}. \end{aligned} \tag{24}$$

so that $y_c(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t}$ with the substution of $\omega_n = \sqrt{c}$. The complementary solution is

$$Y_c(t) = A\cos\omega_n t + B\sin\omega_n t = y(0)\cos\omega_n t + y'(0)\sin\omega_n t$$

in which ω_n is called the *natural radian frequency* of oscillation.

MATLAB Solutions to Differential Equations In solving a differential equation and indeed any type of equation, MATLAB gives us the advantage of being able to vary the parameters in the equation and solve the equation repeatedly with different parameters. The changes in the solutions as the parameters are varied are often best shown by plotting the results.

MATLAB Symbolic Differential Equation Solution

Consider the second-order differential equation

$$\frac{d^2y(t)}{dy^2} + b\frac{dy(t)}{dt} + y(t) = 0$$

subject to the conditions

$$y(0) = 1 \quad \text{and} \quad \dot{y}(0) = 0.$$

For this equation, $b = 2$ is the critical damping value since $c = 1$ in Equation 17.

The accompanying MATLAB script solves the general differential equation and plots the solutions for different values of b with $c = 1$. When $b = 3$, the solution in the figure represents overdamped behavior. The underdamped case with $b = 1$ is also shown.

```
% EX5_4.M  Harman Page 224 Solve symbolically the second order equation
%  D2y+b*Dy+c*y=0  and plot for b=1, b=3 with c=1.
%
sym('b')
y = dsolve('D2y+b*Dy+1*y=0','y(0)=1','Dy(0)=0','t');
y=simple(y)                    % Simplify the solution
% Substitute values b=1 and b=3
clf                            % Clear any figures and
hold on                        %  plot multiple graphs
ezplot(subs(y,'b',3.0),[0,10])
gtext('b=3')                   % Annotate text with mouse
ezplot(subs(y,'b',1.0),[0,10])
gtext('b=1')
title('Solution to D2y+b*Dy+y=0, y(0)=1,Dy(0)=0')
ylabel('y(t)')
grid
hold off                        % Default setting
```

The symbolic MATLAB command **dsolve** is used to solve the differential equation subject to the given initial condition. Notice the use of the **subs** command to substitute various values for b. The command **gtext** is used to annotate the graph at points designated by the mouse cursor position.

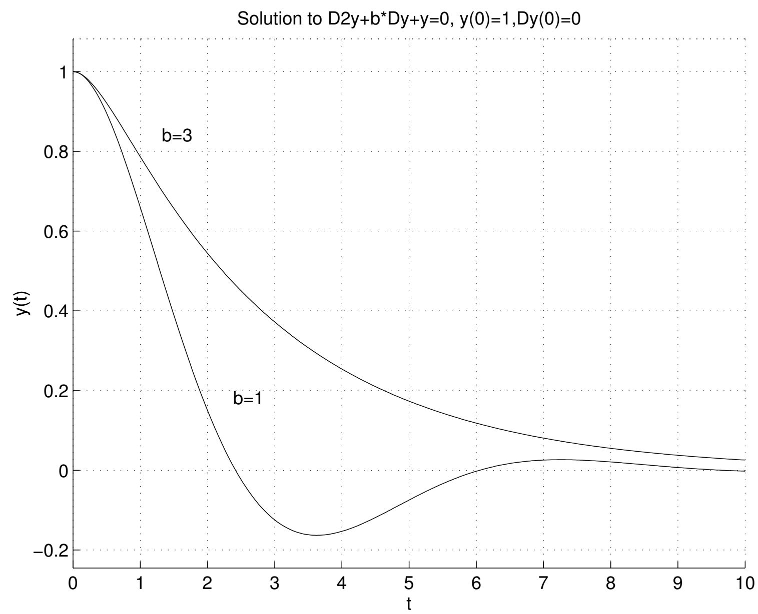

Figure 1: Response of Homogeneous 2nd order System

Particular Solutions of Differential Equations Various techniques are used to find particular solutions to ordinary differential equations. In this section, the techniques of solution described are the following:

