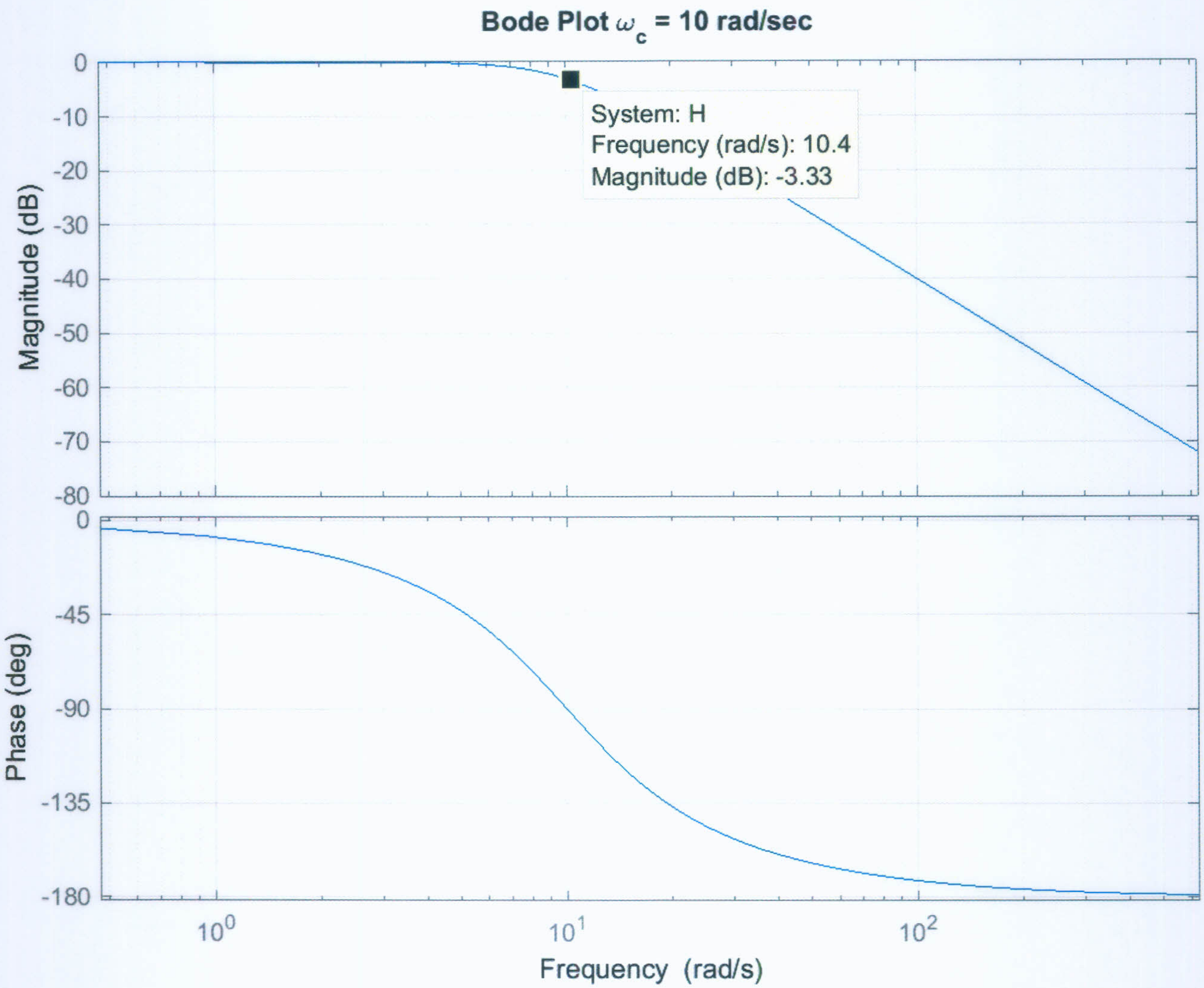
```
% Example 2pole butter  tlh
%  Analog Butterworth filter design
%  design an 2-pole filter with a bandwidth of 10 rad/sec
%  Prototype  H(s) =    1
%                    -----------------        wb = 1 rad/sec
%                    s^2 + 2^(1/2)s +1
%
[z,p,k] = buttap(2);      % 2 pole filter
[b,a] = zp2tf(z,p,k);     % convert the zeros and poles to polynomials
wb = 10;                 % new bandwidth in rad/sec
[b,a] = lp2lp(b,a,wb);    % transforms to the new bandwidth
%                          By hand replace s by s/wb
f = 0:15/200:100;       % define the freq. in Hz for plotting
w = 2*pi*f;
H = tf(b,a)
%  Continuous-time transfer function.
% H =            100
 %      ---------------------         wb = 10 r/s
 %        s^2 + 14.14 s + 100

figure(1)
bode(H,w);
grid, title('Bode Plot \omega_c = 10 rad/sec')
```

