Harman Outline Pre-Fourier Functions CENG 5131

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Functions - orthogonal 2.10. (Note: For more detail see Section 6.9 P337, Section 7.4, P361) Equation 2.73 and Equation 2.74

Vector Spaces of Functions We now come to perhaps the most important generalizations of vector theory in the chapter as they pertain to techniques in advanced mathematics. Reaching this point has involved extending the theory of vectors in $\mathbf{R^2}$ and $\mathbf{R^3}$ to those in $\mathbf{R^n}$ and then considering abstract vector spaces. The key points to be considered here include

1. Generalizing the dot product for vectors to the inner product for functions;
2. Extending the idea of length of a vector to define the norm of a function;
3. Introducing the concept of expressing a function in terms of a linear combination of orthogonal functions based on the expansion of a vector in terms of the basis vectors for the vector space.

The *inner product* of $f(x)$ and $g(x)$ is defined on the interval $[a, b]$ as the integral of the product

$$\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx. \tag{1}$$

Based on the definition of the inner product, the norm is written as

$$\|f\| = \langle f, f\rangle^{1/2} = \left[\int [f(x)]^2\, dx\right]^{1/2}. \tag{2}$$

If the inner product of two nonzero functions is zero, the functions are said to be *orthogonal*.

EXAMPLE of Sinusoids

The functions $\sin x$ and $\cos x$ are orthogonal over intervals such as $[0, \pi]$ or $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} \sin kx \sin mx\, dx = \int_{-\pi}^{\pi} \cos kx \cos mx\, dx = 0 \tag{3}$$

for $k \neq m$ and k, m integers. The inner products of $\sin kx$ and $\cos kx$ are also orthogonal over a period.

Also note that

$$\int_0^T \sin^2 mx\, dx = \int_0^T \cos^2 mx\, dx = \frac{T}{2}. \tag{4}$$

This indicates that $\sin nx$ or $\cos nx$ could be normalized to form an orthonormal set since

$$\left\langle \frac{\cos nx}{\sqrt{T/2}}, \frac{\cos nx}{\sqrt{T/2}} \right\rangle = 1.$$

The result for $\langle \sin nx, \sin nx \rangle$ is the same. Thus, the factor $1/\sqrt{T}$ is used to normalize the functions.

Now we come to the main point.

Suppose we wish to expand a function $f(x)$ in terms of simpler (we hope) functions. Suppose a function $f(x)$ is continuous on the interval $[a, b]$. Then, following an approach similar to that of expanding a vector in terms of orthonormal basis vectors, we postulate that $f(x)$ can be expressed as

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x), \tag{5}$$

where the coefficient c_m is determined as

$$c_m = \int_a^b f(x)\phi_m(x)\, dx. \tag{6}$$

Assume that $f(x) = \sum_{k=0}^{n} c_k \phi_k$, then if the function is evaluated at $n+1$ points

$$\begin{aligned}
f(x_0) &= c_0\phi_0(x_0) + c_1\phi_1(x_0) + \cdots + c_n\phi_n(x_0) \\
f(x_1) &= c_0\phi_0(x_1) + c_1\phi_0(x_1) + \cdots + c_n\phi_n(x_1) \\
&\vdots \\
f(x_n) &= c_0\phi_0(x_n) + c_1\phi_1(x_n) + \cdots + c_n\phi_n(x_n)
\end{aligned}$$

which leads to an $(n+1) \times (n+1)$ system of equations to solve for the $n+1$ c_i values. See Harman Section 3.6, P124 for solution methods. Assuming that the ϕ_i are orthonormal, integrating both sides of the equations over a suitable interval leads to the coefficients as

$$\begin{aligned}
\int_a^b f(x)\phi_0(x)dx &= c_0 \\
\int_a^b f(x)\phi_1(x)dx &= c_1 \\
&\vdots \\
\int_a^b f(x)\phi_n(x)dx &= c_n
\end{aligned}$$

since the only terms left on the right hand side of the equation array are of the form

$$\int_a^b f(x)\phi_m(x)dx = \int_a^b c_m\phi_m^2(x)dx = c_m$$

Considering Harman Page 362 in Chapter 7, it can be shown that the coefficients in Equation 6 are the *Fourier coefficients*. The Fourier coefficients yield a smaller square error than any other choice of coefficients in the approximation using a linear combination of the ϕ_i's. This fact is stated in Theorem 7.1, Harman page 362.

