HARMAN ADU ENGR MATH

The relationship between vector spaces and linearity for the points in $\mathbf{R}^2$ is that if a set of points represented by vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ all lie on one line through the origin, any linear combination of them lies in the same line since they are all multiples of the same vector. Also, the zero vector is in the set. We say that a subset S of a vector space V is a *linear subspace* if every linear combination of elements of S is also in S. Usually, the linear subspace is simply called a *subspace*, as was the case in our discussions in Chapter 2.

TRANSFORMATIONS IN THE PLANE AND THREE-DIMENSIONAL SPACE

In computer graphics, a typical problem is to display a view of a 2D or 3D object on a video screen. Using matrix algebra, new views of the object can be generated by rotation, translation, and scaling. Describing the motion (kinematics) of a robot manipulator is an important problem in robotics. Matrices can be used to define the position and orientation of the manipulator at any time with respect to the coordinates of the robot's reference frame.

In general, matrices can be used to allow points and vectors to be rotated about coordinate axes, translated in space, and referenced relative to other reference frames. As shown in a later chapter, we may also use matrices to map the coordinates of spherical or cylindrical reference frames into xyz-coordinates (Cartesian space), or vice versa.

Rotations in the Plane In Figure 3.2, a vector $\mathbf{x} = [x, y]$ is rotated through the angle θ to become the vector $\mathbf{x}' = [x', y']$.

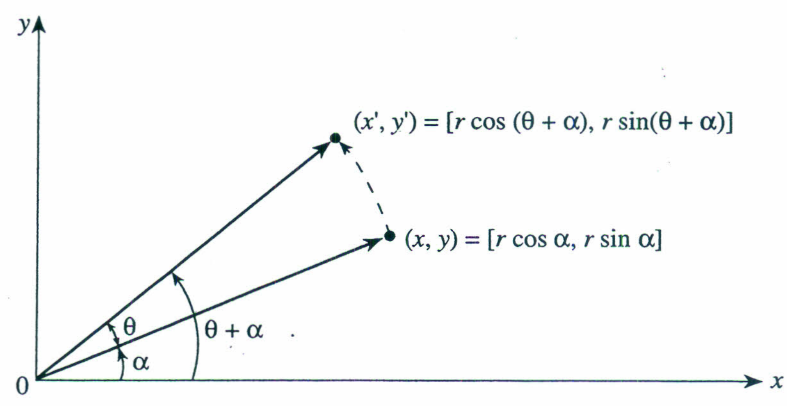

FIGURE 3.2 *Rotation of a two-dimensional vector*

The length r of the vector is not changed by rotation. From the geometry, the coordinates of $\mathbf{x}$ in terms of the angles involved are

$$x = r\cos\alpha \quad \text{and} \quad y = r\sin\alpha.$$

It is desired to obtain the coordinates of the rotated vector in terms of x, y, and the angle θ. Thus, the rotated vector $\mathbf{x}'$ in Figure 3.2 has coordinates

$$\begin{aligned} x' &= r\cos(\theta+\alpha) = r\cos\theta\cos\alpha - r\sin\theta\sin\alpha, \\ y' &= r\sin(\theta+\alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha. \end{aligned}$$

Thus, the relationship between the endpoint coordinates is

$$\begin{aligned} x' &= x\cos\theta - y\sin\theta, \\ y' &= x\sin\theta + y\cos\theta. \end{aligned}$$

Defining the 2D *rotation* matrix as

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \tag{3.28}$$

we see that

$$R_\theta \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

by multiplying $\mathbf{x}^T$ by R_θ. If the angle θ is changed to $-\theta$, the sign of the off-diagonal terms are changed. Of course, the rotation matrix should become the identity matrix if $\theta = 0°$.

Notice that in these operations, the vectors are considered 2×1 column vectors. Also, the series of operations $R_{\theta_1} R_{\theta_2} \mathbf{x}^T$ results in a rotation by angle $\theta_2 + \theta_1$. If the order of matrix multiplications is reversed, the rotation angle remains the same. This is always the case when a series of rotations of a vector from the origin are made in the plane.

