Outline Sampling

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Introduction to Sampling An analog signal $f(t)$ must be *sampled* before it can be processed by a digital system. It may be hard to believe that $f(t)$ can be sampled to yield an accurate representation of the signal. However, there are strict conditions for sampling that must be recognized when sampling an analog signal. In the following, it is assumed that $f(t)$ is a signal in time that must be sampled. The sampled signal is called a *digitized* signal.

In general, an analog signal will contain a number of frequency components given by a Fourier analysis of the signal. Only a pure sinusoid contains one frequency. The highest frequency in the signal of interest determines the sampling rate.

A few definitions and theorems are given in the table for sampling a signal $f(t)$:

T_s	The *sampling interval* is the time in seconds between samples of $f(t)$.
$F_s = S$	The sampling frequency or rate in samples per second (Hertz).
$f_{\max}$	The highest frequency in the signal in Hertz.
$F = f/S$	The *digital frequency* as the ratio of an analog frequency divided by the sampling frequency. The range of F is $-0.5S \leq F \leq 0.5S$.
Sampling theorem	The analog signal must be sampled at $S > 2f_{\max}$ to avoid *aliasing*. This is called the *Nyquist rate*.
Aliasing	If $S < 2f_{\max}$, the analog signal is *aliased* to a lower frequency .

We shall see that ideal sampling in time results in a *periodic* function of time in both time and frequency.

The Result of Sampling in Time In this section, the term *sampling* refers to replacement of a function $f(t)$ by the function $f(nT_s)$. Here we consider *sampled-data* systems in which input and output functions are considered at only discrete values of t, usually at values $nT_s, n = 0, 1, 2, \ldots$, where T_s is a positive time in seconds.

By sampling the continuous function $f(t)$ at every T_s seconds, we obtain the discrete sequency of points $f_d(t)$ with values defined at $t = 0, T_s, 2T_s, \ldots$. This discrete sequence can be written in terms of the unit impulse function

$$\delta(t - nT_s) = \begin{cases} 1, & t = nT_s \\ 0, & t \neq nT_s \end{cases} \tag{1}$$

where $n = 0, 1, \ldots$ in the following manner:

$$f_d(t) = \sum_{n=0}^{\infty} f(nT_s)\delta(t - nT_s). \tag{2}$$

See Harman Page 481.

1 In the frequency domain, the spectrum of $f(t)$ is periodic with period

$$S = \frac{1}{T_s} \text{ Hertz}$$

2 If the sampling rate $S < 2f_{\text{analog}}$, the signal is aliased to a lower frequency

$$|f_{\text{aliased}}| < 0.5S$$

1. The aliased frequencies are many and have values

$$f_{\text{aliased}} = f_{\text{analog}} - MS \text{ where } -0.5S < f_{\text{aliased}} \leq 0.5S.$$

Example - not Aliased
A 100 Hz sinusoid is sampled at S=240 Hertz. Here

$$1/S = T_s = 1/240 = 0.0042 \text{ sec}$$

so the valid range of positive frequencis goes to 120 Hertz. Thus,

$$f_d(t) = \sum_{n=0}^{\infty} \cos(2\pi 100\, nT_s)\, \delta(t-nT_s) = \sum_{n=0}^{\infty} \cos(2\pi n \frac{100}{240})\, \delta(t-nT_s). \quad (3)$$

Since $F = 5/12 < 0.5$, the result is a sampled sinusoid with frequency $f = (5/12) \times 240 = 100$ Hz. Note that the number of samples is infinite! In theory, using the sampled points, the cosine wave could be exactly reconstructed. In practice, the A/D converter would only sample N points and reconstruction by a D/A converter would only use N points. This would lead to some inaccuracies but *aliasing* would not be the cause of them.

Example - Aliased
Consider the sinusoid again

$$f(t) = \cos(200\pi t + \theta).$$

The sampling frequency must be $S > 200$ Hz to avoid aliasing. Suppose that

$$S = 140 \text{ Hz so that } F = 100/140 = 0.71.$$

Then, the spectrun is aliased at frequencies

$$f_{\text{aliased}} = 100 - M \times 140 \text{ Hz}$$

where $M = 1, 2, \ldots$ as long as the aliased frequencies fall within the range $(-70, 70)$ Hz. The only one in the range is

$$f_{\text{aliased}} = 100 - 140 = -40\text{Hz}.$$

The signal represented by this sampling is

$$\cos(-80\pi t + \theta) = \cos(80\pi t - \theta)$$

which is in error. Note the phase reversal which is the result of a negative frequency.

When a sampled signal has a finite range of frequencies, it may be easy to determine the highest frequency and sample at twice that frequency. Alternatively, the signal may be filtered to limit the frequency range as described in Harman Page 523. An *anti-aliasing* low-pass filter restricts the freqencies in a signal by attenuating any frequencies above the cutoff of the filter. Also, only a finite number of points can be sampled in practice.
Thus the questions for practical sampling could be posed as follows:

Sampling of a signal Three of the most important questions in the specification of a data acquisition system such as that shown in Figure 11.7, Page 523, are the following:

1. How often should the analog signal be sampled -How to choose T_s.
2. How long should the signal be sampled - How to choose N as the number of points?
3. What should be done if the frequency range of the signal is not limited?

