Using Equation 5.78, the complete solution becomes

$$\Phi(t)\mathbf{c} + \mathbf{x}_p(t) = \begin{bmatrix} 2e^t + te^t \\ \dfrac{te^t}{2} + \dfrac{5e^t}{4} + \dfrac{e^{3t}}{12} - \dfrac{1}{3} \end{bmatrix}.$$

Why are the terms like te^t necessary in the solution? □

HARMAN

5.10 TRANSFORMING DIFFERENTIAL EQUATIONS

In this section, the nth-order linear differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y(t)}{dt^{n-1}} + \cdots + a_1(t)\frac{dy(t)}{dt} + a_0(t)y(t) = f(t) \quad \textbf{(5.79)}$$

with initial conditions

$$y(t_0), \dot{y}(t_0), \ldots, y^{(n-1)}(t_0)$$

will be converted into a set of n first-order differential equations. The main reason for the conversion to a first-order system is that a number of methods are available to solve the problem completely using numerical techniques. Under fairly general conditions, we are certain that a solution to Equation 5.79 exists and is unique.

■ THEOREM 5.8

Uniqueness

Let the coefficients $a_i(t), i = 0, 1, \ldots, n-1$ and $f(t)$ be continuous in some common interval containing the point t_0. If $y(t)$ is found that satisfies the equation and the initial conditions for

$$y(t_0), \dot{y}(t_0), \ldots, y^{(n-1)}(t_0),$$

the resulting solution is unique.

■

As has been the case for the other equations studied in this chapter, the problem will be first formulated assuming the coefficients in Equation 5.79 are variable. Then, the equation with constant coefficients will be studied.

To convert Equation 5.79 into a first order system, we replace the variable $y(t)$ and its derivatives by n new variables defined as follows:

$$\begin{aligned} x_1(t) &= y(t), \\ x_2(t) &= \dot{x}_1(t) = \dot{y}(t), \end{aligned}$$

$$
\begin{aligned}
x_3(t) &= \dot{x_2}(t) = \ddot{y}(t), \\
&\vdots \\
x_n(t) &= \dot{x}_{n-1}(t) = y^{(n-1)}(t).
\end{aligned}
$$

The key to writing the n first-order equations is to notice that

$$
\dot{x}_i(t) = x_{i+1}(t), \qquad i = 1, 2, \ldots, n-1,
$$

and that $\dot{x}_n(t) = y^{(n)}(t)$. Rearranging the nth-order equation yields the last equation in terms of the coefficients and the forcing function $f(t)$ in the form

$$
\begin{aligned}
\dot{x}_n &= y^{(n)} \\
&= -a_{n-1}(t)y^{(n-1)} - a_{n-2}(t)y^{(n-2)} - \cdots - a_1(t)\dot{y} - a_0(t)y + f(t) \\
&= -a_{n-1}(t)x_n - a_{n-2}(t)x_{n-1} - \cdots - a_1(t)x_2 - a_0(t)x_1 + f(t).
\end{aligned}
$$

In matrix form, the equations become

$$
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -a_0(t) & -a_1(t) & \ldots & -a_{n-1}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} f(t).
$$

These equations can be written as

$$
\dot{\mathbf{x}} = A(t)\mathbf{x} + \mathbf{f}(t), \tag{5.80}
$$

where $\mathbf{f}(t)$ is understood to be the scalar function $f(t)$ multiplied by the column vector with zero entries except for the last entry which is 1.

Thus, the system of Equation 5.80 is equivalent to Equation 5.79 in the sense that if $\mathbf{x}$ is a solution of Equation 5.80, then the first component $x_1(t) = y(t)$ is a solution of Equation 5.79. The initial conditions result in the equation $\mathbf{x}(t_0) = \mathbf{c}$, where $\mathbf{c}$ is the constant vector

$$
\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y(t_0) \\ \dot{y}(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{bmatrix} = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix}.
$$

☐ EXAMPLE 5.15 ***Reduction of a Second-Order Equation***

Consider the second-order equation

$$
m\ddot{x}(t) + b\dot{x}(t) + kx(t) = f(t). \tag{5.81}
$$

Using the principles just defined, set $x_1(t) = x(t)$ and $x_2(t) = \dot{x}(t)$, so the first-order system becomes

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= -\frac{k}{m}x_1(t) - \frac{b}{m}x_2(t) + \frac{f(t)}{m}.
\end{aligned}$$

The matrix equation is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{f(t)}{m}.$$

This method could be applied to convert the second-order equations that were discussed in Example 5.3 and Example 5.6.