Three-dimensional Rotations Figure 3.3 illustrates the 3D reference frame we will use for rotations.

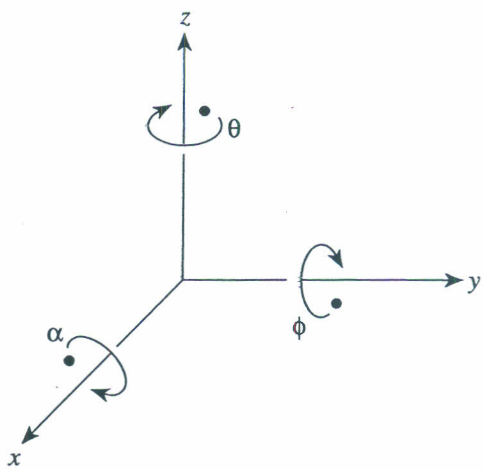

FIGURE 3.3 *Cartesian reference frame in three dimensions*

In three dimensions, the axis of rotation must be specified. The subscripts for a rotation matrix will indicate the axis of rotation and the angle. The rotation matrices for the three axes are as follows:

1. Rotation by an angle α about the x-axis:

$$R_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}; \tag{3.29}$$

2. Rotation by an angle ϕ about the y-axis:

$$R_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}; \tag{3.30}$$

3. Rotation by an angle θ about the z-axis:

$$R_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.31}$$

When performing a series of 3D rotations about several axes by matrix multiplications, the order in which the rotations are performed is important, since the rotation matrices do not commute in general. Except for special cases, applying the rotation matrices to a vector in different order will generate a different result. As a check on the results, the rotation matrices should become identity matrices when the angles involved are set to zero. Also, the rotation matrices are orthogonal matrices, so the columns (or rows) must be orthonormal.

Homogeneous Transformations (Optional) The 3×3 rotation matrix does not provide for translation or scaling of a vector. The concept of *homogeneous coordinate representation* is introduced to develop matrix transformations that include rotation, translation, scaling, and perspective transformation. Using homogeneous transformations, the transformation of an n-dimensional vector is performed in an $(n+1)$-dimensional space.

In homogeneous transformations, the true vectors in $\mathbf{R}^3$ are written as vectors in $\mathbf{R}^4$ with a scaling factor as the last component. Let the 3D column vector be defined as

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Then, the homogeneous representation is

$$\mathbf{x}_h = \begin{bmatrix} sx \\ sy \\ sz \\ s \end{bmatrix},$$

where s is a numerical scaling value. We will assume that this scaling factor is 1 for our present purposes. The rotation matrices that multiply this vector must be 4×4. This is accomplished by adding another row and column to the rotation matrices previously defined. For example, the homogeneous rotation matrix about the z-axis is

$$Rh_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A homogeneous translation matrix in effect adds a translation vector to the vector being transformed. Assume that the translation values are $[t_1, t_2, t_3]$. Then, a homogeneous translation matrix has the form

$$T = \begin{bmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

☐ **EXAMPLE 3.19** *MATLAB Homogeneous Transformations*

Linear transformations are easily accomplished using MATLAB. The accompanying scripts show the M-file (CLXROTZ.M) as well as the function (`clxrotzf`) that perform a homogeneous transformation to rotate a 3D vector by an arbitrary angle in degrees around the z-axis. A test case is shown from the diary file created when the M-file was executed. The test vector is input as [1 2 3] and the rotation angle is 60°. The diary file showing the results is included with the M-file script.

