Version:

TEX'd:
Apr. 19, 2014

1 *Fourier Analog & Digital Summary*

Fourier Relationships Analog & Digital Harman

Apr. 19, 2014

PREVIEW

The main Fourier analysis techniques are the Fourier transform (FT), Fourier Series (FS), Discrete Fourier transform (DFT), and the Fast Fourier Transform (FFT). The FT is used to determine the spectrum of an aperiodic continuous signal and the FS is used for periodic signals. The DFT computes the spectrum of discrete signals and the FFT is the algorithm to compute the DFT.

[-T/2, T/2] INTERVAL

On the interval $[-T/2, T/2]$, the limits of integration for the Fourier series can be changed from $[-\pi, \pi]$ by assigning to the integration variable t the value $2\pi t/T$. The period of the function is thus T.

Assuming that $f(t)$ is continuous on the interval $-T/2 \le t \le T/2$, the coefficients a_n and b_n can be computed by the formulas

$$\begin{aligned} a_0 &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt, \\ a_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2n\pi t}{T}\right)\,dt, \\ b_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2n\pi t}{T}\right)\,dt, \end{aligned} \tag{1.1}$$

where $n = 1, 2, \ldots$ is any positive integer.

The Fourier series on the interval $[-T/2, T/2]$ is thus

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right]. \tag{1.2}$$

The dc term is written $\dfrac{a_0}{2}$ so that the equation for the a_n coefficients holds for $n = 0, 1, 2, \ldots$.[1]

Frequency Components Assuming the variable t represents time, the function $f(t)$ repeats every T seconds. The *frequency* associated with the fundamental sinusoid in the series of Equation 1.2 is $f_0 = 1/T$, measured in cycles per second, or hertz. The parameter

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T}$$

is the frequency in radians per second.

Since $2n\pi/T = 2n\pi f_0 = n\omega_0$, the series in Equation 1.2 can be written

$$\begin{aligned} f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(2\pi n f_0 t) + b_n \sin(2\pi n f_0 t)] \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos(n\omega_0 t) + b_n\sin(n\omega_0 t)], \end{aligned} \tag{1.3}$$

which emphasizes the components in terms of their frequencies.

[1] Sometimes the series is written with A_0 as the dc term. In this case, the constant multiplying the integral for a_0 replaced by A_0 would be multiplied by $1/T$.

The first term in cosine or sine is called the *fundamental* component, and the other terms are the *harmonics* with frequencies that are integer multiples of the fundamental component's frequency. Thus, the frequencies of the Fourier series terms are

$$f_0, 2f_0, 3f_0, \ldots,$$

although some of the components may be zero for a particular Fourier series. However, $f(t)$ is a continuous function of time, and this aspect of the Fourier series is emphasized when the series is used to approximate $f(t)$. In other applications, the frequencies of the components are of primary interest.

The Fourier series can also be written in *cosine with phase* or *shifted-cosine* form as

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(2\pi n f_0 t + \theta_n) \tag{1.4}$$

where for $n = 1, 2, \ldots$
$c_n = \sqrt{a_n^2 + b_n^2}, \quad \theta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$ when $a_n > 0$.

Sometimes a function of a spatial variable x is of interest. If the function has period λ meters, the function repeats as

$$f(x + \lambda) = f(x).$$

Then, the variable t in Equation 1.3 is replaced by x, and the frequency components are defined by replacing f_0 with $1/\lambda$. The spatial equivalent of ω_0 is

$$k = \frac{2\pi}{\lambda},$$

measured in inverse units of length. Such a formulation of Fourier series is used frequently in problems involving optics. In optical applications, the values nk are called *spatial frequencies*. Thus, λ represents the wavelength of the light wave being analyzed.

COMPLEX SERIES

The trigonometric Fourier series contains a series of sines and cosines and thus involves real functions. It is often convenient to write the series for a function $f(t)$ with period T as a sum of exponential functions in the form

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\omega_0 t}, \tag{1.5}$$

where $\omega_0 = 2\pi/T$ as before and the coefficients α_n are the complex Fourier coefficients.

