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1 Z TRANSFORMS S-PLANE

1.1 THE Z-PLANE AND THE S-PLANE

From the relationship

$$z = e^{sT_s} = e^{(\sigma+i\omega)T_s} = e^{\sigma T_s} e^{i\omega T_s} \quad (1.1)$$

we map the s-plane into the z-plane. The $i\omega$ axis maps into the unit circle

$$z = e^{\imath\omega T_s}$$

which has magniude $|z| = 1$. The values of $i\omega T_s$ determine the position on the circle. As the argument increases in the positive direction , points on the circle wrap around starting at $z = 1$ when $\omega T_s = 0$ corresponding to $\omega = 0$ in the s-plane. At the angle $\omega T_s = \pi$, $z = -1$. The region of the $j\omega$ in the s-plane axis from $\omega = 0$ to $-\omega T_s = -\pi$ map to the lower half of the unit circle and again $z = -1$ when $-\omega T_s = -\pi$.

In terms of sampling theory, the limits used to preserve the uniqueness of the mapping correspond to the Nyquist frequencies $\omega_s = \pm\pi/T_s$.

DAMPING AND THE Z-PLANE

The left-hand side of the s-plane, for values $s = \sigma + i\omega$ with $\sigma < 0$ and $|\omega| < \pi/T_s$ maps into the interior of the unit circle in the z-plane. Since poles in the left-hand s-plane correspond to a BIBO stable continuous system, the corresponding poles for stable discrete systems must lie within the unit circle in the z-plane. Note that the negative real axis in the s-plane maps into the real axis from 0 to 1 in the z-plane. Thus, a digital system with a pole at -0.5, for example, has no corresponding continuous system. (Shahian p 263).

If $\sigma > 0$, the points in the right-hand s-plane map to the exterior of the unit circle in the z-plane.

Vertical lines in the s-plane such that $\pi/T_s \leq \omega \leq \pi/T_s$ and $\sigma < 0$, map into a circle in the z-plane centered at $z = 0$ with radius $r = \exp \sigma T_s$.

☐ EXAMPLE 1.1 ***Mapping the s-plane to the z-plane***

Consider the s-domain function

$$G(s) = \frac{1}{(s+1)(s+2)(s^2+1)}$$

with poles at $s = -1, -2, \pm i$. For $T_s = 1$, the poles in the z-plane given by $\exp(sT_s)$ appear at

$$z = 0.3679, 0.1353, 0.5403 + 0.8415i, 0.5403 - 0.8415i$$

as computed by the MATLAB script below and shown in Figure 1.1.

The inverse Laplace transform of $G(s)$ leads to time functions such as e^{-t},e^{-2t} and $e^{\pm it}$ or $\cos t$ and $\sin t$. Thus, the oscillations have frequency $f = 1/2\pi$ Hertz or 1 rad/sec. With $T_s = 1$, $\omega_s = 2\pi$ and the maximum digital frequencies are $F = .5$ or $\Omega = \pi$ radians according to Equation **??**.

In the z-plane, the pole at $0.5403 + 0.8415i$ has magnitude 1 since it lies on the unit circle and angle

$$\theta_z = \tan^{-1} \frac{0.8415}{0.5403} = 1 \text{ radian.}$$

With $T_s = 1$, the maximum digital frequency $\Omega = \pi$ rad occurs at the point $z = -1$.

If $T_s = 0.1$, the poles in the z plane are changed as indicated in the results of the MATLAB calculation. The maximum digital frequency is $F = 5$ or $\Omega = 10\pi$ radians. The angle of the pole $z = 0.9950 + 0.0998i$ is $\theta_z = 0.1000$ radians as expected.

```
Example 1.1
%s2zplane.m
% Plot z-plane poles for G(s)=1/[(s+1)(s+2)(s^2+1)
% See Taylor p252
%
% Let Ts=1.0
Ts=1.0
poless=[-1 -2 +i -i]
polesz=exp(poless*Ts)
%
% Define zeros and poles as column vectors
zplane(polesz')      % There are no zeros
title('Z-plane for s-plane poles -1,-2,+1=-i')
grid
%
% Results
%
%Ts =  1
%poless =-1.0000    -2.0000      0 + 1.0000i        0 - 1.0000i
%polesz =0.3679      0.1353 0.5403 + 0.8415i   0.5403 - 0.8415i
%
% Change sampling time
%
%Ts1 = 0.1000
Ts1=0.1
poless1=[-1 -2 +i -i]
polesz1=exp(poless1*Ts1)
% poless1 =-1.0000  -2.0000      0 + 1.0000i        0 - 1.0000i
% polesz1 = 0.9048   0.8187   0.9950 + 0.0998i   0.9950 - 0.0998i
```

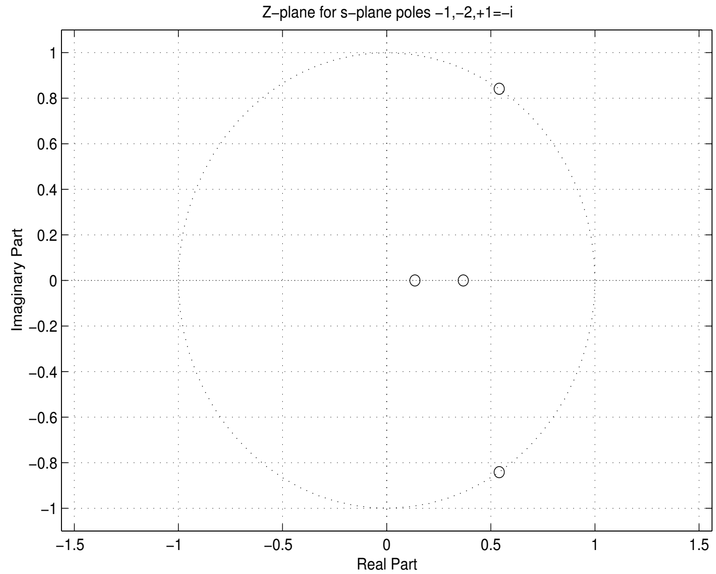

FIGURE 1.1 *z-plane poles from s-plane*

□

Second-order systems Consider the second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with poles at

$$s_1 = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \qquad s_2 = -\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2}.$$

The term $\sigma = -\zeta\omega_n$ is the real part of a pole in the s-domain that corresponds to damping of the time response of the system $G(s)$. The pole in the z-plane lies on a circle centered at $z = 0$ with radius

$$|z| = \exp(-\zeta\omega_n T_s).$$