MATLAB Script

```
Example 3.19
% CLXROTZ.M  Rotate a vector around the z axis
%  Input the vector [x y z] and the angle in degrees.
%  Function clxrotzf is called to perform rotation
v1=input(' Vector [x y z]= ')
theta=input(' Input rotation angle (degrees)= ')
v11=[v1 1]';                   % Form homogeneous vector
vrotz=clxrotzf(v11,theta);  % Rotate
fprintf('Rotated vector\n')
vrotz                          % Display result
%
% -------------------------------------------
```

```
% Results (Rotation matrix and rotated vector}
%
>>clxrotz
Vector [x y z]= [1 2 3]

v1 =
     1     2     3

 Input rotation angle (degrees)= 60

theta =
    60
Rotated vector
vrotz =
   -1.2321
    1.8660
    3.0000
    1.0000
>>quit
```

The function clxrotzf rotates the vector passed to it by the specified angle in degrees about the z-axis. The homogeneous transformation yields a 4×1 vector as the result. In space, the vector $[1, 2, 3]^T$ is rotated to a new vector with the coordinates

$$\begin{bmatrix} -1.2321 \\ 1.8660 \\ 3.0000 \end{bmatrix}.$$

MATLAB Script

```
Example 3.19
function yh3rotz=clxrotzf(xto_rot,theta_rot)
% CALL: yh3rotz=clxrotzf(xto_rot,theta_rot)
%  Rotate the vector xto_rot by the angle theta_rot
%    around the z axis
%  xto_rot must be a 4 x 1 column vector, theta_rot in degrees.
theta_rot=theta_rot*pi/180;         % Convert to radians
Arotz=[cos(theta_rot) -sin(theta_rot) 0 0;sin(theta_rot)...
 cos(theta_rot) 0 0
0 0 1 0;0 0 0 1];
yh3rotz=Arotz*xto_rot;
```

□

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functions to be studied in this section and in later chapters are mostly vectors defined on $\mathbf{R}^2$ or $\mathbf{R}^3$. In $\mathbf{R}^3$, the vector function would have the form

$$\mathbf{F}(x_1, x_2, x_3) = [\,F_1(x_1, x_2, x_3), F_2(x_1, x_2, x_3), F_3(x_1, x_2, x_3)\,]$$

written in terms of the components. As a vector in rectangular coordinates, $\mathbf{F}$ could be written

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k},$$

using the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ defined in Chapter 2. Such vector functions will now be studied in the cylindrical and spherical coordinate systems.

CURVILINEAR COORDINATES

Many formulas in engineering and physics can be simplified by choosing the most convenient system of coordinates. Mathematically, the coordinates of the formula may originally be designated as $(\,x_1, \ldots, x_n\,)$ and a transformation is sought to assign new coordinates $(\,u_1, \ldots, u_n\,)$. Although there are many possible coordinate systems, this section discusses only the cylindrical and spherical systems shown in Figure 12.12. These systems are of primary interest in mathematical physics.