By substituting the identities

$$\begin{aligned} \cos(n\omega_0 t) &= \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}, \\ \sin(n\omega_0 t) &= \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}, \end{aligned} \tag{1.6}$$

in the trigonometric form of the series, the relationship between the trigonometric and exponential coefficients is found to be

$$\begin{aligned} \alpha_0 &= \frac{a_0}{2}, \\ \alpha_n &= \frac{a_n - ib_n}{2} \quad \text{for } n > 0 \;, \\ \alpha_{-n} &= \frac{a_n + ib_n}{2}. \end{aligned} \tag{1.7}$$

Notice that α_{-n} is the complex conjugate of the term α_n and $|\alpha_n| = |\alpha_{-n}|$.[2] Thus, the series in Equation 1.5 becomes

$$f(t) = \alpha_0 + \sum_{n=1}^{\infty} [\alpha_n e^{in\omega_0 t} + \alpha_{-n} e^{-in\omega_0 t}]. \tag{1.8}$$

Also, note that a_0 in the trigonmetric series is twice the value of α_0. When the series is written in either form, the dc component represented by $a_0/2$ or α_0 must be the same number.

When the positive frequency spectrum is plotted, the magnitude of the frequency component at $f_n = nf_0, n = 1, 2...$ must be

$$c_n = \sqrt{a_n^2 + b_n^2}, \quad c_0 = \frac{a_0}{2}$$

from the shifted cosine form of Equation 1.4. If the complex spectrum is plotted at frequencies $-nf_0, -(n+1)f_0, ..., -f_0, 0, f_0, 2f_0, ..., nf_0$, the spectral values are

$$|\alpha_n| = |\alpha_{-n}| = \frac{1}{2}c_n = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

[2] These results hold when $f(t)$ is a real-valued function.

☐ **EXAMPLE 1.1** ***Fourier series square wave example***

A square wave of amplitude A and period T shown in Figure 1.1 can be defined as

$$f(t) = \begin{cases} A, & 0 < t < \dfrac{T}{2}, \\ -A, & -\dfrac{T}{2} < t < 0, \end{cases}$$

with $f(t) = f(t+T)$, since the function is periodic.

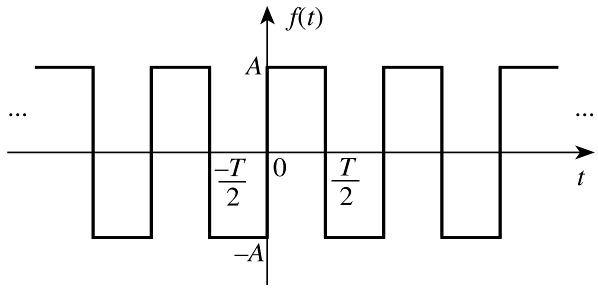

FIGURE 1.1 *Square wave of Example 1.1*

The first observation is that $f(t)$ is odd, which yields the result that $a_0 = 0$ and $a_i = 0$ for every coefficient of the cosine terms. Letting $\omega_0 = 2\pi/T$, the coefficients b_n are

$$b_n = 2\left(\frac{2}{T}\right)\int_0^{T/2} A\sin(n\omega_0 t)\,dt.$$

The result is

$$f(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

where $(2n-1)$ is introduced to assure that only odd terms are included in the summation. The sine waves that make up the Fourier series for the odd square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right],$$

so the series consists not only of sine terms, as expected, but also odd harmonics appear. This is due to the rotational symmetry of the function since the wave shapes on alternate half-cycles are identical in shape but reversed in sign. Such waveforms are produced in certain types of rotating electrical machinery.

☐

☐ EXAMPLE 1.2 ***Complex Series Square Wave Example***

Consider the odd square wave of Example 1.1 and the complex Fourier coefficients

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{0}(-A)e^{-in\omega_0 t}\,dt + \frac{1}{T}\int_{0}^{T/2}(A)e^{-in\omega_0 t}\,dt, \qquad (1.9)$$

which leads to the series

$$f(t) = \frac{2A}{i\pi}\sum_{n=-\infty}^{\infty}\frac{e^{i(2n-1)\omega_0 t}}{(2n-1)}, \qquad (1.10)$$

as defined in Equation 1.5.

This form contains complex coefficients, but the series can be written in terms of sine waves by combining the corresponding terms for positive and negative arguments. To determine the coefficients, the amount of difficulty is about the same for the trigonometric series and the complex series. However, the complex series perhaps has an advantage when the magnitude of the coefficients are of interest.

