DSP Linear Systems, Fourier, Laplace, Z

April 22, 2014

Consider a system defined by a differential equation:

Figure 1: SystemDifferentialEq Time $h(t)$

Consider a Linear Time Invariant (LTI) system defined by a Transfer Function:

Figure 2: SystemTransferFunction Laplace $H(s)$ or Fourier $H(j\omega)$ or $H(z)$.

System Relationships Analog LTI System No I.C.s

Input	*System*	*Output*
$x(t)$	$h(t)$	$y(t) = \int_0^t x(\tau)h(t-\tau)\, d\tau.$
$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\, dt$	$H(j\omega)$	$Y(j\omega) = X(j\omega)H(j\omega)$
$X(s) = \int_0^{\infty} x(t)e^{-st}\, dt \quad s = \sigma + j\omega$	$H(s)$	$Y(s) = H(s)X(s)$

To make this more concrete, consider an input $x(t)$ with output $y(t)$ in the time domain and $Y(s) = H(s)X(s)$ in the Laplace domain.

Assuming that $x(\tau) = 0$ for $\tau < 0$, adding the response to all the past inputs leads to the integral

$$y(t) = \int_0^t x(\tau)h(t-\tau)\, d\tau. \tag{1}$$

This integral is called the *convolution* or *superposition* integral and the operation is said to be the convolution of x, the input, and h, the impulse response of the system. The convolution is often written $x * h$.

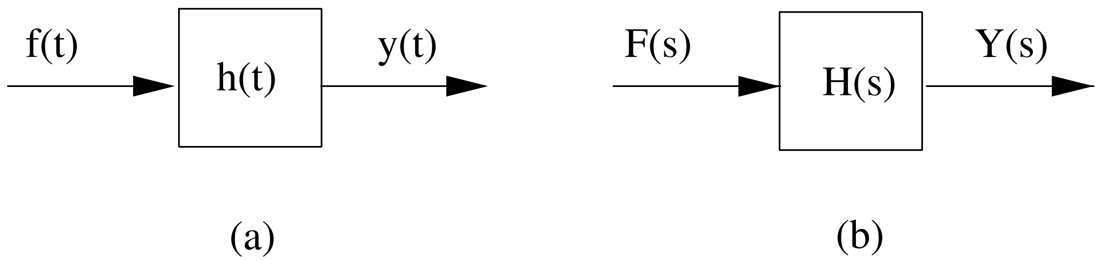

Figure 3: System views in time and Laplace variable s

Let the Laplace Transform of $f(t)$ be $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[h(t)] = H(s)$. Then,

$$\mathcal{L}[f * h] = \mathcal{L}\left[\int_0^t f(\tau)h(t-\tau)\, d\tau\right] = F(s)H(s). \quad (2)$$

In words, the Laplace transform of the convolution $f(t) * h(t)$ is the product $F(s)H(s)$. (Reference Harman page 444)

If the Fourier Transform of $f(t)$ exists, we can use the Laplace Transform to find

$$F(j\omega) = F(s)|_{s=j\omega}$$

which is used to determine the frequency response of the system as $F(j\omega)$.

Here is a diagram of a closed loop system:

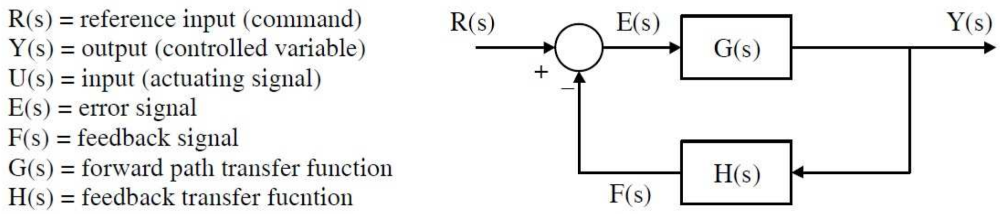

Figure 4: ClosedLoopwithNotes

In this figure $U(s) = E(s)$ since there is no compensator in the loop. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}.$$

System Relationships Digital

Input	*System*	*Output*
$x[n]$	$h[n]$	$\sum_{i=0}^{n} h[i]x[n-i]$
$X(k) = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$		DFT
$X(Z) = \sum_{n=0}^{\infty} x[n]z^{-n}$	$H(Z)$	$Y(Z) = X(Z)H(Z)$

USEFUL FORMULAS Fourier

$$\mathcal{F}[f(t)] = F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt. \qquad (3)$$

The DFT yields the frequency spectrum at N points by the formula

$$F_k = F\left(\frac{k}{NT_s}\right) = \sum_{n=0}^{N-1} f(nT_s)e^{-i2\pi nk/N} \qquad (4)$$

for $k = 0, \ldots, N-1$. The frequency components in the discrete spectrum are spaced at intervals $\Delta f = 1/(NT_s)$.

Sampling and Aliasing

1 In the frequency domain, the spectrum of $f(t)$ is periodic with period

$$S = \frac{1}{T_s} \text{ Hertz}$$

2 If the sampling rate $S < f_{\text{analog}}$, the signal is aliased to a lower frequency

$$|f_{\text{aliased}}| < 0.5S$$

3 The aliased frequencies could be many and have values

$$f_{\text{aliased}} = f_{\text{analog}} - MS \text{ where } -0.5S < f_{\text{aliased}} \leq 0.5S.$$

4 T_s determines the highest frequency in the spectrum as

$$f_{\text{max}} = \frac{S}{2} = \frac{1}{2T_s} \text{ Hz.}$$

5 The resolution in frequency is determined by the number of points chosen as

$$f_1 = \Delta f = \frac{1}{NT_s}\ (f_{\text{DFT}}).$$

The frequencies in the DFT are thus

$$(f_{\text{dc}}, f_1, 2f_1, 3f_1, \ldots, f_{\text{max}} = \frac{N}{2}\Delta f)$$

Z-transforms If the transfer function $H(z)$ is evaluated for values of

$$z = \exp(i2\pi F) = \exp(i\Omega)$$

we obtain the *frequency response*, $H(z)$, of the system. The *digital frequency*

$$F = \frac{f_{\text{analog}}}{f_s} = fT_s,$$

where T_s is the sampling time or time between samples. The analog frequency $f = 1/(2T_s)$ corresponds to the digital frequency $F = 1/2$ or $\Omega = \pi$.