1. Undetermined coefficients,
2. Variation of parameters,
3. MATLAB solution.

Method of Undetermined Coefficients A particular solution to the nth-order linear differential equation

$$\mathcal{L}_n[y_p(t)] = f(t)$$

can be found in principle using a number of techniques. When the equation has constant coefficients, as in Equation 10, the simplest approach is the method of *undetermined coefficients*. This amounts to making an educated "guess" as to the form of y_p from the form of the forcing function $f(t)$. This method is appropriate when $f(t)$ is a constant, a polynomial in t, exponential or trigonometric functions of t, or finite sums or products of these functions.

The Table lists the form of $f(t)$ in the left column and the assumed undetermined coefficient solutions in the right column. The solution method consists of substituting the assumed solution in the equation and finding the unknown constants.

$f(t)$	*Choice for* y_p
$Ke^{\alpha t}$	$ce^{\alpha t}$
Kt^n, $(n = 0, 1, \ldots)$	$c_n t^n + c_{n-1}t^{n-1} + \cdots + c_1 t + c_0$
$K \sin \omega t$ or $K \cos \omega t$	$c_1 \cos \omega t + c_2 \sin \omega t$
$Ke^{\alpha t} \cos \omega t$ or $Ke^{\alpha t} \sin \omega t$	$e^{\alpha t}(c_1 \cos \omega t + c_2 \sin \omega t)$

In the table, it is assumed that $e^{\alpha t}$, $\sin \omega t$, or $\cos \omega t$ are not solutions of the homogeneous equation, as described in Equation 15 and the Table of Homogeneous solutions. If the chosen particular solution happens to be a solution of the homogeneous equation $L_n[y(t)] = 0$, multiply the trial particular solution by t or powers of t if the homogeneous solution results from multiple roots of the characteristic equation.

Summary The solution approach using undetermined coefficients can be made more rigorous by stating explicitly the conditions under which a particular solution can be found by this method. The following theorem defines the solutions to Equation 10,

$$L_n[y_p(t)] = f(t),$$

when $f(t)$ has the form of a polynomial times an exponential function.

Theorem for Particular solutions *Consider the nth-order, linear, nonhomogeneous equation with constant coefficients*

$$L_n[y_p(t)] = p(t)e^{\gamma t}, \tag{25}$$

in which $p(t)$ is a polynomial of degree r in t and γ is a complex number. In the following, let $k = 0$ if γ is not a root of the characteristic equation ($g(\gamma) \neq 0$), and let k be the multiplicity of the root γ if $g(\gamma) = 0$. Then, there exists a particular solution of Equation 25 of the form

$$y_p(t) = t^k h(t)e^{\gamma t}, \tag{26}$$

where $h(t)$ is an rth-*degree polynomial. The coefficients of $h(t)$ can be found by the method of undetermined coefficients.*

The theorem covers all the cases just shown in the Table of Undetermined Coefficients for particular solutions and also the case that $f(t)$ would solve the homogeneous equation. The notation $g(\lambda)$ means the characteristic equation

$$g(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0,$$

as in Equation 11.

To interpret Theorem , we consider the cases involving the solutions to the homogeneous equation by the exponential term in the forcing function as follows:

1. $e^{\gamma t}$ does not solve $L_n[e^{\gamma t}] = 0$, so $k = 0$.
2. $e^{\gamma t}$ solves $L_n[e^{\gamma t}] = 0$ and k is the multiplicity of the root γ in the characteristic equation.

Examples of repeated roots with a solution to the complementary equation as a forcing function.