Each Fourier Series (FS) coefficient has the form

$$\alpha_n = F[n] = \frac{2A}{in\pi} = \frac{2A}{n\pi}e^{-i\pi/2}, \qquad n = \pm 1, \pm 3, \ldots,$$

and the coefficients for even values, $n = 0, \pm 2, \ldots$, are zero. Notice that the coefficients decrease as the index n increases. The use of these coefficients to compute the *frequency spectrum* of $f(t)$ is considered later. The trigonometric series is derived from the complex series by expanding the complex series of Equation 1.10 as

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty}\alpha_n e^{in\omega_0 t} \\ &= \cdots - \frac{2A}{3\pi i}e^{-i3\omega_0 t} - \frac{2A}{\pi i}e^{-i\omega_0 t} + \frac{2A}{\pi i}e^{i\omega_0 t} + \frac{2A}{3\pi i}e^{i3\omega_0 t} + \cdots \end{aligned}$$

and recognizing the sum of negative and positive terms for each n as $2\sin(n\omega_0 t)$. The trigonometric series becomes

$$f(t) = \frac{4A}{\pi}\left(\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\omega_0 t]}{(2n-1)},$$

which is the result of Example 1.1.

☐

Orthogonality To find the coefficients in Equation 1.5, each side is multiplied by $e^{-im\omega_0 t}$ and integrated over the period to yield

$$\int_{-T/2}^{T/2} f(t)e^{-im\omega_0 t}\,dt = \sum_{n=-\infty}^{\infty} \alpha_n \int_{-T/2}^{T/2} e^{i(n-m)\omega_0 t}\,dt. \tag{1.11}$$

Since the terms with different exponents are orthogonal, all terms but that for which $m = n$ are zero for the integral on the right-hand side. Thus,

$$\int_{-T/2}^{T/2} f(t)e^{-im\omega_0 t}\,dt = \int_{-T/2}^{T/2} e^{-in\omega_0 t}e^{in\omega_0 t}\,dt = \alpha_n T,$$

so that dividing both sides T yields the coefficients

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-in\omega_0 t}\,dt. \tag{1.12}$$

The coefficients of the exponential series yield the frequency components at each discrete frequency $n\omega_0 = n2\pi f_0 = n2\pi/T$. Comparing the Fourier Transform FT if the Fourier transform of $f(t)$ exists

$$\mathcal{F}[f(t)] = F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt. \tag{1.13}$$

we see that the integral is very similar to the one for α_n but the variable ω in the FT is continuous. The transform $F(i\omega)$ represents the *frequency spectrum* of $f(t)$, and it may be complex even though $f(t)$ is real. The magnitude $|F(i\omega)|$ is called the amplitude spectrum of $F(i\omega)$. Using the notation $F(i\omega)$ emphasizes the fact that the Fourier transform is a function of a complex variable.

By comparing the two results, it is clear that designating the transform $F(i\omega) = \mathcal{F}[f(t)]$,

$$\frac{F(n\omega_0)}{T} = \alpha_n. \tag{1.14}$$

Thus, we conclude that the Fourier series coefficients are obtained by *sampling* the Fourier transform at the points $n\omega_0$ and dividing by the period T of the time function. However, the Fourier series itself is a continuous function of time, but the Fourier transform is a function of ω in the frequency domain.

Comparing the coefficients of the Fourier series of Harman Example 8.7 for a periodic pulse train of even rectangular pulses of width τ, amplitude A, and period T and the Fourier transform of Example 8.11 for a single even pulse of width τ, amplitude A, shows that the series coefficients are

$$\alpha_n = \frac{1}{T}\int_{-\tau/2}^{\tau/2} f(t)e^{-in\omega_0 t}\,dt = \frac{A\tau}{T}\frac{\sin(n\omega_0\tau/2)}{n\omega_0\tau/2}$$

and the transform is

$$\mathcal{F}[f(t)] = F(i\omega) = \int_{-\tau/2}^{\tau/2} f(t)e^{-i\omega t}\,dt = A\tau\frac{\sin(\omega\tau/2)}{\omega\tau/2}.$$

Thus, for a given $f(t)$ such as a pulse and its periodic version $f_p(t)$, the FS coefficients for $f_p(t)$ are *samples* of the FT spectrum scaled by the period $T = 1/f_0$ as

$$F[k] = \frac{1}{T}\; F(f)|_{f=kf_0}$$

where $F[k]$ represents the Fourier Series coefficient at the frequency kf_0 with $f_0 = 1/T$ and $F(f)$ represents the Fourier Transform with $f = \omega/2\pi$.

