Z plane & S to Z and H(z) 4/15

Introduction to Z-Transforms We now turn to a transform that is called the Z*-transform* due to the use of the complex variable $z = x + iy$ in the transform. The Z-transform plays the same role for discrete systems as the Laplace transform does for continuous systems. The table summarizes some of the applications of the Z- transform.

Definition	The Z-transform is defined as a sum that transforms discrete signals to the complex frequency (Z) domain.
System analysis	The Z-transform converts convolutions to a product and difference equations to algebraic equations.
Stability	Stability of a discrete linear system can be determined by analyzing the transfer function $H(z)$ given by the Z-transform.
Frequency response	The transfer function $H(z)$ can be evaluated to determine the frequency response of a discrete system.
Digital filters	Digital filters can be analyzed and designed using the Z-transform.
Control	Digital control systems can be analyzed and designed using Z-transforms.

We shall see that there is a close connection between the Z and Laplace transforms.

The Z-plane and the s-plane Laplace and Z-Transforms In this section, the term *sampling* refers to replacement of a function $f(t)$ by the function $f(nT)$. Here we consider *sampled-data* systems in which input and output functions are considered at only discrete values of t, usually at values $nT, n = 0, 1, 2, \ldots$, where T is a positive constant.

By sampling the continuous function $f(t)$ at every T_s seconds, we obtain the discrete function $f_d(t)$ with values defined at $t = 0, T_s, 2T_s, \ldots$. This discrete function can be written in terms of the unit impulse function

$$\delta(t - nT_s) = \begin{cases} 1, & t = nT_s \\ 0, & t \neq nT_s \end{cases} \tag{1}$$

where $n = 0, 1, \ldots$ in the following manner:

$$f_d(t) = \sum_{n=0}^{\infty} f(nT_s)\delta(t - nT_s). \tag{2}$$

The discrete time function $f_d(t)$ has as its Laplace transform

$$\begin{aligned} \mathcal{L}[f_d(t)] &= \int_0^{\infty} \sum_{n=0}^{\infty} f(nT_s)\delta(t - nT_s)e^{-st}\,dt \\ &= \sum_{n=0}^{\infty} f(nT_s)e^{-nT_s s} \qquad (3) \end{aligned}$$

using the definition of the one-sided Laplace transform and the properties of the impulse function.

Defining a new complex variable

$$z = e^{T_s s}$$

leads to the definition of the Z-transform as the Laplace transform of the discrete function $f(nT_s)$

$$\mathcal{L}[f(nT_s)] = \sum_{n=0}^{\infty} f(nT_s)z^{-n} = F(z). \tag{4}$$

ZtoS From the relationship

$$z = e^{sT_s} = e^{(\sigma+i\omega)T_s} = e^{\sigma T_s}e^{i\omega T_s} \tag{5}$$

we map the s-plane into the z-plane. The $i\omega$ axis maps into the unit circle

$$z = e^{\imath\omega T_s}$$

which has magniude $|z| = 1$. The values of $i\omega T_s$ determine the position on the circle. As the argument increases in the positive direction , points on the circle wrap around starting at $z = 1$ when $\omega T_s = 0$ corresponding to $\omega = 0$ in the s-plane. At the angle $\omega T_s = \pi$, $z = -1$. The region of the $j\omega$ in the s-plane axis from $\omega = 0$ to $-\omega T_s = -\pi$ map to the lower half of the unit circle and again $z = -1$ when $-\omega T_s = -\pi$.

In terms of sampling theory, the limits used to preserve the uniqueness of the mapping correspond to the Nyquist frequencies $\omega_s = \pm\pi/T_s$.

Damping and the z-plane The left-hand side of the s-plane, for values $s = \sigma + i\omega$ with $\sigma < 0$ and $|\omega| < \pi/T_s$ maps into the interior of the unit circle in the z-plane. Since poles in the left-hand s-plane correspond to a BIBO stable continuous system, the corresponding poles for stable discrete systems must lie within the unit circle in the z-plane. Note that the negative real axis in the s-plane maps into the real axis from 0 to 1 in the z-plane. Thus, a digital system with a pole at -0.5, for example, has no corresponding continuous system. (Shahian p 263).

If $\sigma > 0$, the points in the right-hand s-plane map to the exterior of the unit circle in the z-plane.

Vertical lines in the s-plane such that $\pi/T_s \leq \omega \leq \pi/T_s$ and $\sigma < 0$, map into a circle in the z-plane centered at $z = 0$ with radius $r = \exp(\sigma T_s)$.

Mapping the s-plane to the z-plane Consider the s-domain function

$$G(s) = \frac{1}{(s+1)(s+2)(s^2+1)}$$

with poles at $s = -1, -2, \pm i$. For $T_s = 1$, the poles in the z-plane given by $\exp(sT_s)$ appear at

$$z = 0.3679, 0.1353, 0.5403 + 0.8415i, 0.5403 - 0.8415i$$

as computed by the MATLAB script below and shown in Figure **??**.

The inverse Laplace transform of $G(s)$ leads to time functions such as e^{-t},e^{-2t} and $e^{\pm it}$ or $\cos t$ and $\sin t$. Thus, the oscillations have frequency $f = 1/2\pi$ Hertz or 1 rad/sec. With $T_s = 1$, $\omega_s = 2\pi$ and the maximum digital frequencies are $F = .5$ or $\Omega = \pi$ radians.