If the system represented by Equation 5.81 is a mechanical system, as previously discussed, we associate $x(t) = x_1(t)$ as the position of a mass. Then, the variables $\dot{x}_1(t)$ and $x_2(t)$ represent the velocity. Thus, the first matrix equation defines the velocity of the mass. The second equation is the acceleration.

Assuming that m, b, and k are positive values, the eigenvalues of the matrix determine the frequency of oscillation (or rate of decay) of the complementary solution. If $b = 0$, there is no damping, and the frequency of the harmonic motion caused by any initial conditions would be $\omega = \sqrt{k/m}$.

□

WHAT IF? Compare Example 5.6 with Example 5.13. These two examples solve the second-order differential equation without damping.

SOLUTION METHODS

If the equation has constant coefficients, the homogeneous solution of $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$ could be found by using the eigenvalues and eigenvectors for the matrix of coefficients, as previously discussed. A particular solution to the equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ could also be found by various methods presented earlier in the text.

Another approach is to use an algorithm to perform numerical integration to solve $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{f}(t)$. As stated before, MATLAB employs a technique called *Runge-Kutta* integration to solve such problems. The details of the algorithm will be discussed in Chapter 6.

Linear differential equations were introduced in this chapter by writing the equation as an nth-order differential equation. When the coefficients are constant, the homogeneous solution was found by first solving for the roots of the nth degree characteristic equation. These roots led to the homogeneous solution as a linear combination of exponential functions with n arbitrary constants. The particular solution, called the *forced solution*, could be found by various methods, such as undetermined coefficients or variation of parameters. The complete solution is formed by adding the homogeneous solution and a particular solution.

It is important to remember that the values of the arbitrary constants that occur in the homogeneous solution must be determined by applying

A typical example of a linearized equation is Ohm's approximation describing the voltage E across the terminals of a resistor due to a current I flowing through the resistor. For a constant current,

$$E = RI,$$

where R is the resistance. Mathematically, the voltage is proportional to current, with R as the proportionality constant. In fact, the resistance of resistors made of carbon and similar substances varies with temperature. The temperature of the resistor is, in turn, a function of the current flowing through it since the power involved is $P = I^2 R$ and the temperature increases with the power. Fortunately, the dependence of resistance on current is slight in most electronic devices as long as the current does not cause an excessive temperature rise in the resistor. Therefore, a linear relationship between E and I is justified in many cases.

Even though Ohm's approximation is usually called Ohm's law, such approximations are *not* fundamental in physics, as are Newton's laws. Newton's laws are a general statement about nature. Ohm's law is an approximation to the behavior of resistors subject to a limited range of voltage and current.[3]

Hooke's law is a fundamental approximation in mechanics. A spring with a restoring force governed by the equation $F = kx$ is called a *linear spring* since the displacement is proportional to the force involved. An actual spring force may be governed by a more complicated equation, such as

$$F(x) \approx kx + k_1 x^3,$$

where a cubic nonlinear term is used since we expect that $F(x) = -F(-x)$ if the spring is symmetric. Notice that the motion of the spring will be approximately linear if $k_1 x^3 << kx$. If either k_1 or the displacement x is small enough, the second term can be neglected compared with the linear term. The expression $dF = k\,dx + 3k_1 x^2\,dx$ shows the dependence of F on displacement for small displacements dx.