SAMPLED DATA AND FOURIER ANALYSIS

When a signal is sampled and the DFT is used to compute the spectrum, the first question is how does the DFT ($F_{\text{DFT}}[k]$) result relate to the FT ($F(f)$) or FS ($F[k]$) results? When the FFT algorithm is used to compute the DFT, the second is how to determine the FT or FS from the FFT result?

Two important points must be kept in mind when signal sampling and the DFT are used to analyze the spectrum of the signal as follows:

1. Sampling in time of the signal results in a *periodic* spectrum for the frequency result.

2. The DFT or FFT do not include any information about "real world" time or frequencies.

Definition of DFT and IDFT Assume that a function $f(t)$ is defined at a set of N points, $f(nT_s)$ for $n = 0, \ldots, N-1$ values, as shown in Figure 1.2. The DFT yields the frequency spectrum at N points by the formula

$$F_k = F\left(\frac{k}{NT_s}\right) = \sum_{n=0}^{N-1} f(nT_s)e^{-i2\pi nk/N} \tag{1.15}$$

for $k = 0, \ldots, N-1$. Thus, N sample points of the signal in time lead to N frequency components in the discrete spectrum spaced at intervals $f_s = 1/(NT_s)$. The Inverse DFT (IDFT) is defined as

$$f_n = f(nT_s) = \frac{1}{N}\sum_{k=0}^{N-1} F\left(\frac{k}{NT_s}\right)e^{i2\pi nk/N} \tag{1.16}$$

for $n = 0, \ldots, N-1$. The IDFT is used to re-create the signal from its spectrum.

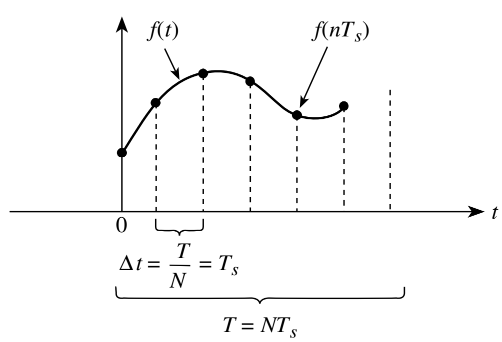

FIGURE 1.2 *Approximation of a signal by sampling*

The FFT algorithms take advantage of the symmetry in the exponential functions $\exp(-i2\pi nk/N)$ to reduce the number of computations while computing the DFT. For example, a direct calculation of the DFT requires N^2 multiplications. The basic FFT requires approximately $N \log_2 N$ multiplications. If $N = 4096$ points, the FFT reduces the number of multiplications from more than 16 million to less than 50,000.

Figure 1.3 shows the important parameters for the DFT and the common FFT algorithms when applied to physical signals. In the figure, the magnitude of the spectrum computed by the FFT is plotted as

$$|F(f)| = \sqrt{[F_r(f)]^2 + [F_i(f)]^2}$$

since the FFT yields N *complex* values. In the upper plot, the DFT components are plotted for the index $k = 0$ to $k = N - 1$ which emphasizes the symmetry about the index value $k = N/2$. The k-th positive digital frequency has the value $F = k/N$.

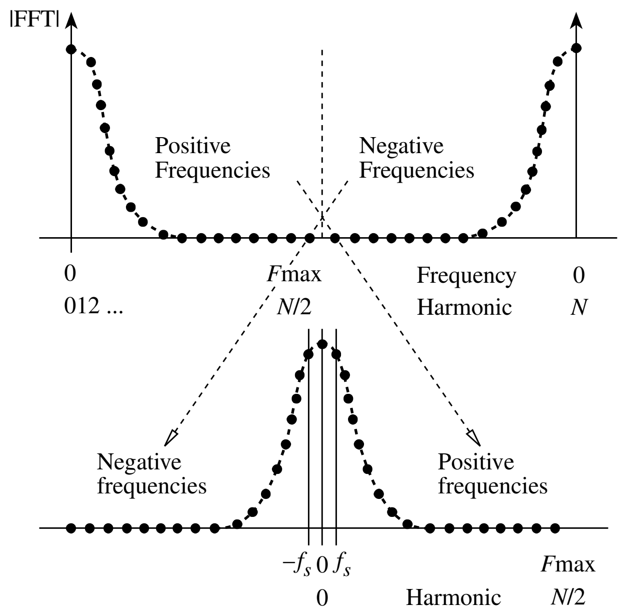

FIGURE 1.3 *Discrete Fourier transform spectrum*

The output of most FFT algorithms is folded in frequency, as shown in the upper plot of Figure 1.3 and the DFT spectrum is symmetrical around the frequency $F_{\max}$. This is called the *folding* frequency when the symmetry of the spectrum is being discussed.