For example, consider the equation with repeated roots and a forcing function with the same exponent

$$L_2[y(t)] = \ddot{y}(t) + 2\dot{y}(t) + y(t) = e^{-t}. \tag{27}$$

Since the characteristic equation is $(\lambda+1)^2 = 0$, a root is $\lambda = -1$, but with multiplicity two. The complementary solution is thus

$$y_c(t) = (c_1 + c_2 t)e^{-t}.$$

The exponential forcing term e^{-t} is a solution of $L_2[y](t) = 0$, $k = 2$ in the theorem, and a solution is sought in the form

$$y_p(t) = t^2 h(t)e^{-t}.$$

Suppose that the polynomial in the forcing function of Equation 25 is of the form

$$p(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0.$$

Then, we search for a polynomial $h(t)$ in the form

$$h(t) = b_n t^n + b_{n-1}t^{n-1} + \cdots + b_0.$$

It is now possible to substitute the solution of Equation 26 into the differential Equation 25 and solve for the coefficients of $h(t)$, treating them as unknowns. The result will be a system of linear equations by equating coefficients of like powers on each side of Equation 25. In Equation 27, $p(t) = 1$ so the degree is $r = 0$. In this case, $h(t) = C$ in the particular solution, where C is a constant to be determined. Thus, the particular solution to the equation

$$\ddot{y}(t) + 2\dot{y}(t) + y(t) = e^{-t}$$

is $y_p(t) = Ct^2e^{-t}$. You should substitute $y_p(t)$ in the differential equation and show that the undetermined coefficient is $C = 1/2$. The complete solution is thus

$$y(t) = y_c(t) + y_p(t) = (c_1 + c_2 t)e^{-t} + \frac{1}{2}t^2e^{-t}.$$

Now apply initial conditions to determine c_1 and c_2. (Harman Page 228).

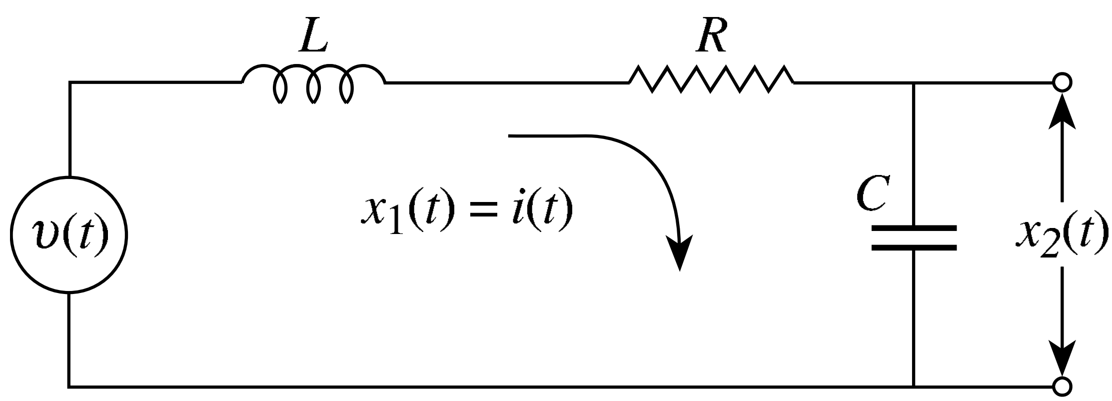

Figure 2: Second Order System

General form of second order equation. Writing

$$\ddot{y}(t) + 2\zeta\omega_n\,\dot{y}(t) + \omega_n^2\,y(t) = \alpha f(t). \tag{28}$$

allows the equation to be a model for any second-order differential equation with constant coefficients if the coefficients are interpreted properly. The characteristic equation

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

has the roots

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{29}$$

The Greek letter ζ (zeta) is used to designate the damping ratio.

An example would be Newton's law for a mass, spring, dashpot system

$$m\ddot{y}(t) + \beta\dot{y}(t) + ky(t) = f(t) \tag{30}$$

which dividing both sides by m yields $\omega_n^2 = k/m$ and

$$\zeta = \text{damping ratio } = \frac{\beta}{2\sqrt{km}},$$

which has the interpretation of the actual damping β coefficient divided by the critical damping value.