NOTE $$\left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s/10 \to s} \to \frac{100}{s^2+10\sqrt{2}\,s+100}$$

CHECK $H(0) = 1$ ✓

$$H(10) = \frac{100}{-100+j\,100\sqrt{2}+100} = \frac{100}{j\,100\sqrt{2}}$$

$$|H(10)| = \frac{1}{\sqrt{2}} \qquad \angle H(10) = -\pi/2$$



Bode Plot ω_c = 10 rad/sec
System: H
Frequency (rad/s): 10.4
Magnitude (dB): -3.33
Magnitude (dB)
0
-10
-20
-30
-40
-50
-60
-70
-80
Phase (deg)
0
-45
-90
-135
-180
10^0
10^1
10^2
Frequency (rad/s)

Of the four classical filter types based on magnitude specifications, the Butterworth filter is monotonic in the passband and stopband, the Chebyshev I filter displays ripples in the passband but is monotonic in the stopband, the Chebyshev II filter is monotonic in the passband but has ripples in the stopband, and the elliptic filter has ripples in both bands.

REVIEW PANEL 13.2

Magnitude Characteristics of Four Classical Filters

Butterworth: Monotonic in both bands **Chebyshev I:** Monotonic passband, rippled stopband
Elliptic: Rippled in both bands **Chebyshev II:** Rippled passband, monotonic stopband

The design of analog filters typically relies on frequency specifications (passband and stopband edge(s)) and magnitude specifications (maximum passband attenuation and minimum stopband attenuation) to generate a minimum-phase filter transfer function with the smallest order that meets or exceeds specifications. Most design strategies are based on converting the given frequency specifications to those applicable to a **lowpass prototype** (LPP) with a cutoff frequency of 1 rad/s (typically the passband edge), designing the lowpass prototype, and converting to the required filter type using frequency transformations.

13.1.1 Prototype Transformations

The lowpass-to-lowpass (LP2LP) transformation converts a lowpass prototype $H_P(s)$ with a cutoff frequency of 1 rad/s to a lowpass filter $H(s)$ with a cutoff frequency of ω_x rad/s using the transformation $s \Rightarrow s/\omega_x$, as shown in Figure 13.2. This is just linear frequency scaling.

$H_1(\omega) = \frac{1}{s+1} \Big|_{s \to s/10}$

$H_{10}(s) = \frac{10}{s+10}$

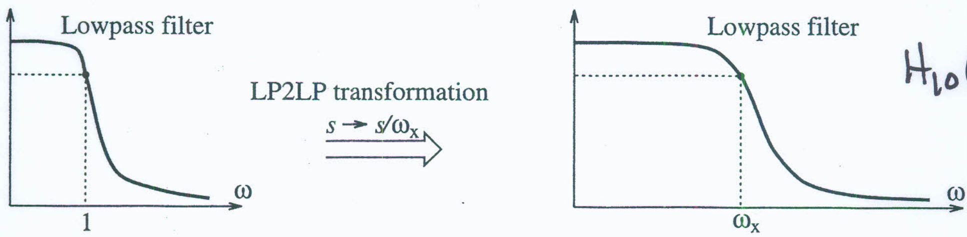


Figure 13.2 The lowpass-to-lowpass transformation

The lowpass-to-highpass (LP2HP) transformation converts a lowpass prototype $H_P(s)$ with a cutoff frequency of 1 rad/s to a highpass filter $H(s)$ with a cutoff frequency of ω_x rad/s, using the *nonlinear* transformation $s \Rightarrow \omega_x/s$. This is illustrated in Figure 13.3.