Folding Frequency The theory of the folding shown in Figure 1.3 is that N points in time produce N points in the frequency domain. However, for a real signal, there are N complex numbers in the transform with N real parts and N imaginary parts. The real part is even and the imaginary part is odd around the folding point. Including the component at $f = 0$, there are really only $N/2 + 1$ unique points on the positive frequency axis.

Symmetry about the origin It is often convenient to plot the DFT spectrum showing the symmetry about the origin as in the lower plot of Figure 1.3. In terms of the components, the DFT of a real sequence possesses *conjugate symmetry* about the origin.

The procedure for plotting is to shift the upper half of the spectral components from $k > N/2$ to $k = N - 1$ to negative values $k - N$ so that the origin represents the zero of frequency. The results for $k > N/2$ are thus negative frequency results. The component at $N/2$ corresponds to the maximum positive frequency.

DFT summary Table 1.1 summarizes the DFT or FFT parameters when a real signal is sampled every T_s seconds for $(N-1)T_s$ seconds.

TABLE 1.1 *DFT parameters*

Parameter	*Notation*
Time domain:	
Sample interval Δt	T_s (s) $= 1/S$; S is the sampling rate samp/sec
Sample size	N points
Length	$(N-1)T_s$ (s)
Period (from IDFT)	$T = NT_s$ (s)
Frequency domain:	
Frequency Spacing	$f_s = \dfrac{1}{T} = \dfrac{1}{NT_s}$ (Hz)
Spectrum size	N components
Maximum frequency	$\dfrac{N}{2} f_s = F_{\max} = \dfrac{1}{2T_s}$ (Hz)
Frequency period	$F_p = N f_s = \dfrac{1}{T_s}$ (Hz)

Note that the important parameters for sampling a signal are the sampling interval T_s and the length of time the signal is sampled T with the result that only two of the three parameters can be selected in the equation

$$T = NT_s.$$

The digital frequency, described in Section 10.9 of Harman, is defined by the ratio of the analog frequency of the signal being sampled to the sampling frequency as

$$F = \frac{f_{analog}}{f_{sample}} = \frac{f_{analog}}{S}$$

where S samples per second is often quoted in Hertz and is defined by the sampling time as

$$S = \frac{1}{T_s}.$$

To avoid aliasing, S in Hertz must be greater than twice the highest frequency signal being sampled so that the maximum allowable digital frequency is

$$F_{\max} = \frac{1}{2}.$$

Looking now at the frequency domain results in Table 1.1, the time parameters determine the results in the frequency spectrum of the signal as follows:

1. The frequency $Resolution$ is given by the inverse of the total sampling time $1/T$ just as in a Fourier series

2. The $Maximum\ frequency$ is given by one-half the sampling frequency as

$$F_{\max} = \frac{1}{2T_s} = \frac{S}{2}$$

SAMPLING

Thus the two most important questions in the specification of a data acquisition system are the following:

1. How often should the analog signal be sampled?

2. How long should the signal be sampled?

☐ **EXAMPLE 1.3** ***Sampling Example***

This example defines the relationship between sampling interval, frequency resolution, and number of samples for the DFT. In terms of previous notation, T_s is the sampling interval in seconds, $\Delta f = f_s$ is the frequency resolution, and N is the number of sample points in time and in frequency.

Consider an analog signal with frequencies of interest up to 1200 Hertz. The desired frequency resolution is 0.5 Hertz. Thus, the signal should be filtered so that frequencies above 1200 Hertz are eliminated from the signal. This filtering removes frequencies and noise in the signal above 1200 Hertz. The noise consists of unwanted signals added to the desired signal that are the result of environmental effects as the signal is transmitted to the data acquisition system.