In the z-plane, the pole at $0.5403 + 0.8415i$ has magnitude 1 since it lies on the unit circle and angle

$$\theta_z = \tan^{-1}\frac{0.8415}{0.5403} = 1 \text{ radian.}$$

With $T_s = 1$, the maximum digital frequency $\Omega = \pi$ rad occurs at the point $z = -1$.

If $T_s = 0.1$, the poles in the z plane are changed as indicated in the results of the MATLAB calculation. The maximum digital frequency is $F = 5$ or $\Omega = 10\pi$ radians. The angle of the pole $z = 0.9950 + 0.0998i$ is $\theta_z = 0.1000$ radians as expected.

```
%s2zplane.m
% Plot z-plane poles for G(s)=1/[(s+1)(s+2)(s^2+1)
% See Taylor p252
%
% Let Ts=1.0
Ts=1.0
poless=[-1 -2 +i -i]
polesz=exp(poless*Ts)
%
% Define zeros and poles as column vectors
zplane(polesz')      % There are no zeros
title('Z-plane for s-plane poles -1,-2,+1=-i')
grid
%
% Results
%
%Ts =  1
%poless =-1.0000    -2.0000      0 + 1.0000i        0 - 1.0000i
%polesz =0.3679      0.1353 0.5403 + 0.8415i   0.5403 - 0.8415i
%
```

```
% Change sampling time
%
%Ts1 = 0.1000
Ts1=0.1
poless1=[-1 -2 +i -i]
polesz1=exp(poless1*Ts1)
% poless1 =-1.0000  -2.0000       0 + 1.0000i        0 - 1.0000i
% polesz1 = 0.9048   0.8187   0.9950 + 0.0998i   0.9950 - 0.0998i
```

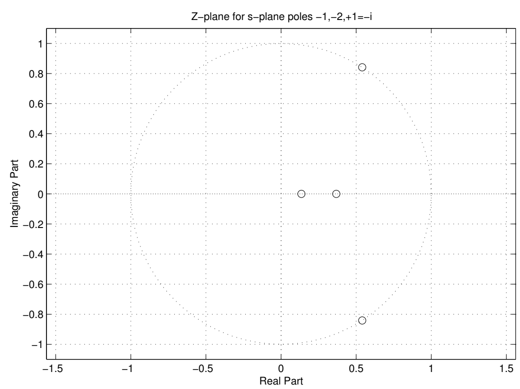

Figure 1: z-plane poles from s-plane

Second-order systems Consider the second-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with poles at

$$s_1 = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \qquad s_2 = -\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2}.$$

The term $\sigma = -\zeta\omega_n$ is the real part of a pole in the s-domain that corresponds to damping of the time response of the system $G(s)$. The pole in the z-plane lies on a circle centered at $z = 0$ with radius

$$|z| = \exp(-\zeta\omega_n T_s).$$

Z-transforms and Frequency Response We wish to determine the frequency response from the transfer function given by the Z-transform of $y(n)$.

If the transfer function $H(z)$ is evaluated for values of

$$z = \exp(i2\pi F) = \exp(i\Omega) \tag{6}$$

we obtain the *frequency response*, $H(i2\pi F)$, of the system. This is equivalent to evaluating $H(z)$ on the unit circle in the z-plane. Note that the function

$H(i2\pi F)$ is periodic with period 1 since $\exp(i2\pi F)$ is periodic with period 1. The *digital frequency* is defined as

$$F = \frac{f}{f_s} = fT_s,$$

where T_s is the sampling time or time between samples. The sampling rate $S = f_s = 1/T_s$ samples per second is often quoted in Hz since then it is easier to compare to the analog frequency given in Hertz.

The analog frequency $f = 1/T_s$ corresponds to the digital frequency $F = 1$ or $\Omega = 2\pi$. However considering sampling theory, the range of F is limited as

$$-0.5 \leq F \leq 0.5 \text{ and } -\pi \leq \Omega \leq \pi$$

since $F = 0.5$ indicates that the sampling rate is twice the analog frequency being sampled. From Equation 6, we have the following correspondences

f=dc	$F = 0$	$z = 1$
$f = S/4$	$F = 1/4$	$z = i$
$f = S/2$	$F = 1/2$	$z = -1$

Note for the equation

$$H(z) = \frac{0.1z}{z - 0.9} \tag{7}$$

for example,

$$H(z = 1) = 0.1(1)/0.1 = 1$$

is the attenuation at the dc value of $f = 0$. Notice that S is not relevant here. However, if $f = S/2$, the result becomes

$$H(z = -1) = \frac{0.1(-1)}{-1 - 0.9} = 0.0526$$

which indicates a low-pass filter characteristic. Writing Equation 7 as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.1}{1 - 0.9z^{-1}},$$

the difference equation becomes

$$y[n] = 0.9\, y[n-1] + 0.1x[n].$$

Check this with $x[n] = 1$ for $n \geq 0$ comparing the step response of $H(z)$ as

$$y[n] = 1 - (0.9)^{n+1} \tag{8}$$

with the difference equation solution Equation 8 for various values of n as for example

$$y[0] = 0.9y[-1] + 0.1x[0] = 0.1 \tag{9}$$
$$y[1] = 0.9[0.1] + 0.1 = 0.19 \tag{10}$$