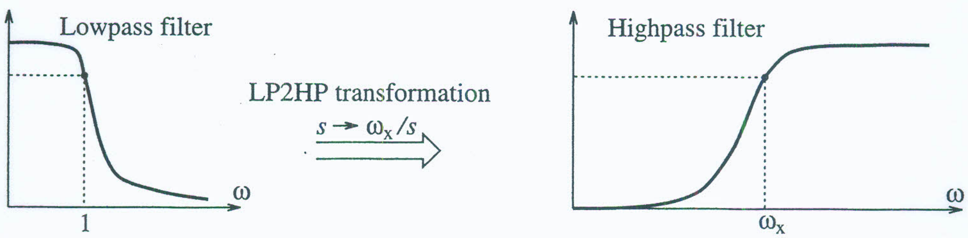


Figure 13.3 The lowpass-to-highpass transformation

The lowpass-to-bandpass (LP2BP) transformation is illustrated in Figure 13.4. It converts a lowpass prototype $H_P(s)$ with a cutoff frequency of 1 rad/s to a bandpass filter $H(s)$ with a center frequency of

ω_0 rad/s and a passband of B rad/s, using the nonlinear, quadratic transformation

$$s \Rightarrow \frac{s^2 + \omega_0^2}{sB} \qquad \omega_{\rm LP} \longrightarrow \frac{\omega_{\rm BP}^2 - \omega_0^2}{\omega_{\rm BP} B} \tag{13.1}$$

Here, ω_0 is the *geometric* mean of the band edges ω_L and ω_H, with $\omega_L \omega_H = \omega_0^2$, and the bandwidth is given by $B = \omega_H - \omega_L$. Any pair of geometrically symmetric bandpass frequencies ω_a and ω_b, with $\omega_a \omega_b = \omega_0^2$, corresponds to the lowpass prototype frequency $(\omega_b - \omega_a)/B$. The lowpass prototype frequency at infinity is mapped to the bandpass origin. This quadratic transformation yields a transfer function with *twice* the order of the lowpass filter.

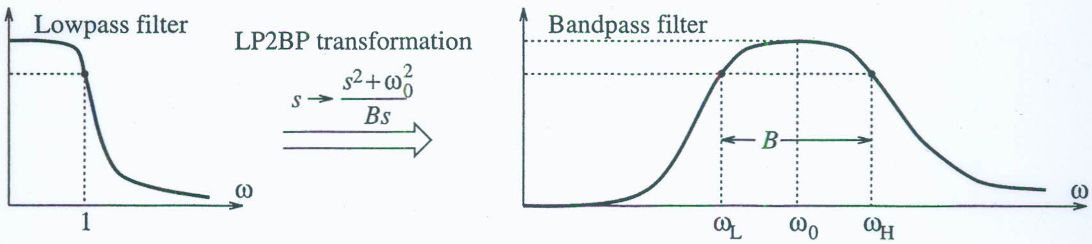


Figure 13.4 The lowpass-to-bandpass transformation

The lowpass-to-bandstop (LP2BS) transformation is illustrated in Figure 13.5. It converts a lowpass prototype $H_P(s)$ with a cutoff frequency of 1 rad/s to a bandstop filter $H(s)$ with a center frequency of ω_0 rad/s and a stopband of B rad/s, using the nonlinear, quadratic transformation

$$s \longrightarrow \frac{sB}{s^2 + \omega_0^2} \qquad \omega_{\rm LP} \longrightarrow \frac{\omega_{\rm BS} B}{\omega_0^2 - \omega_{\rm BS}^2} \tag{13.2}$$

Here $B = \omega_H - \omega_L$ and $\omega_0^2 = \omega_H \omega_L$. The lowpass origin maps to the bandstop frequency ω_0. Since the roles of the passband and the stopband are now reversed, a pair of geometrically symmetric bandstop frequencies ω_a and ω_b, with $\omega_a \omega_b = \omega_x^2$, maps to the lowpass prototype frequency $\omega_{\rm LP} = B/(\omega_b - \omega_a)$. This quadratic transformation also yields a transfer function with *twice* the order of the lowpass filter.