See Harman Page 524 for answers to the first two questions. With proper sampling, the frequency spectrum of the sampled signal has the following characteristics using the DFT to compute the spectrum:

1. T_s determines the highest frequency in the spectrum as
$$F_{\text{max}} = \frac{S}{2} = \frac{1}{2T_s} \text{ Hz.}$$
2. The resolution in frequency is determined by the number of points chosen as
$$F_1 = \Delta F = \frac{1}{NT_s} \ (F_{\text{DFT}}).$$
The frequencies in the DFT are thus
$$(F_{\text{dc}}, F_1, 2F_1, 3F_1, \ldots, F_{\text{max}} = \frac{N}{2}\Delta F)$$
3. What should be done if the frequency range of the signal is not limited?

Example Harman Page 526
Consider an analog signal with frequencies of interest up to 1200 hertz. Thus, the signal should be filtered so that $f_{\text{max}} = 1200$ hertz. This filtering removes frequencies in the signal above 1200 hertz and *noise* above f_{max} hertz. The noise consists of unwanted signals added to the desired signal.

By the sampling theorem, the sampling interval in time must be
$$T_s < \frac{1}{2f_{\text{max}}} = \frac{1}{2400} \quad \text{seconds,}$$
so that at least 2400 samples per second are needed.

Further, suppose that the desired digital frequency resolution is to be at least $\Delta F = 0.5$ Hertz or less.

For a resolution of 0.5 hertz,

$$\Delta F = \frac{1}{NT_s} = 0.5\text{Hz}.$$

The total time of sampling is thus $NT_s = T = 1/0.5 = 2$ seconds or longer. The total number of points required is thus

$$N = \frac{T}{T_s} = \frac{2}{(2400)^{-1}} = 4800.$$

If N is to be a power of 2 for the FFT algorithm, $2^{13} = 8192$ samples would be taken. With the same sampling rate, the resolution would now be $\Delta F = 0.2928$ Hz.

The sampling rate could be increased to 4096 samples per second, which is sampling at a rate corresponding to about 3.4 times the highest frequency of interest. The resolution would remain the same since $8192 \times 1/4096 = 2$ sec of sampling time.

What should be done if the frequency range of the signal is not limited?

In practice, there is some upper limit to the frequency spectrum of any real signal. The solution is to sample fast enough that the aliased frequencies are so small that they do not effect the results.

Consider the square wave (Harman P 379) of period T seconds. In the Fourier series there is no dc term and the fundamental frequency is $f_1 = 1/T$ Hz. The square wave is often described as a $1/T$ Hertz square wave, although $1/T$ Hz only represents the fundamental frequency.

The sine waves that make up the Fourier series for an odd square wave are

$$f(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{\sin(3\omega_0 t)}{3} + \cdots\right],$$

where $\omega_0 = 2\pi/T$, so the series consists not only of sine terms, as expected, but also odd harmonics appear. For example, if $T = 1$ millisecond, the square wave is often called 1 kHz square wave and the frequencies in the wave are

$$1, 3, 5, 7, 9, 11, 13, 15, 17, \ldots \text{Hertz}$$

Thus, there is no sampling rate to avoid aliasing. However, the n^{th} harmonic is reduced by $1/n$ compared to the amplitude of f_1. For the 1 kHz square wave, the amplitude of the 21^{st} harmonic is less than 5% of the amplitude of the fundamental. Sampling at $S = 42$ kHz would keep the aliasing error below 5%.

To accurately sample the wave so that aliasing would not occur with the 65^{th} would require sampling at $130,000$ samples per second!

Fourier Transform Example The Fourier spectrum of

$$f(t) = \begin{cases} 0, & t < 0, \\ e^{-t}, & t \geq 0. \end{cases}$$

is

$$|F(i\omega)| = \frac{1}{\sqrt{1+\omega^2}},$$

as shown in Example 8.9, Harman page 397, with $A = 1$ and $\alpha = 1$. The Figure shows the plot of $f(t)$ and $|F(i\omega)|$ created by the accompanying MATLAB script. The spectrum is always symmetric around $\omega = 0$ when $f(t)$ is real.

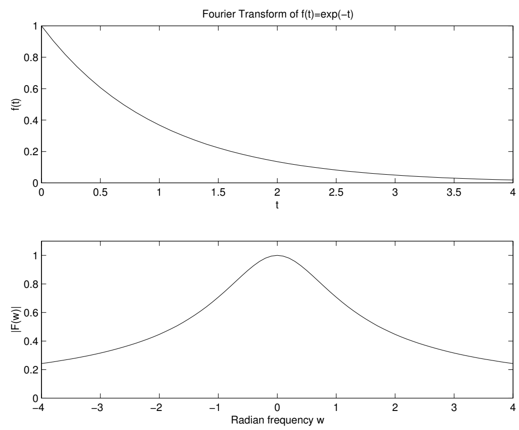

Figure 1: Time signal and Fourier Transform

Since the spectrum has no "largest value", we must approximate the results. A reasonable approach is to consider the frequency values to be zero when they are diminished below some percentage of the maximum value.

Suppose the maximum frequency is considered to be 5% of the maximum of the spectrum. Thus, the sampling frequency S should be twice the frequency at $0.5S$ found as

$$|F(2\pi 0.5S)| = \frac{1}{\sqrt{1+(2\pi 0.5S)^2}} = 0.05\,|F(0)| = .05 \times 1 = 1/20.$$

The solution for S yields $1 + \pi^2 S^2 = 400$ or $S \geq 7$ Hz. This is $\omega_s = 44$ rad/sec. At $\omega_{\max} = 22$ rad/sec, $|F(i\omega)| = 1/22 < .05$ as required.