☐ EXAMPLE 12.6 ***Linearization***

Suppose the acceleration of an automobile can be described by the equation of motion

$$M\frac{dv(t)}{dt} = cu(t) - \alpha v^2(t), \tag{12.33}$$

where the first term represents the acceleration caused by the engine at a throttle setting u and the second term is the drag caused by air resistance. Since this force is proportional to the square of the speed, the equation is nonlinear. Solving this by numerical techniques would not be difficult if the constants

[3]The purist may comment that Newton's laws are also approximations. This is so, but the range of applicability is enormous. Only at velocities approaching the speed of light (relativity), in the quantum world (quantum mechanics), or in other very special cases would we abandon Newton's laws and substitute a more complicated theory.

were known. However, we can analyze the motion of the car for small changes in throttle position by linearizing the equation.

Take (U_0, V_0) to be the "operating point," so that the car travels at a constant speed V_0 for a constant throttle position U_0. Inserting this condition into Equation 12.33 yields

$$M\frac{dV_0}{dt} = cU_0 - \alpha V_0{}^2(t) = 0,$$

or $V_0 = \sqrt{cU_0/\alpha}$. Let us assume that a small change in the throttle position leads to a small change in speed. Thus, we set

$$u(t) = U_0 + \Delta u, \quad v(t) = V_0 + \Delta v$$

and substitute again into the equation of motion. The result is

$$M\frac{d}{dt}[V_0 + \Delta v] = c[U_0 + \Delta u] - \alpha[V_0 + \Delta v]^2.$$

Using the result that $cU_0 = \alpha V_0^2$ and expanding the terms but neglecting the second-order term $-\alpha\Delta v^2$ leads to the approximation

$$M\frac{d}{dt}[\Delta v] \approx c[\Delta u] - 2\alpha V_0 \Delta v.$$

Writing $\Delta v = v_a(t)$ and $\Delta u = u_a(t)$ to indicate the linearized variables, the original differential equation of motion described by Equation 12.33 becomes the linear differential equation

$$M\frac{dv_a(t)}{dt} + 2\alpha V_0 v_a(t) = cu_a(t). \tag{12.34}$$

This is a linear differential equation with constant coefficients, which can be solved by the techniques presented in Chapter 5.

□

12.4 TWO-DIMENSIONAL TAYLOR SERIES

The notion of sequences and series of functions of a single variable as described in Chapter 6 can be extended to functions of several variables. For example, the power series expansion for a function of two variables $F(x, y)$ is

$$F(x,y) = \sum_{n=0}^{\infty} f_n(x,y) = f_0(x,y) + f_1(x,y) + \cdots + f_n(x,y) + \cdots, \tag{12.35}$$

with the terms

$$f_n(x,y) = c_{n,0}\, x^n + c_{n,1}\, x^{n-1}y + \cdots + c_{n,n-1}\, xy^{n-1} + c_{n,n}\, y^n.$$

Control of Mobile Robots | Week 3

https://www.youtube.com/watch?v=Tsc5q-jQwfY

State Space – Play first about 10 Minutes.

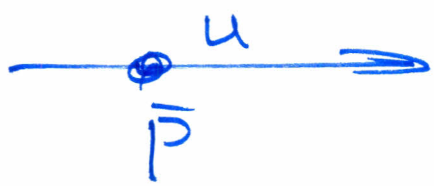
Control of Mobile Robots
Dr. Magnus Egerstedt Professor School of Electrical and Computer Engineering
Module 3 Linear Systems

https://kot.rogacz.com/Science/Studies/14/Conrob/lecture/Module3.pdf youtube.com/watch?v=Tsc5g-sQwt4

MAGNUS Lecture 3

YOUTUBE week 3

ABOUT 5-6 min in video



$$\ddot{\bar{p}} = u$$

$x_1 = p \qquad \dot{x}_1 = x_2 \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$x_2 = \dot{p} \qquad \dot{x}_2 = u$

$$\dot{\vec{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = p = x_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}$$

$$\boxed{\text{So} \quad \begin{aligned} \dot{\vec{x}} &= A\bar{x} + B\bar{u} \\ y &= Cx \end{aligned}}$$

Then 2d $\bar{p} = (p_x, p_y)$

So x_1, x_2, x_3, x_4

Do same

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$