In a series RLC circuit, resistance plays the role of damping and inductance is analogous to mass. For this circuit, the electrical analog of Equation 33 takes the form

$$L\ddot{q}(t) + R\dot{q}(t) + \frac{1}{C}q(t) = v(t). \tag{31}$$

In this equation, $q(t)$ is the electrical charge and the current in the circuit is $i(t) = x_1(t) = dq(t)/dt$. If the output voltage $v_0(t) = x_2(t)$ is the voltage across the capacitor, then

$$i(t) = C\frac{dv_o}{dt}$$

and the series equation becomes

$$L\frac{di(t)}{dt} + Ri(t) + v_o(t) = V_{in}(t).$$

Substituting for $i(t)$ as defined by the change in capacitor voltage and dividing by LC yields the equation

$$\ddot{v_o} + \frac{R}{L}\dot{v_o} + \frac{1}{LC}v_o = \frac{1}{LC}v_{in}(t) \tag{32}$$

Second-order parameters. The parameters listed in the Table relate mechanical and electrical systems in terms of damping for a second-order differential equation model in which mass and inductance are analogs. The parameters apply for the mechanical system of Equation 33 and the series (RLC) electrical circuit of Equation 32 that is analogous.

Parameter	*Mechanical*	*Electrical*
ω_n	$\sqrt{\frac{k}{m}}$	$\sqrt{\frac{1}{LC}}$
ζ	$\frac{\beta}{2\sqrt{km}}$	$\frac{R}{2}\sqrt{\frac{C}{L}}$
Critical damping	$\beta^2 = 4km$	$R^2 = 4\frac{L}{C}$
Analog	m, mass	L, inductance

Between the low-damping condition for a system and the overdamped condition, critical damping occurs for the values given in the table. In the mechanical circuit, $\beta = 2\sqrt{km}$. For a series electrical circuit, the critical value of the resistance must be $R = 2\sqrt{L/C}$.

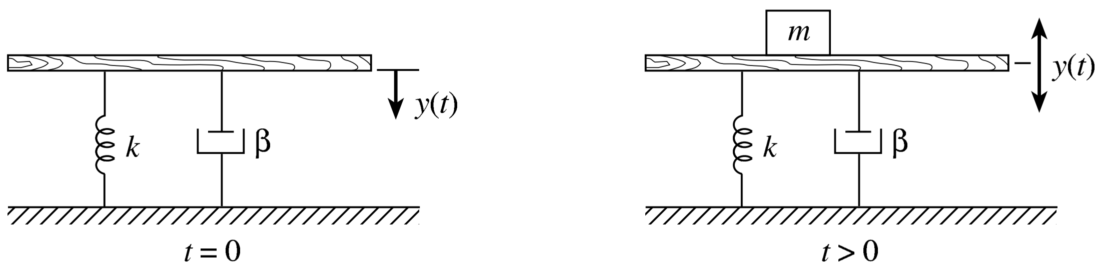

Figure 3: Second Order System

Example with complex roots and step forcing function - Harman P 256

Solutions of Second-Order Equations

In this example, a second-order differential equation will be solved in several ways. It is good practice to study an equation to be solved before applying numerical techniques. If an analytical solution can be found, this serves as a check on the results of the computer solution. The computer solution then might be used to solve the problem with the variables changed to simulate changes in a physical system or to determine the optimum design for a specified purpose. For example, it may be necessary to assure that oscillations do not occur in response to a step disturbance. By selecting the proper amount of damping, a nonoscillatory response can be achieved.

The figure 3 shows a simple physical system consisting of a massless beam supported by a spring and a damper. This system can be modeled by the equation

$$m\ddot{y}(t) + \beta\dot{y}(t) + ky(t) = f(t). \tag{33}$$

The system is assumed to be at rest before $t = 0$, but a mass is placed on the beam at $t = 0$. The problem is to find the motion of the system measured as the motion of the beam and mass $y(t)$ and the final position of the beam after any oscillations have died out.

The response of a system to a forcing function can be divided into a *transient solution* and a *steady-state solution*. Mathematically, the steady-state position is defined by the solution $y(t)$ as t approaches infinity. The transient response of a mechanical system will generally exhibit damped vibrations before the system reaches a steady state. In this example, we search for values of damping β that will eliminate the vibrations as the system responds to a step function input.