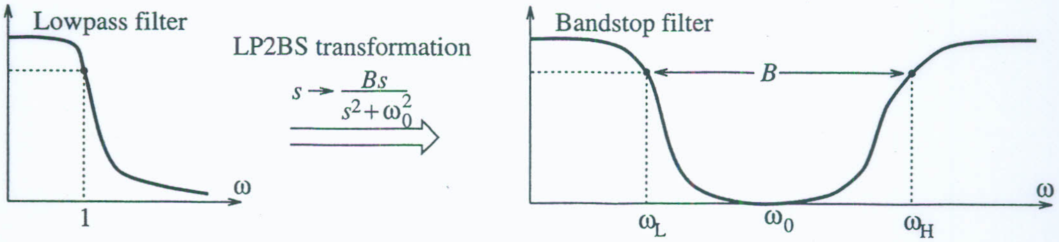


Figure 13.5 The lowpass-to-bandstop transformation

13.1.2 Lowpass Prototype Specifications

Given the frequency specifications of a lowpass filter with band edges ω_p and ω_s, the specifications for a lowpass prototype with a passband edge of 1 rad/s are $\nu_p = 1$ rad/s and $\nu_s = \omega_s/\omega_p$ rad/s. The LP2LP transformation is $s \rightarrow s/\omega_p$. For a lowpass prototype with a stopband edge of 1 rad/s, we would use $\nu_p = \omega_s/\omega_p$ rad/s and $\nu_s = 1$ rad/s instead.

8.6.1 Butterworth Filters

KAMEN + HECK 3RD

For the two-pole system with the transfer function

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$

it follows from the results in Section 8.5 that the system is a lowpass filter when $\zeta \geq 1/\sqrt{2}$. If $\zeta = 1/\sqrt{2}$, the resulting lowpass filter is said to be *maximally flat,* since the variation in the magnitude $|H(\omega)|$ is as small as possible across the passband of the filter. This filter is called the two-pole *Butterworth filter.*

The transfer function of the two-pole Butterworth filter is

$$H(s) = \frac{\omega_n^2}{s^2 + \sqrt{2}\omega_n s + \omega_n^2}$$

$\zeta = \frac{\sqrt{2}}{2}$

$2\zeta = \sqrt{2}$

Factoring the denominator of $H(s)$ reveals that the poles are located at

$$s = -\frac{\omega_n}{\sqrt{2}} \pm j\frac{\omega_n}{\sqrt{2}}$$

Note that the magnitude of each of the poles is equal to ω_n.

Setting $s = j\omega$ in $H(s)$ yields the magnitude function of the two-pole Butterworth filter:

$$|H(\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 2\omega_n^2\omega^2}}$$

$$= \frac{\omega_n^2}{\sqrt{\omega_n^4 - 2\omega_n^2\omega^2 + \omega^4 + 2\omega_n^2\omega^2}}$$