Example Legendre Polynomials

The Legendre polynomials are designated $P_n(x)$, with the following properties:

1. $P_n(x)$ is a polynomial of degree n.
2. $P_n(1) = 1$ for each n.
3. The set of polynomials is orthogonal on the interval $[-1, 1]$.

It can be shown that these properties uniquely determine the $P_n(x)$ and that the resulting polynomials are the *Legendre polynomials* introduced in Harman Chapter 6 as the solutions to Legendre's differential equation. The results presented there showed that $P_n(x)$ can be derived as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n, \tag{7}$$

where n is an integer. The first six Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1, & P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\ P_1(x) &= x, & P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), \\ P_2(x) &= \frac{1}{2}(3x^2 - 1), & P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x). \end{aligned}$$

The Legendre polynomials with even-numbered subscripts are even functions and the polynomials with odd subscripts are odd functions. The polynomials can be calculated by the recursion relation

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x), \tag{8}$$

and the square of the norm of $P_n(x)$ is

$$||P_n(x)||^2 = \int_{-1}^{1} P_n^2(x)dx = \frac{2}{2n + 1}. \tag{9}$$

Legendre Series The *Legendre series*, sometimes called the Fourier-Legendre series, to approximate $f(x)$ on the interval $[-1, 1]$ has the form

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x). \tag{10}$$

Using Equation 9 and the orthonormal properties of the polynomials results in the Fourier (Legendre) coefficients,

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x)\, dx \tag{11}$$

If $f(x)$ is square integrable on the interval $[-1, 1]$, the series will converge to $f(x)$. The next section presents convergence and other properties of such series.

Legendre Series Example. Consider expanding the function $f(x)$ defined as

$$f(x) = \begin{cases} -1, & -1 \le x < 0, \\ 1, & 0 < x \le 1, \end{cases}$$

in a Legendre series of the form

$$f(x) \approx c_0 + c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x).$$

We assume that $f(0) = [f(0^-) + f(0^+)]/2 = 0$. This is convergence in the mean at a point of discontinuity.

Since $f(x)$ is odd, we immediately set $c_0 = c_2 = 0$. Then, integrating in Equation 11 over the half-integral yields

$$\begin{aligned} c_1 &= 2 \cdot \frac{3}{2} \int_0^1 1 \cdot x\, dx = \frac{3}{2}, \\ c_3 &= 2 \cdot \frac{7}{2} \int_0^1 1 \cdot \left(\frac{5}{2} x^3 - \frac{3}{2} x \right) dx = -\frac{7}{8}. \end{aligned}$$

The resulting series is thus

$$\begin{aligned} f(x) &\approx \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) \\ &= \frac{45}{16} x - \frac{35}{16} x^3. \end{aligned}$$

Power in a Signal. It is an important result of alternating current theory that the power associated with a periodic wave of voltage or current $f(t)$ with period T is proportional to the mean-square value of $f(t)$. The mean-square formula is

$$\overline{f^2(t)} = \frac{1}{T} \int_{-T/2}^{T/2} [f(t)]^2\, dt, \tag{12}$$

which is seen to be the average of the square of $f(t)$. For a pure sinusoid, the average value of its square is one-half the peak value. The *root-mean-square* or *rms* value for the sinusoid $V_0 \sin \omega t$ is thus

$$V_{rms} = \sqrt{\frac{V_0^2}{T} \int_0^T \sin^2 \omega t dt} = \frac{V_0}{\sqrt{2}}.$$

Thus, a 120 volt ac line has a peak voltage of $V_0 = \sqrt{2} \times 120 \approx 170$ volts. The rms value indicates the value that a direct-current voltage would have to cause the same amout of heating to a resistor. It is sometimes called the *effective value* of the voltage.

The instantaneous value $v(t)i(t)$ in a circuit has units volt-amperes and represents both the *real* and *reactive* components of the result. In terms of the power in a resistive circuit in which the voltage and current are in phase, the reactive component is zero and the real power is

$$\langle P \rangle = V_{rms}\ I_{rms}$$

usually measured in watts. If the voltage and current are out of phase by an angle, say ϕ, the power becomes

$$\langle P \rangle = V_{rms}\ I_{rms}\ \cos\phi$$

with the term $\cos\phi$ called the *power factor* that ranges from 0 to 1. Notice that when the voltage and current are orthogonal, the power is zero indicating that no work is done since charging and discharging an ideal capacitor in a complete cycle, for example, is conservative work.