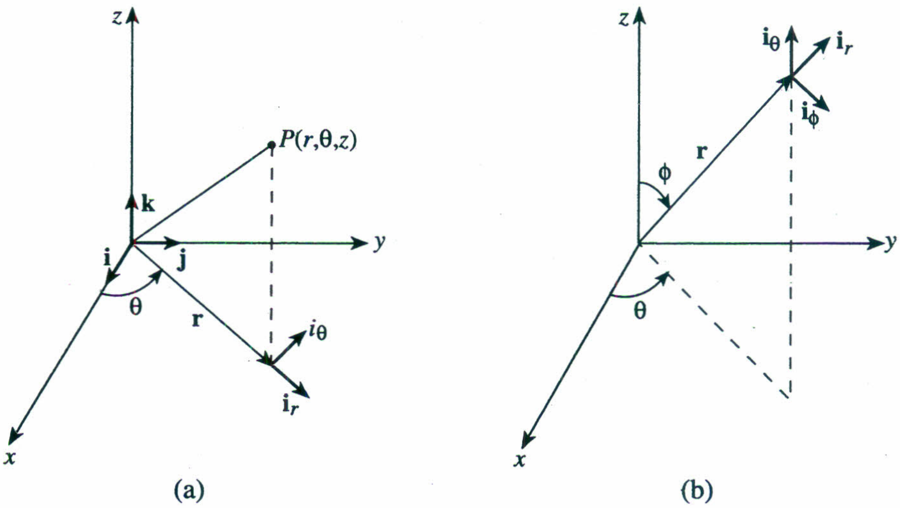

FIGURE 12.12 *The cylindrical and spherical coordinate systems*

Figure 12.12 shows the cylindrical and spherical coordinate systems and the unit vectors associated with each system. These two systems are examples of coordinate systems called *curvilinear* because if all but one of the nonrectangular coordinates are held fixed and the remaining one is varied, the coordinate transformation describes a curve in space. An important consequence of this is that the unit vectors defining a point in the curvilinear coordinate system may not be constant as the point moves in space since their direction changes as the point changes position. In Figure 12.12, this is true for all of the unit vectors, except for $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in rectangular coordinates and $\mathbf{i}_z = \mathbf{k}$ in cylindrical coordinates.

Notation. In this section, the vector $\mathbf{R}$ will be used as the vector from the origin to a point in space. The vector $\mathbf{r}$ is used as in Figure 12.12 to indicate the vector in cylindrical coordinates from the z-axis to the projection of the point of interest in the xy-plane. Some textbooks make no distinction and others use ρ in place of the vector $\mathbf{r}$. In any case, care should be taken not to confuse the two position vectors in cylindrical coordinates.

The dot notation,

$$\dot{f}(t) \equiv \frac{df(t)}{dt},$$

is used to indicate the derivative with respect to t for any function of time. The second derivative with respect to time is often designated $\ddot{f}(t)$.

Consider the position vector of a point moving in rectangular coordinates,

$$\mathbf{R}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \tag{12.54}$$

where $x(t)$, $y(t)$, and $y(t)$ define the positions of the point at each moment along the x, y, and z axes, respectively. The time derivative of $\mathbf{R}$ represents the velocity of the point and is computed as

$$\begin{aligned} \frac{d\mathbf{R}}{dt} &= \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j} + \dot{z}(t)\,\mathbf{k} + x(t)\frac{d\mathbf{i}}{dt} + y(t)\frac{d\mathbf{j}}{dt} + z(t)\frac{d\mathbf{k}}{dt} \\ &= \dot{x}(t)\,\mathbf{i} + \dot{y}(t)\,\mathbf{j} + \dot{z}(t)\,\mathbf{k} \end{aligned} \tag{12.55}$$

since the rectangular unit vectors are constants.

CYLINDRICAL COORDINATES

Referring to Figure 12.12, the cylindrical coordinates (r, θ, z) are related to rectangular coordinates by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z. \tag{12.56}$$

Writing the unit vectors in cylindrical coordinates in terms of the rectangular set yields

$$\begin{aligned} \mathbf{i}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}, \\ \mathbf{i}_\theta &= -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}, \\ \mathbf{i}_z &= \mathbf{k}. \end{aligned} \tag{12.57}$$

Both the scalar transformation of Equation 12.56 and the transformations for the unit vectors of Equation 12.57 can be solved to define the inverse transformation. Table 12.5 indicates both transformations.

TABLE 12.5 *Cylindrical coordinates*

Cylindrical $r\theta z$	*Rectangular* xyz
$r = \sqrt{x^2 + y^2}$	$x = r\cos\theta$
$\theta = \arctan y/x$	$y = r\sin\theta$
$z = z$	$z = z$
$\mathbf{i}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}$	$\mathbf{i} = \cos\theta\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta$
$\mathbf{i}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$	$\mathbf{j} = \sin\theta\,\mathbf{i}_r + \cos\theta\,\mathbf{i}_\theta$
$\mathbf{i}_z = \mathbf{k}$	$\mathbf{k} = \mathbf{i}_z$

DISTANCE FORMULA

The distance between endpoints of two vectors in curvilinear coordinates can be computed by transforming to rectangular coordinates. Thus, the distance between two points in cylindrical coordinates $P_1(r_1, \theta_1, z_1)$ and $P_2(r_2, \theta_2, z_2)$ is

$$d = \left[(r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2 + (z_1 - z_2)^2\right]^{1/2}.$$

Expanding and collecting terms results in the distance formula

$$d = \left[r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) + (z_1 - z_2)^2\right]^{1/2}, \qquad \textbf{(12.58)}$$

for the distance between two points in cylindrical coordinates.