$$= \frac{\omega_n^2}{\sqrt{\omega_n^4 + \omega^4}}$$

$$= \frac{1}{\sqrt{1 + (\omega/\omega_n)^4}} \qquad (8.55)$$

BUTTER 2ND

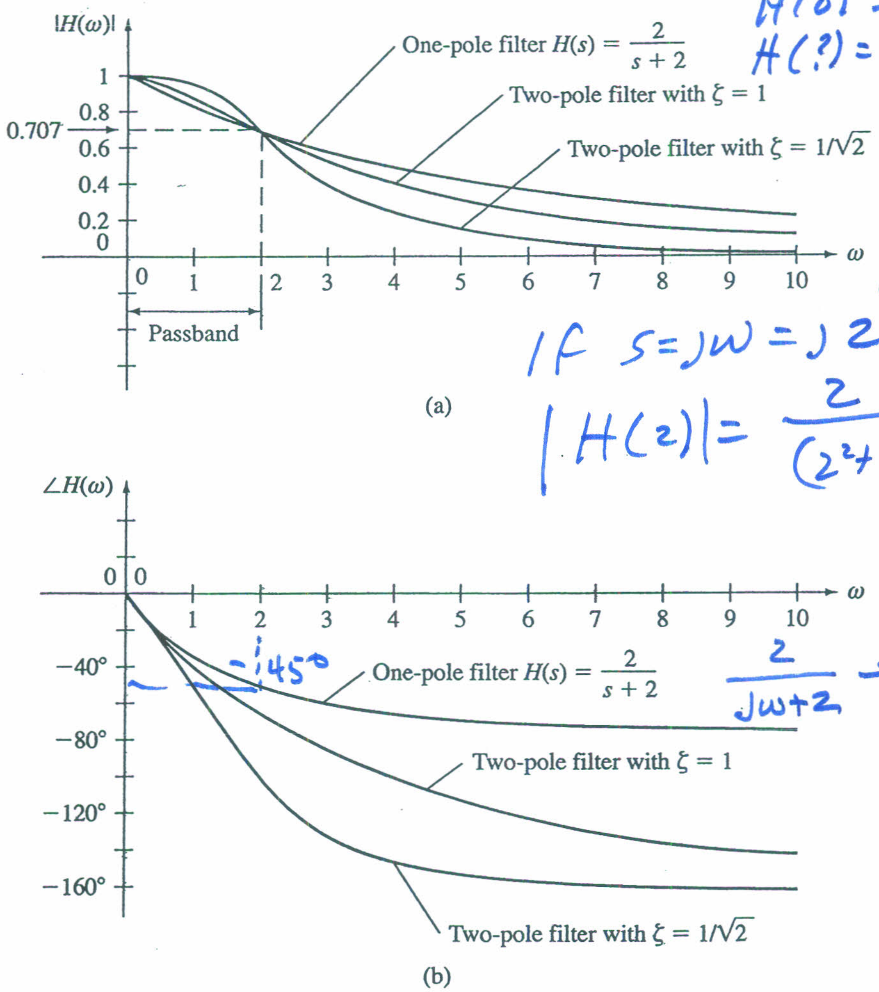
From (8.55) it is seen that the 3-dB bandwidth of the Butterworth filter is equal to ω_n; that is, $|H(\omega_n)|_{\text{dB}} = -3$ dB. For a lowpass filter, the point where $|H(\omega)|_{\text{dB}}$ is down by 3 dB is often referred to as the *cutoff frequency*. Hence, ω_n is the cutoff frequency of the lowpass filter with magnitude function given by (8.55).

For the case $\omega_n = 2$ rad/sec, the frequency response curves of the Butterworth filter are plotted in Figure 8.33. Also displayed are the frequency response curves for the one-pole lowpass filter with transfer function $H(s) = 2/(s + 2)$, and the two-pole

$H(0) = 1$

$H(?) = 1/\sqrt{2}$

NOT BODE



LINEAR

If $s = j\omega = j2$

$|H(2)| = \frac{2}{(2^2+2^2)^{1/2}}$

$\frac{2}{j\omega+2} \rightarrow -\tan^{-1} \omega/2$

FIGURE 8.33
Frequency curves of one- and two-pole lowpass filters: (a) magnitude curves; (b) phase curves.

lowpass filter with $\zeta = 1$ and with cutoff frequency equal to 2 rad/sec. Note that the Butterworth filter has the sharpest transition of all three filters.

***N*-pole Butterworth filter.** For any positive integer N, the *N-pole Butterworth filter* is the lowpass filter of order N with a maximally flat frequency response across the passband. The distinguishing characteristic of the Butterworth filter is that the poles lie on a semicircle in the open left-half plane. The radius of the semicircle is equal to ω_c, where ω_c is the cutoff frequency of the filter. In the third-order case, the poles are as displayed in Figure 8.34.

The transfer function of the three-pole Butterworth filter is

$$H(s) = \frac{\omega_c^3}{(s + \omega_c)(s^2 + \omega_c s + \omega_c^2)} = \frac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}$$