**Discrete Homework 5 CENG 5431 Due Feb 25**

A *geometric series* is a series with each term after the first being a fixed multiple of the preceding term. The multiplier is a real number $r$, called the *ratio*, so that $a_{n+1} = ra_n$. If the sum is taken from $n = 0$, the geometric series is represented as

$$\sum_{n=0}^{\infty} ar^n = a + ar + \cdots \quad (a \neq 0). \tag{1}$$

The series converges to the sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \tag{2}$$

if $-1 < r < 1$ but diverges if $|r| > 1$.

The $n$th partial sum for the geometric series is found by subtracting the terms

$$\begin{aligned} S_n - rS_n &= a + ar + \cdots + ar^n - \left(ar + ar^2 + \cdots + ar^{n+1}\right) \\ &= a - ar^{n+1}, \end{aligned}$$

so that $S_n - rS_n = a(1 - r^{n+1})$. Thus, solving for $S_n$ leads to the result

$$S_n = \frac{a(1-r^{n+1})}{1-r}$$

for the sum of the first $n+1$ terms. Taking the limit as $n$ goes to infinity with $|r| < 1$ shows that the sum of the series is $a/(1-r)$, as shown in Equation 2.

Power series are used extensively for computing or approximating values of functions. Suppose that a power series converges to the value of $f(x)$, so that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots. \tag{3}$$

Such a series is called a *power series representation*, or a *power series expansion*, of $f(x)$. Alternatively, the function is said to be *represented*, or *expressed*, by the series. Although many functions of interest can be represented by a power series, there are many functions that cannot. If $f(x)$ can be represented by a power series in an interval, the function is termed *analytic* in the interval. The Taylor series is an important example of a power series.

Mathematically, the $Z$-transform is a rule by which a sequence of numbers is transformed into a function of the complex variable $z = x + iy$, which can be written

$$z = \rho \exp(i2\pi F) = \rho \exp(i\Omega).$$

If $\{f(n)\}$ is a sequence, then we write

$$\mathcal{Z}[f(n)] = F(z) \quad \text{and} \quad f(n) = \mathcal{Z}^{-1}[F(z)]. \tag{4}$$

The $Z$-transform is defined by the series

$$\begin{aligned}
\mathcal{Z}[f(n)] &= \sum_{n=0}^{\infty} f(n) z^{-n} \\
&= f(0) + \frac{f(1)}{z} + \frac{f(2)}{z^2} + \cdots \qquad (5) \\
&= f(0) + f(1)z^{-1} + f(2)z^{-2} + \cdots. \qquad (6)
\end{aligned}$$

This transform is typically used to analyze sequences for which $f(n) = 0$ for $n < 0$ so this transform is called the *unilateral*, or *single-sided* $Z$ transform since the sum begins at zero.

Do the problems by hand unless otherwise indicated and **check** the results whenever possible. However, you may wish to verify your results with MATLAB solutions to the problems when appropriate. You can use symbolic MATLAB to check results if you have access to it.

**Problem 1 10 Points**

Sum the following series (Do a little research into series expansions)

(a) $1 + \frac{1}{2} + \frac{1}{4} + \cdots$

(b) $\pi + \frac{\pi}{\sqrt{2}} + \cdots + \frac{\pi}{\sqrt{2^n}} + \cdots$

**Problem 2 20 Points**

Write the power series for the following fractions

(a)

$$\frac{1}{1+x}$$

by direct polynomial division of 1 by $1 + x$

(b) Using the result of Part a write the power series for

$$\frac{1}{1-x}$$

(c) Using the result of Part a write the power series for

$$\frac{1}{(1-x)^2}$$

by using the fact that this expression is the derivative of the fraction in Part b.

(d) Using the result of Part a write the power series for

$$\frac{1}{2-x}.$$

**Problem 3 20 Points**

(a) Solve the difference equation by classical methods as in Harman Example 10.2 and by z-transforms if the input is $\delta(n)$

$$y(n) = ay(n-1).$$

(b) Let $a = 0.9$ and $y(0) = 1$ and use MATLAB to plot the solution. Use stem plot and plot 25 values, but label the x-axis from 0 to 25.

**Problem 4 20 Points**

Determine the step response $Y(z)$ and by inversion of the Z-transform find $y(n)$ for the smoothing filter (Expand Y(z)/z for partial fractions and see Harman Chapter 10, Table 10.4 for help)

$$H(z) = \frac{0.1z}{z - 0.9}.$$

What is $H(z = 1)$ and what does it mean? What is $(z-1)Y(z)$ as $z \to 1$ ?

**Problem 5 10 Points**

Show that the $Z$-transform of the sequence $\{A \cos \beta n\}$ is

$$F(z) = \frac{Az(z - \cos \beta)}{z^2 - 2z \cos \beta + 1}.$$

**Problem 6 20 Points**

Consider the transfer function

$$H(z) = \frac{0.1z}{z - 0.9}.$$

Find $H(z)$ and the digital frequency $F$ and $\Omega$ for the following cases (See Harman Section 10.9):

(a) $z = e^{i0.01}$.

(b) $z = e^{i3}$

(c) Plot $H(z)$ as in Harman Example 10.17 and pick off the points (Tools, data cursor) in (a) and (b). Note that freqz plots the dB versus $\pi$*radian digital frequency.