☐ **EXAMPLE 12.16** ***Cylindrical Coordinates***

Suppose the position of a particle is defined by the vector

$$\mathbf{R}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

in rectangular coordinates, and it is desired to determine the velocity of the particle in cylindrical coordinates. The position vector in cylindrical coordinates is easily determined by writing $\mathbf{R}(t)$ with x and y converted to cylindrical coordinates and using the conversions of the unit vectors in Table 12.5. The result is

$$\begin{aligned} \mathbf{R}(t) &= r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + z\,\mathbf{k} \\ &= r\,\mathbf{i}_r + z\,\mathbf{i}_z, \end{aligned}$$

in which r, θ, z, and the unit vector $\mathbf{i}_r$ are functions of time.

The velocity of the particle is found by differentiating the position vector with respect to time, including the unit vector $\mathbf{i}_r$, to yield

$$\mathbf{v} = \frac{d\mathbf{R}(t)}{dt} = \frac{dr}{dt}\,\mathbf{i}_r + r\frac{d\mathbf{i}_r}{dt} + \frac{dz}{dt}\,\mathbf{i}_z.$$

Differentiating the unit vector $\mathbf{i}_r$ with respect to time leads to the expression

$$\begin{aligned}\frac{d\mathbf{i}_r}{dt} &= \frac{d}{dt}(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) \\ &= \frac{d\theta}{dt}[-\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}] = \frac{d\theta}{dt}\mathbf{i}_\theta = \dot{\theta}\,\mathbf{i}_\theta,\end{aligned}$$

so that the expression for $d\mathbf{R}(t)/dt$ becomes

$$\mathbf{v} = \dot{r}\,\mathbf{i}_r + r\dot{\theta}\,\mathbf{i}_\theta + \dot{z}\,\mathbf{i}_z.$$

The speed is determined by combining the radial velocity, angular velocity, and the velocity in the z-direction as

$$|\mathbf{v}| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2}.$$

□

SPHERICAL COORDINATES

Referring to Figure 12.12, the spherical coordinates (r, θ, ϕ) are related to rectangular coordinates by the equations

$$x = r\cos\theta\sin\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\phi. \tag{12.59}$$

Notice that if $\phi = 90^\circ$, the spherical relationships become those for cylindrical coordinates in the xy-plane. If $\phi = 0^\circ$, $r = z$. The transformations between rectangular and spherical coordinates, including the unit vectors, are listed in Table 12.6.

TABLE 12.6 *Spherical coordinates*

Rectangular xyz	*Spherical rθφ*
$x = r\cos\theta\sin\phi$	$r = \sqrt{x^2 + y^2 + z^2}$
$y = r\sin\theta\sin\phi$	$\theta = \arctan y/x$
$z = r\cos\phi$	$\phi = \arctan\sqrt{x^2 + y^2}/z$
Spherical unit vectors:	
$\mathbf{i}_r = \cos\theta\sin\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\phi\,\mathbf{k}$	
$\mathbf{i}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}$	
$\mathbf{i}_\phi = \cos\theta\cos\phi\,\mathbf{i} + \sin\theta\cos\phi\,\mathbf{j} - \sin\phi\,\mathbf{k}$	
Rectangular unit vectors:	
$\mathbf{i} = \cos\theta\sin\phi\,\mathbf{i}_r - \sin\theta\,\mathbf{i}_\theta + \cos\theta\cos\phi\,\mathbf{i}_\phi$	
$\mathbf{j} = \sin\theta\sin\phi\,\mathbf{i}_r + \cos\theta\,\mathbf{i}_\theta + \sin\theta\cos\phi\,\mathbf{i}_\phi$	
$\mathbf{k} = \cos\phi\,\mathbf{i}_r - \sin\phi\,\mathbf{i}_\phi$	