By the sampling theorem, the sampling interval in time must be

$$T_s < \frac{1}{2 \times 1200} = \frac{1}{2400} \quad \text{seconds},$$

so that at least 2400 samples per second are needed. This corresponds to

$$F = \frac{1200}{2400} = 0.5.$$

For a resolution of 0.5 Hertz, $T = 1/0.5 = 2$ seconds. The total number of points required is thus

$$N = \frac{T}{T_s} = \frac{2}{(2400)^{-1}} = 4800.$$

If N is to be a power of 2 for the FFT algorithm, $2^{13} = 8192$ samples would be taken. The sampling rate could be increased to 4096 samples per second, which is sampling at a rate corresponding to about 3.4 times the highest frequency of interest or $F = 1200/4096$.

☐

Fourier Series and DFT To get the FS values from the DFT or FFT of $f(t)$ with period T sampled at times $nT_s = n/S,\ n = 0, 1, \ldots, N-1$

$$F[k] = \frac{1}{N}\, F_{\mathsf{DFT}}[k] \tag{1.17}$$

where $N = T/T_s = TS$.

Fourier Transform and DFT The DFT can be used to approximate the continuous Fourier transform. The continuous Fourier transform is

$$\mathcal{F}[f(t)] = F(f) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi ft}\, dt. \tag{1.18}$$

The frequency f in hertz is used as the parameter in this integral. The function $F(i\omega)$, where $\omega = 2\pi f$ is the frequency in radians per second, could be calculated as well.

Using the sampled $f(t)$ with $t = nT_s$ and replacing f by the discrete frequencies $f_s = k/(NT_s)$ leads to the approximation of the Fourier transform as

$$F\left(\frac{k}{NT_s}\right) = T_s \sum_{n=0}^{N-1} f(nT_s)e^{-i2\pi nk/N} \tag{1.19}$$

for $k = 0, \ldots, N-1$. The factor $\Delta t = T_s$ replaced dt in the integral and is used as a multiplier of the DFT defined by Equation 1.15 in order to approximate the continuous Fourier transform.

Finally, to get the Fourier transform values from the DFT or FFT,

$$F(f)|_{f=kS/N=k/T} = \frac{1}{S}\, F_{\mathrm{DFT}}[k] = T_s\, F_{\mathrm{DFT}}[k]. \tag{1.20}$$

To get the positive spectrum of a periodic signal, the result is

$$F[k] = \frac{2}{N} F_{\mathrm{DFT}}[1 : N/2].$$

Note the factor 2. Comparing Equation 1.17 and Equation 1.20 and noting that the Fourier Transform can be derived for the Fourier Series by multiplying by the period T as in Equation 1.14, the derivation of the Fourier Transform from the Fourier Series computed by the DFT is

$$F(f)|_{f=kS/N=k/T} = T \times F_{\mathrm{DFT}}[k] = \frac{T}{N}\, F_{\mathrm{DFT}}[k] = T_s\, F_{\mathrm{DFT}}[k] \tag{1.21}$$

as given in Equation 1.20.

TABLE 1.2 *Relations*

Type	*Time Signal*	*Spectrum*
Fourier Transform $F(f) = \mathcal{F}(x(t))$	Continuous in t-Aperiodic	Continuous in f
Fourier Series	Continuous in t	Discrete with Spacing
$F[k]\|_{k=0,1,\cdots}$	Period T sec	$kf_0 = k\frac{1}{T}$
Discrete Fourier Transform	Sampled $S = (1/T_s)$	Discrete with Spacing[1,2]
$F_{\mathrm{DFT}}[k]\|_{k=0,1,\cdots,N-1}$	Period $T = NT_s$	$kf_0 = k\frac{1}{T} = k\frac{1}{NT_s} = k\frac{S}{N}$
Fast Fourier Transform	Sampled $S = (1/T_s)$	Discrete with Spacing[1,2]
$F_{\mathrm{DFT}}[k]\|_{k=0,1,\cdots,N-1}$	Period $T = NT_s$	$kf_0 = k\frac{1}{T} = k\frac{1}{NT_s} = k\frac{S}{N}$

Notes:
[1] For real $f(t)$, conjugate symmetry about point $k = N/2$.

Highest frequency is $F_{\mathrm{MAX}} = \frac{1}{2T_s} = k\frac{S}{2}$, i.e. $F_{\mathrm{DFT}}[k]|_{k=N/2}$. The values for $k > N/2$ up to $k = N-1$ are *negative* frequency components.

[2] The two-sided spectrum of $f(t)$ will show one-half magnitude of each frequency component in the positive frequency spectrum.