As a preliminary design, let the physical components have the following properties

$$\begin{aligned} m &= 1 \text{ kilogram}, \\ \beta &= 4 \text{ newton-seconds/meter}, \\ k &= 40 \text{ newtons/meter}, \end{aligned}$$

where the International System of Units (SI) is used to define the mass m in kilograms, the viscous friction β in newton-seconds/meter, and the spring constant k in newtons/meter. The forcing function is $f(t) = mg$, where g is the force of gravity (9.81 meters per second^2). The solution will be given as displacement measured in meters.

First, we will solve the equation by the traditional methods described in previous sections of this chapter. Dividing the equation by the value of m yields

$$\ddot{y}(t) + 4\dot{y}(t) + 40y(t) = 9.81, \qquad t \geq 0. \tag{34}$$

The characteristic equation is $\lambda^2 + 4\lambda + 40 = 0$, with solutions $\lambda = -2 \pm 6i$. Since the real part of the eigenvalues are negative, the complementary solution exhibits damped sinusoidal oscillation (vibration) decaying to zero with time. Such a system is said to be *stable*. Since there is an imaginary part to the eigenvalues, vibration of the beam will occur with this damping constant.

Since the forcing function is constant, assume a particular solution

$$y_p(t) = K.$$

Because the derivatives of y_p are zero, substituting $y_p(t) = K$ in the equation gives the simple result $40K = 9.81$, or $K = 0.2453$ meters. Thus, the complete solution to Equation 34 has the form

$$y(t) = 0.2453 + e^{-2t}\left[c_1 \sin(6t) + c_2 \cos(6t)\right], \qquad t \geq 0.$$

Forming $y(0) = \dot{y}(0) = 0$ and solving for the constants gives the final result

$$y(t) = 0.2453\left[1 - \frac{1}{3}e^{-2t}\sin(6t) - e^{-2t}\cos(6t)\right]. \tag{35}$$

Notice that the particular solution provides a check on the overall solution. It is required that

$$\lim_{t\to\infty} y(t) = 0.2453$$

as the *steady-state* solution. The nonconstant part of the solution in Equation 35 is the *transient* solution that decays to zero with time as it must in any real system with damping.

```
>> help symbolic
  Symbolic Math Toolbox
  Version 5.10 (R2013a) 13-Feb-2013

  Calculus.
    diff        - Differentiate.
    int         - Integrate.
    limit       - Limit.
    poles       - Poles of a function.
    taylor      - Taylor series.
    symsum      - Summation of series.
    symprod     - Product of series.

  Simplification.
    simplify    - Simplify.
    expand      - Expand.
    factor      - Factor.
    collect     - Collect.
    simple      - Search for shortest form.

    dsolve      - Symbolic solution of differential equations.


odeToVectorField - Convert higher-order ODEs to systems of first-order ODEs.

  Integral Transforms.
    fourier     - Fourier transform.
    laplace     - Laplace transform.
    ztrans      - Z transform.
    ifourier    - Inverse Fourier transform.
    ilaplace    - Inverse Laplace transform.
    iztrans     - Inverse Z transform.

    sym         - Create symbolic object.
    syms        - Short-cut for constructing symbolic objects.
    findsym     - Determine symbolic variables.
    pretty      - Pretty print a symbolic expression.
    latex       - LaTeX representation of a symbolic expression.

    heaviside   - Step function.
    dirac       - Delta function.

rectangularPulse - Rectangular pulse function.
    triangularPulse - Triangular pulse function.
    sign        - Sign function.
```

```
  Trigonometric Functions.
    sin         - Sine function.
    cos         - Cosine function.   Etc.


    ezplot      - Easy to use function, implicit, and parametric curve plotter.

    symcalcdemo - Calculus demonstration.



    Examples:
      >> dsolve('Dx = -a*x')

             ans =  C2*exp(-a*t)


     >>  x = dsolve('Dx = -a*x','x(0) = 1','s')

x = exp(-a*s)    (Note x(0)=1)
```